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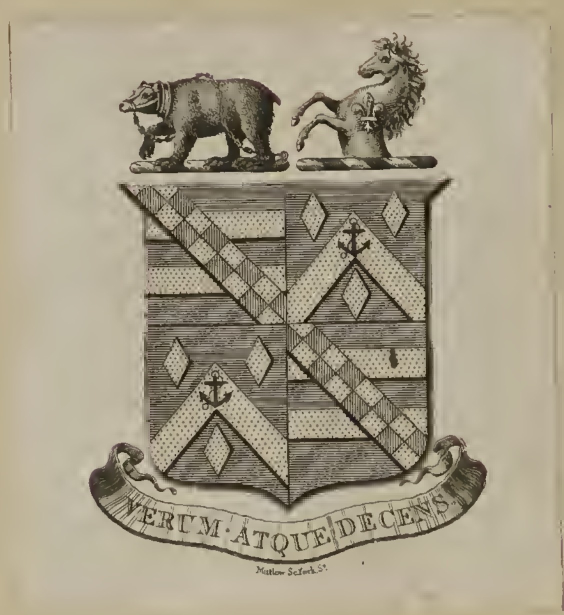

2 vols 3/-

In 2. Vol. $\frac{2.2}{2.3}$
Feb. 12. 17
Vol. 1. Collat. & perfect.
P. Wright

THE ELEMENTS OF THAT Mathematical Art COMMONLY CALLED ALGEBRA, Expounded in Four BOOKS.

By *JOHN KERSEY.*

Nil tam difficile eſt, quod non ſolertia vincat.
Dimidium facti, qui benè cœpit, habet.

LONDON:

Printed by WILLIAM GODBID, for *Thomas Paſſinger* at the ſign of the Three Bibles on *London*-Bridge, and *Benjamin Hurlock* over-againſt St. *Magnus* Church, near *London*-Bridge. M.DC.LXXIII.

TO

ALEXANDER DENTON

Of *Hillesdon* in the County of *Bucks*, Esquire,

AND

Mr EDMUND DENTON

His Brother;

The hopeful Blossoms, and only Offspring of the Truly Just and Vertuous

EDMUND DENTON Esq;

Son and Heir of

Sr *ALEXANDER DENTON* Knt.

A faithful Patriot, and eminent Sufferer in our late Intestine Wars, for his Loyalty to His late MAJESTY

King CHARLES the First,

Of Ever-Blessed Memory:

JOHN KERSEY,

In testimony of his Gratitude, for signal Favours conferr'd on him by that truly Noble Family;

Which also gave both Birth and Nourishment to his *Mathematical* Studies,

HUMBLY DEDICATES

His Labours in this Treatise of the ELEMENTS OF THE

ALGEBRAICAL ART.

THE PREFACE.

IT is an undoubted truth, that among all Humane Arts and Sciences, *ARITHMETICK* and *GEOMETRY* have obtain'd the greateſt evidence of Certainty. This Prerogative reſults from the Verity and Perſpicuity of their Principles, which conſiſt of *Definitions*, *Poſtulates*, (or *Petitions*,) and *Axioms*; for theſe being intelligible, reaſonable and certain, are univerſally aſſented to as pure Fountains of Knowledge, and ſure Foundations of right Reaſoning, by all judicious and impartial Students in Sciences. Hence it is, that all Propoſitions which are proved by thoſe certain Principles are likewiſe certain, and called Demonſtrative Truths, by which are meant ſtrictly and properly, infallible Conſequences, or Concluſions, deduced from clear and undeniable Premiſſes. For which cauſe, divers Philoſophers have endeavour'd, as far as the quality of their Diſcourſes would admit, to make the force of their Arguments amount to Mathematical Demonſtration, which, by univerſal conſent of the Learned, is the cleareſt and moſt convincing Proof of the Truth of a Propoſition, that can poſſibly be given by Humane reaſoning. Nor was it without Reaſon, that the Ancients, (as many of the Learned affirm,) taught their Scholars *Arithmetick* and *Geometry*, next after the Rudiments of Letters, as expedients to take off their minds from Levity, and to render them capable of ſound Judgement, before they entred upon the Study of Philoſophy: Which Method of Schooling was in great eſteem with *Plato*, (as his Book of *Common-weal* teſtifies;) who was of Opinion, That ingenious and pregnant Proficients in *Arithmetick* were apt to learn any Arts whatſoever, and he permitted no Student that was ignorant of *Geometry* to enter into his School.

This alſo may be added concerning the Excellency of thoſe Twin-like Arts or Sciences, That they depend not upon any other Sciences, either for Help or Demonſtration; nor do they owe their Dignity to the Suffrage or Vote of our Senſes, which oft-times deceive us; but ſince *Quantity*, about which *Arithmetick* and *Geometry* are converſant, may be conſider'd abſtractively, and ſeparate from all kind of Matter, the Verity of their Propoſitions is examined and proved in the Mind only; where, among all the Exerciſes that conduce to the ſearch of Truth, none are found ſo pure, clear and comprehenſible, (right Reaſon being Judge,) as *Arithmetick* and *Geometry*; thence they are called Pure Mathematicks, and are properly to be learnt before any of the reſt of the Mathematical Arts.

Nor are *Arithmetick* and *Geometry* excellent in themſelves only, but highly eſteem'd alſo for their manifold Utility, as well in the Employments of Men about Accompts, Trade, Building, Meaſuring of Land, and divers other common Affairs, as in facilitating and enlivening divers other Noble Arts; for how can Harmonical Compoſition in *Muſick*, or exact Meaſure and Proportion in *Painting* be perform'd, without the aſſiſtance of *Arithmetick* and *Geometry*. Beſides, theſe Sciences, (as the Mathematician very

well knows,) like the two Pillars, *Jachim* and *Booz*, in the Porch of *Solomon*'s Temple, are the ſtability and ſupport of all the reſt of the Mathematical Arts; for if *Aſtronomy*, *Navigation*, *Dyalling*, *Opticks*, *Fortification*, and the reſt of the Arts called Mixed Mathematicks, be ſtript of the Demonſtrations and Operations imparted to them by *Geometry* and *Arithmetick*, that which remains will be as barren as the Earth without the Influence of the Sun, and as unactive as a humane Body without a reaſonable Soul.

The premiſſes may ſuffice to give a hint of the Excellency and Utility of *Arithmetick* and *Geometry*, whence we may reaſonably inferr; Firſt, that ſo great and ſo profitable a Subject is worthy of the Study of all ingenious Minds, in a degree proportional to their reſpective Stations or Employments, as well for promoting their own, as the Publick Good. Secondly, that that Art which by a more eaſie, and not leſs ſure Method than that called *Synthetick*, finds out the Solutions and Demonſtrations of the more knotty Propoſitions, as well *Geometrical* as *Arithmetical*; (and oftentimes by the way too, diſcovers unexpected and admirable Speculations,) may very well deſerve the Enquiry of ſuch Lovers of Art as have hours to ſpare, and are deſirous to be acquainted with the choiceſt pieces in the Common-wealth of Learning: But ſuch an Art is that commonly called *ALGEBRA*, which firſt aſſumes the Quantity ſought, whether it be a Number or a Line in a *Queſtion*, as if it were known, and then, with the help of one or more Quantities given, proceeds by undeniable Conſequences, until that Quantity which at firſt was but aſſumed or ſuppoſed to be known, is found equal to ſome Quantity certainly known, and is therefore known alſo.

Which Analytical way of Reaſoning produceth in Concluſion, either a *Theorem* declaring ſome Property, Proportion or Equality, juſtly inferr'd from things given or granted in a *Propoſition*, or elſe a *Canon* directing infallibly how that may be found out or done which is deſired; and diſcovers Demonſtrations of the certainty of the reſulting *Theorem* or *Canon*, in the *Synthetical* Method, or way of Compoſition, by the Steps of the *Analyſis*, or *Reſolution*. Theſe are but glances of the many Rare Effects produced by the *Analytick* or *Algebraick* Art, which is an inexhauſtible Fountain of *Theorems*, a Key truly golden for the unlocking of *Problems* as well *Geometrical* as *Arithmetical*; and not only a ſure, but delightful Guide to ſuch Students, who not being ſatisfied with a bare knowledge of the Truth or practical Uſe of thoſe ſublime Inventions that have rendred the antient Mathematicians ſo venerable, are deſirous to know how they were found out, and how to proſecute their ſearch of Truth, ſo, as to advance Knowledge upon Solid Foundations.

But the Excellency of the *Algebraical* Art is beſt known to thoſe that are acquainted with the moſt eminent Writers upon that Subject; among which, theſe are deſervedly Famous, namely, *Diophantus* of *Alexandria*, (the firſt Inventor of this rare Art, as ſome by his Preface to *Dionyſius* do conjecture; but others give the Honour of that Noble Invention to *Geber* an Arabian Aſtronomer, whence, as is conceived, the word *Algebra* took riſe,) *Cardanus, Tartaglia, Clavius, Stevinus, Vieta*, (the firſt Inventor, or at leaſt the happy Reſtorer of *Specious*, or *Literal Algebra*, ſo called, becauſe it operates chiefly by Alphabetical Letters,) Mr. *William Oughtred*, (our learned Countrey-man,) whoſe *Clavis Mathematicæ*, for Solid matter,

neat

neat Contractions, and succinct Demonstrations, is hardly to be parallel'd,) Mr. *Thomas Harriot*, (another learned Mathematician of our Nation,) *Ghetaldus*, *Andersonus*, *Bachetus*, *Herigonius*, *Cartesius*, *Fran. van. Schooten*, *Florimond de Beaune*, *Hugenius*, *Huddenius*, *Slusius*, *Fermatius*, *Billius*, *Rhenaldinus*, and many others too numerous to be here recited; but to bring up the Rear of these renowned Analysts, I shall mention four more of our own Nation, and now living, (whose pardon I humbly begg for this my boldness,) namely, the Right Reverend Father in God, *Seth*, Lord Bishop of *Sarum*, Dr. *John Wallis*, Professor of *Geometry* in the University of *Oxford*, Dr. *Isaac Barrow* Master of *Trinity*-Colledge in *Cambridge*, and one of His Majesties Chaplains, and Dr. *John Pell*; the learned Works of which four Worthies proclaim their rare Talents in Universal Mathematicks.

Now because this excellent Art is but very sparingly treated of in our native Language, and since according to the old Maxim, *Bonum quò communius eò melius*, Good the more common the better it is; I have, in imitation of the industrious Bee that gathers Honey from various Flowers, yet without any diminution either of their Beauties or Virtues, extracted out of the before-mentioned Authors, this Tractate consisting of Four Books, (the Two first of which are Printed, a good progress made in the Third, and the Fourth ready for the Press,) and have design'd it chiefly to give such of my Mathematical Countrey-men as are altogether strangers to, and desirous to be acquainted with the so much celebrated Art called *Algebra*, a plain and intelligible Introduction to its Doctrine, as also a considerable taste of its Use, in finding out *Theorems* and solving *Problems*, as well *Arithmetical* as *Geometrical*.

And here, to avoid the stain of Ingratitude, I cannot but declare to the World, that my old and much respected Friend, Mr. *John Collins*, a person well known to be both singularly skilfull in, and an industrious Promoter of the Mathematicks in general, hath been a principal Instrument of bringing this Work to light, as well by animating me to Compile it, as by endeavouring to procure it to be well Printed.

To conclude, I have earnestly endeavoured to render the Fundamentals, and most important Rules of the *Algebraical* Art in both kinds, to wit, Numeral and Literal, very clear and easie to capacities competently exercis'd in the Elements of *Arithmetick* and *Geometry*. And the favourable acceptance, which my Additions to Mr. *Wingat*'s Treatise of Common *Arithmetick* have found, with divers eminent Mathematicians and other Lovers of Art, doth encourage me to hope, that the younger Students of Symbolical *Arithmetick* and *Analytical* Doctrine, will be well pleas'd with the following Discourse, and that my Labours therein will be as candidly accepted, as they have been cordialy intended to serve my Native Countrey.

From my House at the Sign of the Globe *in* Shandois-*Street*, *in* Covent-garden, *the* 15th *day of* April, 1673.

John Kersey.

A TABLE OF The Contents of the First and Second BOOKS of this TREATISE.

The Contents of the First Book.

The

The CONTENTS.

The Contents of the Second Book.

ERRATA.

ERRATA.

Such hath been the exact care of the Printer, *that the Faults of* importance *escaped in this* Impreſſion *of the First and Second* Books *are only these fourteen.*

Page.	*Line.*	Faults,	thus to be corrected.
25	13	— 3*ddde*	— 3*ddee*
31	16	(By $a+b$)	(By $a-b$)
52	7	whole	wholes, or totals.
64	22	19	9
64	32	not exceed	be leſs than
77	37	Squares the Terms	Squares of the Terms
114	23	*Sect.* 12. *Chap.* 2, 3, 5.	*Sect.* 2, 3, 5. *Chap.* 12.
147 152	34 6	a ſingle Character	the greateſt ſingle Character
227	4	ſecond	firſt
253	37	Pronportionals	Proportionals
281	20	*Chap.* 15.	*Chap.* 16.
300	17	$4aa \left\{ \begin{array}{l} \sqsupset\ 40-\frac{36}{100} \\ \sqsubset\ 10 \end{array} \right.$	$4aa \left\{ \begin{array}{l} \sqsupset\ 40 \\ \sqsubset\ 10-\frac{36}{100} \end{array} \right.$
317	34	ſeventh	ſeventeenth

THE

A Treatise of the

ELEMENTS

OF THE

Algebraical Art.

BOOK I.

CHAP. I.

Concerning the Nature, Scope, and Kinds of ALGEBRA: *The Construction of* Cossick *Quantities, or* Powers; *with the manner of expressing them by Alphabetical Letters: The signification of* Characters *used in the First Book.*

I. THE Mathematical Arts or Sciences are exercis'd about *Quantity*, which is compris'd under Numbers, Lines, Superficies, and Solids: These, if they be considered abstractively, and separate from all kind of Matter, are the proper Objects of *Arithmetick* and *Geometry*, which are called *Pure Mathematicks*.

II. The *Method* which Mathematicians are wont to use in searching out truth about Quantity, is twofold; *viz.* 1. *Synthetical*, or by way of Composition: 2. *Analytical*, or by way of Resolution.

III. Mathematical Composition, or the *Synthetical method*, argues altogether with known Quantities to search out unknown; and then demonstrates that the Quantity found out will satisfie the Proposition.

IV. Mathematical Resolution, or the *Analytical Art*, commonly called *Algebra*, is that way of reasoning which assumes or takes the Quantity sought as if it were known or granted; and then with the help of one or more Quantities given or known, proceeds by Consequences, until at length the Quantity first only assumed or feigned to be known, is found equal to some Quantity or Quantities certainly known, and is therefore likewise known.

V. The Scope, Drift or Office of the Analytick or Algebraick Art, is to search out three kinds of Truths, *viz.*

1. *Theorems*; which are nothing else but Declarations, or Affirmations of certain Properties, Proportions, or Equalities, justly inferr'd from some Suppositions or Concessions about Quantity: Which Theorems are to be reserved in store, as ready helps to find out new, and to confirm old Truths. This kind of Resolution when it rests in a bare Invention of Truth, is called *Contemplative*, or *Notional*.

2. *Canons*, or infallible Rules, to direct how to ſolve knotty Queſtions, by the help of Quantities given or known; this kind of Reſolution is called *Problematical.*

3. *Demonſtrations*, or evident and indubitable Proofs, to manifeſt the truth of ſuch Theorems and Canons as are Analytically found out.

VI. *Algebra* is by late Writers divided into two kinds; to wit, *Numeral*, and *Literal*, (or *Specious.*)

VII. *Numeral Algebra* is ſo called, becauſe in this Method of reſolving a Queſtion, the Quantity ſought or unknown is ſolely deſign'd or repreſented by ſome Alphabetical Letter, or other Character taken at pleaſure, but all the Quantities given are expreſt by Numbers.

VIII. *Literal*, or *Specious Algebra* is ſo called, becauſe in this method of reſolving a Queſtion, as well the given or known Quantities as the unknown are all ſeverally expreſſed or repreſented by Alphabetical Letters. Whence it comes to paſs, that at the end of the Reſolution of a Queſtion, every Quantity appearing diſtinct under the ſame Letter or Form by which it was at firſt expreſſed, a *Canon* is diſcovered to direct how the Queſtion propos'd may be ſolved, not only by the quantities firſt given, but by any other whatſoever that are capable of ſolving the Queſtion. In this reſpect therefore *Literal Algebra* far excels the *Numeral*; for this latter ſerves only to ſolve *Arithmetical Queſtions*, and produceth not a *Canon* without much difficulty, in regard the numbers firſt given, by reiterated Multiplications, Diviſions and other Arithmetical operations, will for the moſt part be ſo confounded and interwoven, that their footſteps can hardly be traced out: But *literal* or *Specious Algebra* is applicable to the ſolving of *Geometrical Problems*, as well as *Arithmetical.*

IX. The *Doctrine* of *Algebra* is principally grounded upon the knowledge of certain Quantities called by ſome Authors *Coſſick Quantities*, by others, *Powers*; the Conſtruction whereof is explain'd in ſix Sections next following.

X. Numbers are ſaid to be *in Geometrical Proportion continued*, when as the firſt is to the ſecond, ſo is the ſecond to the third, and ſo is the third to the fourth, &c. As, for Example, theſe Numbers, 1, 2, 4, 8, 16, 32, &c. are Continual Proportionals; for, as the firſt Term 1, is the half of the ſecond Term 2; ſo is the ſecond Term 2, the half of the third Term 4; and ſo is 4 the half of 8, &c. Likewiſe theſe Numbers, 3, 9, 27, 81, 243, &c. are in Geometrical Proportion continued; For as the firſt Term 3 is a third part of the ſecond Term 9, ſo is the ſecond Term 9 a third part of the third Term 27; and ſo is 27 one third of 81, &c. Alſo, theſe Numbers are Continual Proportionals, to wit, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. for as the firſt Term 1, is the double of the ſecond Term $\frac{1}{2}$, ſo is $\frac{1}{2}$ the double of $\frac{1}{4}$, and $\frac{1}{4}$ the double of $\frac{1}{8}$, &c.

XI. In any ſeries or rank of Numbers proceeding from Unity in a continued Geometrical proportion, whether aſcending or deſcending, all the Numbers or Terms except the firſt, which is ſuppoſed to be 1, (to wit, Unity,) are called *Coſſick Numbers*, or *Powers*; *viz.* the ſecond Term or Proportional is called the *Root*, or firſt Power; the third Proportional is called the *Square*, or ſecond Power; the fourth Proportional is called the *Cube*, or third Power; the fifth Proportional is called the *Biquadrate*, or fourth Power, the ſixth Proportional, the fifth Power, &c. As for Example, in this rank of Continual Proportionals, 1, 2, 4, 8, 16, 32, &c. the ſecond Term 2 is the Root; the third Term 4 is the ſecond Power, or the Square of the Root 2; the fourth Term 8 is the third Power, or the Cube of the Root 2; the fifth Term 16 is the Biquadrate or fourth Power of the ſame Root 2, &c. In like manner in this rank of Continual Proportionals deſcending from 1, to wit, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c. the ſecond Term $\frac{1}{2}$ is the Root; the third Term $\frac{1}{4}$ is the ſecond Power; the fourth Term $\frac{1}{8}$ is the third Power, &c. The like is to be underſtood of any other Rank of numbers in a continued Geometrical proportion, whoſe firſt Term or Proportional is Unity.

XII. From the two laſt preceding Sections, (which are grounded upon 10. Prop. 8. Elem. *Euclid.*) it is evident that any Number whatſoever being propoſed for a *Root*, the ſecond Power, or the *Square*, is produced by the multiplication of the Root by it ſelf; the third Power, or the *Cube*, is produced by the multiplication of the ſecond Power by the Root; the fourth Power is produced by the multiplication of the third Power by the Root, &c.

As, for Example, if 2 be given for the *Root*, this 2 multiplyed by it ſelf, produceth 4 for the ſecond Power, to wit, the Square of the Root 2: Again, 4 the ſecond Power being

being multiplyed by the Root 2 gives 8 the third Power, or the Cube; which third Power multiplyed by the Root 2, produceth the fourth Power 16, *&c.*

In like manner, if this Fraction $\frac{2}{3}$ be prefcribed for a Root, by multiplying $\frac{2}{3}$ by it felf there comes forth $\frac{4}{9}$ for the fecond Power, or the Square of the Root $\frac{2}{3}$; Again, the fecond Power $\frac{4}{9}$ multiplyed by the Root $\frac{2}{3}$ produceth the third Power $\frac{8}{27}$, or the Cube of the Root $\frac{2}{3}$; and the third Power $\frac{8}{27}$ multiplyed by the Root $\frac{2}{3}$ gives the fourth Power $\frac{16}{81}$, *&c.*

But when the *Root* is 1, to wit, Unity, every one of its Powers will alfo be 1; for multiplication by 1 makes no alteration. All which will be further illuftrated by the Scales of Coffick numbers or Powers in the following Table, which fhews that if the Root be 5, the Square is 25, the Cube 125, the Biquadrate or fourth Power 625, the fifth Power 3125, *&c.*

A Table of Powers *in Numbers.*

The *Root* or firft Power.	1	2	3	4	5
The *Square* or fecond Power.	1	4	9	16	25
The *Cube* or third Power.	1	8	27	64	125
The *Biquadrate* or fourth Power.	1	16	81	256	625
The fifth Power.	1	32	243	1024	3125
The fixth Power.	1	64	729	4096	15625
The feventh Power.	1	128	2187	16384	78125
The eighth Power, *&c.*	1	256	6561	65536	390625

XIII. The *Root* or firft Power being given, the third, fifth, eighth, or any other Power may be found out without refpect to the intermediate Power or Powers, in this manner, *viz.* Suppofe the number 3 be prefcribed for the Root, and that the fifth Power be defired; firft write down the Root 3 five times thus, 3, 3, 3, 3, 3; then multiply thefe five equal numbers one into another according to the Rule of continual Multiplication, fo the laft Product 243 fhall be the defired fifth Power raifed from the Root 3.

In like manner, if the eighth Power of the Root 2 be defired, you may write the Root 2 eight times thus, 2, 2, 2, 2, 2, 2, 2, 2, thefe multiplyed continually produce 256, which is the eighth Power of the Root 2. After the fame manner you may find out any other Power from a number given for the Root.

XIV. If over or under any Series or Rank of Coffick numbers or Algebraick powers, conftituted according to the three laft foregoing Sections, there be placed a rank of Numbers beginning with Unity, and proceeding according to the natural order of numbers, as 1, 2, 3, 4, 5, 6, 7, 8, 9, *&c.* thefe numbers fo placed are ufually called the *Indices*, or *Exponents* of thofe Powers, as well becaufe they fhew the order, feat, or place of each Power, as alfo its number of Degrees or Dimenfions; that is, how many times the Root is involved or multiplyed in producing each Power refpectively: As for Example, let there be a rank or Scale of Algebraick powers raifed from the root 3, as 3, 9, 27, 81, 243, 729, 2187, *&c.* and over them let there be fo many numbers placed in an Arithmetical progreffion, beginning with 1, and proceeding according to the natural order of Numbers, as here you fee:

INDICES.	1	2	3	4	5	6	7	8	*&c.*
POWERS.	3	9	27	81	243	729	2187	6561	*&c.*

I say the Index 4 in the Arithmetical progression, shews that the fourth Power 81, which stands under 4, is produced by the multiplication of the Root 3 four times into it self, *viz.* these four numbers 3, 3, 3, 3, multiplyed continually will produce 81; likewise the Index 7 in the Arithmetical progression shews, that the seventh Power 2187, which stands under 7, is produced by the multiplication of the Root 3 seven times into it self; *viz.* these seven equal numbers, 3, 3, 3, 3, 3, 3, 3, multiplyed continually produce 2187. And so of others.

To that use of *Indices*, this may be added; *viz.* If any two or more Indices be added together, the summ will be an Index shewing what Power will be produced by the multiplication of those Powers one into another which answer to the Indices that were added together: As for Example, if the Indices 3 and 5 be added together, the summ is the Index 8; which shews, that if the third and fifth Powers be multiplyed one by the other the eighth Power will be produced: As in the rank of Powers in the preceding Tabulet, if the third Power 27 be multiplyed by the fifth Power 243, the Product will give the eighth Power 6561. In like manner, forasmuch as the Indices 2 and 6 added together make the Index 8; therefore the second Power 9 multiplyed by the sixth Power 729 will also produce the eighth Power 6561: Again, because the Indices 1, 2, and 5 added together make the Index 8; therefore the first, second and fifth Powers, to wit, 3, 9, and 243 multiplyed continually will likewise produce the eighth Power 6561. And as the Index 3 added to it self makes the Index 6, so the third Power 27 multiplyed by it self, or squared, will produce the sixth Power 729.

And as the Addition of Indices answers to the Multiplication of their correspondent Powers, so the subtraction of Indices answers to the division of their correspondent Powers: As, for example, because the Index 8 lessened by the Index 5, leaves for a Remainder the Index 3; therefore the eighth Power 6561 divided by the fifth Power 243 gives in the Quotient the third Power 27. Likewise, as the Index 7 lessened by the Index 3 leaves the Index 4; so the seventh Power 2187 divided by the third Power 27, gives the fourth Power 81.

XV. From the premisses it is evident, that upon an Arithmetical foundation, a Scale or Rank of Algebraick Powers may be raised and continued as far as you please; the three first of which have an affinity with, and may be expounded by Geometrical dimensions: For first, we may conceive any terminated Right-line to be divided into a number of equal parts at pleasure, suppose 12; then this number 12, or that Right-line, may be esteemed as a Root: Secondly, the said 12 multiplyed by it self produceth 144 the second Power, which is equal to the Area of a square Superficies whose side is 12: Thirdly, the said second Power 144 multiplyed by the Root 12 produceth the third Power 1728, which is equal to the Solid content of a Cube, (to wit, a Solid in the form of a Dye) whose side is 12.

But none of the rest of the Algebraick powers can properly be explain'd by any Geometrical quantity, in regard there are but three dimensions in Geometry, to wit, Length, Breadth, and Dephth (or Thickness.)

XVI. In searching out the solution of a Question by the Algebraick Art, the number or line sought is usually called a *Root*, which so long as it remains unknown cannot be really exprest, annd therefore it must be design'd or represented by some Symbol or Character, at the will of the Artist; also the Powers which may be imagined to proceed from the said Root in such manner as hath before been declared are likewise to be represented by Symbols or Characters; concerning which there is much diversity among *Algebraical* Writers, every one pleasing his fancy in the choice of Characters: But in this matter I shall imitate Mr. *Thomas Harriot* in his *Ars Analytica*, and *Renates des Cartes* in his *Geometry*, but chiefly the former; whose method of expressing Quantities by alphabetical Letters, I conceive to be the plainest for Learners, *viz.*

To design or represent the Root sought, whether it be a Number or a Line in a Question proposed, we may assume any Letter of the Alphabet, as a, b, or c, &c. but for the better distinguishing of known quantities from unknown, some *Analysts* are wont to assume one of the five Vowels, as, a, or e, &c. to represent the quantity sought; and Consonants, as, b, c, d, &c. to represent quantities known or given: Now if the letter a be assumed to represent the Root sought, then (according to Mr. *Harriot*) the second Power, or the Square raised from that Root, may be represented by aa; the third Power, or the Cube, by aaa; the fourth Power by $aaaa$, the fifth Power by $aaaaa$; and after the same manner

manner any higher Power of the Root or number a may be repreſented: For ſo many Dimenſions or Degrees as are in the Power, ſo many times the Letter which at firſt was aſſumed for the Root is to be repeated.

Or after the manner of *Renates des Cartes*, if the letter a be aſſumed to repreſent the Root, the Square may be deſigned thus, a^2. the Cube thus, a^3. the fourth Power thus, a^4. the fifth Power thus, a^5. And ſo any other Power may be expreſt by writing the Index or Exponent of the Power in a ſmall figure next after, and near the head of the letter aſſumed to repreſent the Root. Both which ways will be further illuſtrated by the following Table.

A Table ſhewing two wayes (now moſt in uſe) to expreſs ſimple Powers *by Alphabetical Letters.*

The *Root* or firſt Power,	a.	a
The *Square* or ſecond Power,	aa.	a^2
The *Cube* or third Power,	aaa.	a^3
The fourth Power,	$aaaa$.	a^4
The fifth Power,	$aaaaa$.	a^5
The ſixth Power,	$aaaaaa$.	a^6
The ſeventh Power,	$aaaaaaa$.	a^7
The eighth Power,	$aaaaaaaa$.	a^8.

After the ſame manner, known Quantities and their Powers may be repreſented by Conſonants; as, b may be put for any known number in a Queſtion, and then its Square may be ſignified by bb, the Cube by bbb, the fourth Power by $bbbb$, the fifth Power by $bbbbb$, the ſixth by $bbbbbb$, and ſo forwards: Or the Square of the Root b may be expreſt thus, b^2. the Cube thus, b^3. the fourth Power thus, b^4. the fifth Power thus, b^5. the ſixth Power thus, b^6. and ſo forwards.

XVII. Numbers ſet before, that is, on the left hand of quantities expreſt by letters are called Numbers prefixt; but if no number be prefixt to the letter, then 1 or unity muſt be imagined to be prefixt: As, in theſe quantities a, (or $1a$,) $2a$, $3a$, $\frac{1}{2}a$, $\frac{2}{3}a$, $5bbb$ (or $5b^3$) the numbers prefixt are (as you ſee) 1, 2, 3, $\frac{1}{2}$, $\frac{2}{3}$, and 5, every one of which numbers (and the like ſo prefixt) ſhews how often the quantity repreſented by the letter or letters immediately following the number is taken; ſo a or $1a$ ſignifies ſome number or line once taken, alſo $2a$ repreſents the double, $\frac{1}{2}a$ the half, and $\frac{2}{3}a$ two third parts of the number or line repreſented by a. In like manner $5bbb$, or $5b^3$, ſignifies that the Cube of the number or line repreſented by b is taken five times.

XVIII. All numbers expreſt by figures and cyphers (as in vulgar Arithmetick) not having any letter or letters annexed to them, are for diſtinction ſake called Abſolute numbers; as theſe numbers, 5, 20, 105, $\frac{1}{2}$, $\frac{2}{3}$, and all others when they be not prefixt or annext to any letter or letters are called Abſolute numbers.

XIX. All *Algebraical operations* are perform'd in an Arithmetical manner, partly in the vulgar way by numbers, and partly by Alphabetical letters, in all the parts of Arithmetick, to wit, Addition, Subtraction, Multiplication, Diviſion, and the Extraction of Roots: But ſince letters cannot be diſpoſed like numbers to perform thoſe operations, ſome Characters muſt of neceſſity be uſed to ſignifie ſuch operations. The Characters uſed in this firſt Book are explained in the following Sections.

XX. This Character $+$ is a ſign of *Affirmation*, as alſo of *Addition*, and always belongs to the quantity that follows the ſign; as, $+a$ affirms the quantity denoted by a to be real, or greater than nothing; the like may be ſaid of $+b$, and $+2c$, &c.

When no ſign is prefixt before a quantity, the ſign $+$ is always to be underſtood, and muſt be imagined to be prefixt; ſo a implies $+a$, likewiſe $2b$ ſignifies the ſame thing with $+2b$; the like of others.

But when the ſign $+$ is placed between two quantities, it imports as much as the word *plus*,

plus, or *more*, and ſignifies that thoſe quantities are added or to be added together: As $3+4$ (or 3 more 4) ſignifies the ſumm of 3 and 4; or it hints that 4 is to be added to 3. In like manner $a+b$ ſignifies the ſumm of numbers or quantities repreſented by a and b; and $a+b+c$ ſignifies the ſumm of quantities denoted by a, b, and c.

XXI. This Character — is a ſign of *Negation*, as alſo of *Subtraction*, and alwayes belongs to the following quantity; as for Example, -5 is a fictitious number leſs than nothing by 5; *viz.* as $+5$ *l.* may repreſent five pounds in money, or the Eſtate of ſome perſon who is clearly worth five pounds; ſo -5 *l.* may repreſent a Debt of five pounds owing by ſome perſon who is worſe than nothing by five pounds.

But when the ſign — is placed between two quantities, it imports as much as the word *minus*, or *leſs*; and intimates that the number or quantity following that ſign is ſubtracted or to be ſubtracted from the number or quantity that ſtands next before the ſame ſign: As $8-3$ (or 8 leſs 3) ſignifies that 3 is ſubtracted or to be ſubtracted from 8; or $8-3$ denotes the exceſs of 8 above 3, to wit, 5.

In like manner $a-b$ (or a leſs b) ſignifies that the quantity denoted by b is ſubtracted or to be ſubtracted from the quantity a; or $a-b$ may ſignifie the exceſs of the quantity a above the quantity b.

XXII. This Character $\backsim$ ſignifies the *Difference* of two quantities, to wit, the exceſs of the greater above the leſs, when 'tis not determin'd or known in which of thoſe quantities the exceſs lyeth; ſo $a \backsim b$ ſignifies the difference of two quantities repreſented by a and b, when 'tis not known whether a be greater or leſs than b.

XXIII. This Character $\times$ is a ſign of *Multiplication*, and is put for the word *into*, or *by*; *viz.* when 'tis ſet between two quantities it ſignifies that they are multiplyed, or to be multiplyed mutually one by the other: As, 6×3 (or 6 into or by 3) imports the Product of the multiplication of 6 by 3, to wit, 18.

In like manner $a \times b$ ſignifies that the quantity repreſented by a is multiplied or to be multiplied by the quantity b: alſo $a \times b \times c$ ſignifies the Product made by the continual multiplication of the quantities a, b, and c, one into another.

But for the moſt part the Multiplication of quantities denoted by letters is ſignified by the joyning of letters together, like letters in a word; as ab ſignifies the Product of the multiplication of the quantity a by the quantity b. Alſo abc ſignifies the Product of the continual multiplication of the quantities a, b and c one into another: All which will be further illuſtrated in Chap. 4.

XXIV. Quantities deſign'd or repreſented by letters are either Simple or Compound.

XXV. A Simple quantity is deſigned or expreſſed either by a ſingle letter, or by two or more letters joyned together like letters in a word: As a (or $+a$) is a ſimple quantity; likewiſe $2aa$, $3abc$, and $dddd$ are ſimple quantities.

XXVI. A Compound quantity conſiſteth of two or more ſimple quantities connected or joyned one to another by $+$ or $-$; ſo $a+b$ is a compound quantity, likewiſe $a-c$, alſo $a+b+c$, and $a+b-c$ are compound quantities.

XXVII. Every one of theſe four Characters, to wit, $+$, $-$, $\backsim$, and $\times$, (before defined in Sect. 20, 21, 22, and 23.) may ſometimes have reference to ſuch a Compound quantity as followeth the ſign, and hath a Line drawn over every member of it. As, for Example, by $a+\overline{b \backsim c}$, you are to underſtand that the difference of the quantities b and c (whether the Exceſs be in b or in c) is added or to be added to the quantity a.

In like manner, $a-\overline{b+c}$ ſhews that the Compound quantity $b+c$ is ſubtracted or to be ſubtracted from the quantity a; where in regard of the line drawn over $b+c$, the ſign — hath reference to the ſubtraction of c as well as b from the quantity a. But if that line were omitted, then the ſign — would only refer to the next following ſimple quantity: As, $a-b+c$, (or $a+c-b$) ſignifies the ſubtraction of b only from $a+c$.

Moreover, $a \backsim \overline{b+c}$ ſignifies the difference between the ſimple quantity a, and the compound quantity $b+c$.

And $a \times \overline{b-c}$ ſignifies that the quantity a is multiplyed or to be multiplyed by the exceſs of the quantity b above the quantity c.

XXVIII. This Character $\surd$ is called a Radical ſign, and ſignifies that the Square root of the number or quantity that ſtands next after the ſaid ſign $\surd$, is extracted, or to be extracted; as $\surd 25$ ſignifies the ſquare root of 25, to wit, 5; and $\surd 36$ ſignifies the ſquare root of 36, to wit, 6.

Like-

Likewiſe $\surd ab$ ſignifies the ſquare root of the quantity ab. So that when a number or quantity immediately follows the ſaid radical ſign $\surd$, the ſquare root of that number or quantity is thereby denoted.

But to deſign or repreſent the Root of a Power higher than a Square, ſome *Algebraical* Writers (whom in this matter I ſhall follow) are wont to write the Index of the Power within a Circle next after the ſign $\surd$; As for Example, $\surd(3)27$ ſignifies the Cubick root of 27, to wit, 3. Likewiſe, $\surd(4)16$ denotes the Biquadrate root of 16, to wit, 2; that is, the root from whence 16 conſidered as the fourth Power is produced. Again, $\surd(5)243$ ſignifies the root from whence 243 conſider'd as the fifth Power is raiſed, which Root is 3. And if you pleaſe you may write $\surd(2)81$ to denote the ſquare root of 81, to wit, 9.

Likewiſe $\surd(3)a$ ſignifies the Cubick root of ſome number or quantity repreſented by a. Alſo $\surd(4)bc$ ſignifies the Biquadrate root of the Quantity bc.

Sometimes the Radical Sign belongs to as many of the following Quantities as have a Line drawn over them; as $\surd:\overline{b+c}:$ or, $\surd(2):\overline{b+c}:$ ſignifies the Square root of the ſumm of the Quantities b and c. Likewiſe $\surd:\overline{bb-c}:$ imports the Square root of the Remainder when the quantity c is ſubtracted from the Square of the quantity b. Which Roots, and ſuch like, are called *Univerſal Roots.*

Again, $d+\surd:\overline{bb-c}:$ ſignifies that the Quantity c is firſt to be ſubtracted from the Square bb, and then the Square root of the Remainder is to be added to the quantity d. But that the Learner may the better perceive my meaning in the three laſt Examples concerning Univerſal Roots, let b ſignifie 4; bb, 16; c, 12; and d, 23. Then $\surd:\overline{b+c}:$ ſignifies $\surd:\overline{4+12}:$ that is, $\surd 16$, to wit, 4. Alſo $\surd:\overline{bb-c}:$ ſignifies $\surd:\overline{16-12}:$ that is, $\surd 4$, to wit, 2. And $d+\surd:\overline{bb-c}:$ ſignifies $23+2$, that is, 25. After the ſame manner the Univerſal Square root of $d+\surd:\overline{bb-c}:$ may be expreſt thus;

$$\surd:\overline{d+\surd\overline{bb-c}}: \text{ that is, } 5.$$

XXIX. Four points ſet in this form :: are always in the middle of four Geometrical Proportionals, as, for Example, theſe four numbers $2.4::6.12$ are Geometrical Proportionals, and to be read thus; As 2 is to 4, ſo is 6 to 12; or, (in the Phraſe of *The Rule of Three)* If 2 give 4, then 6 will give 12.

In like manner theſe four Quantities, $b.d::c.a$ are to be read thus; As b is to d, ſo c to a; that is, look what proportion b hath to d, the ſame proportion hath c to a.

Alſo theſe four Quantities $b+c.d-a::f.g$ do intimate that the ſumm of b and c hath ſuch proportion to the Exceſs of d above a, as f hath to g. The like is to be underſtood of others.

XXX. This Character $\div\div$ ſet at the end of three or more Quantities, imports that they are Continual Proportionals Geometrical; ſo by $2.4.8.16.32\div\div$ it is ſignified that ſuch proportion as 2 hath to 4, the ſame hath 4 to 8, 8 to 16, and 16 to 32.

Likewiſe by theſe $a.b.c\div\div$ you are to underſtand that the quantity a hath the ſame proportion to the quantity b, as b to c.

XXXI. This Character $=$ is the ſign of an Equation or Equality, and imports as much as the word *Equal*; as, $8+4=7+5$ ſignifies that the ſumm of 8 and 4 is equal to the ſumm of 7 and 5. Likewiſe $8=12-4$ that 8 is equal to 12, leſs 4; to wit, the exceſs of 12 above 4.

Again, $8\times3=4\times6$ denotes the Product of 8 multiplyed by 3 to be equal to the Product of 4 into 6.

So alſo, $a+b=c+d$ ſignifies that the ſumm of the quantities a and b is equal to the ſumm of the quantities c and d. This will be further explained in the XI. Chapter.

XXXII. This Character $\sqsubset$ ſtands for the word *Greater*, *viz.* it ſignifies that the Quantity which ſtands before, that is, on the left hand of the ſaid Character is greater than the quantity following the ſame; ſo $5\sqsubset4$ muſt be read thus, 5 is greater than 4. Likewiſe $a+b\sqsubset c$ ſignifies that the Compound quantity $a+b$ is greater than the Simple quantity c. And $d\sqsubset a+c$ ſignifies that the quantity d is greater than $a+c$.

XXXIII. This Character $\sqsupset$ ſignifies that the quantity ſtanding before the Character is leſs than the quantity following the ſame; as, $4\sqsupset5$ muſt be read thus, 4 is leſs than 5. Likewiſe, $a+b\sqsupset c+d$ ſignifies that the compound quantity $a+b$ is leſs than the compound quantity $c+d$.

XXXIV. Quan-

XXXIV. Quantities, whether they be Simple or Compound, which are expreſt either wholly by Letters, or partly by Letters and partly by Numbers written upon one Line, are called Algebraical Integers, or whole Quantities; as theſe, $a, ab, cd+ff, a+3$, &c. But theſe quantities, $\frac{b}{c}$, $\frac{aa+bb}{a+c}$, $\frac{a+3}{b}$, and others ſo written, are called Algebraical Fractions, becauſe each of them like a Fraction in Vulgar *Arithmetick* conſiſts of a Numerator placed above a line, and a Denominator underneath.

CHAP. II.

Addition of Algebraical Integers.

I. Algebraical *Addition* finds out the Summ or Aggregate of two or more Quantities expreſt either wholly by Letters, or partly by Letters and partly by Numbers.

II. The Operations in Algebraick Addition depend principally upon a diligent obſervation of three things, *viz.*

Firſt, You muſt obſerve whether the Quantities to be added be Like or Unlike.

Like Quantities are thoſe which are expreſt by the ſame Letters equally repeated in every one of the Quantities; ſuch are theſe, $a, 5a, -2a$, each of which is expreſt by the ſingle letter a. Alſo theſe are Like quantities, $3aa, aa, -2aa$, each of which is expreſt by a double a, to wit, aa. Likewiſe theſe, $2ab, 3ab, -ab$ are called Like quantities becauſe every one of them is expreſt by the ſame letters, to wit, ab.

Unlike Quantities are thoſe which are expreſt by different Letters, or elſe by the ſame letters unequally repeated; as, for Example, b and c are unlike quantities, becauſe they are expreſt by different letters; alſo $2abc$ and $2ab$ are unlike quantities, becauſe the letter c is in the one, but not in the other. Again, a and aa are unlike quantities, in regard the letter a is not equally repeated in both. The like is to be underſtood of others.

Secondly, You muſt obſerve whether the Signs (to wit, $+$ and $-$) belonging to like quantities given to be added be Like or Unlike: As, for example, theſe quantities $+2a$ and $+3a$ have like ſigns, the ſame ſign $+$ being prefixt before each quantity. Alſo theſe quantities, $-2a$ and $-3a$ have like ſigns, the ſame ſign $-$ being prefixt to each quantity; but theſe quantities $+2a$ and $-3a$ have unlike or different ſigns prefixt.

Thirdly, The numbers prefixed before the letters muſt be diligently obſerved, for their ſumm or difference will be concern'd in Algebraical Addition, as will be manifeſt by the following Rules.

III. When two or more ſimple Algebraical Integers (or whole quantities) propos'd to be added or collected into one Summ are like, and have like ſigns, Firſt collect the numbers prefixt into one ſumm; then to that ſumm annex the letter or letters by which any one of the quantities propos'd is expreſt; laſtly, prefix the given ſign whether it be $+$ or $-$, ſo ſhall this new quantity be the ſumm deſired. As, for Example, if it be deſired to add a to a, or $+1a$ to $+1a$, the ſumm will be $2a$ or $+2a$; for (according to the Rule) the ſumm of the prefixed numbers 1 and 1 is 2, to which I annex a and prefix $+$ (or imagine it to be prefixed,) ſo $2a$ or $+2a$ is the ſumm deſired.

Add	a	$+1a$
	a	$+1a$
Summ	$2a$	$+2a$

In like manner, if to $-2b$ you would add $-b$, the ſumm will be $-3b$. For the numbers prefixt are 2 and 1, which added together make 3, to which annexing b, and prefixing the given ſign $-$, there ariſeth $-3b$, the ſumm deſired.

Add	$-2b$
	$-b$
Summ	$-3b$

More

More Examples of the Rule of Addition in the foregoing Sect. III.

To be added,	$5a$ $3a$	$-5aa$ $-2aa$	$+7ab$ $+13ab$
The Summ,	$8a$	$-7aa$	$+20ab$

To be added,	ac $2ac$ $3ac$	$-3bcd$ $-bcd$ $-6bcd$	$+3a^3$ $+2a^3$ $+7a^3$
The Summ,	$6ac$	$-10bcd$	$+12a^3$

IV. When two Simple quantities propos'd to be added together be like, and have equal numbers prefixt, but unlike or contrary Signs, the Summ will be o, or nothing; for the Affirmative quantity will destroy or extinguish the Negative: As, for example, if it be required to add c, or $+c$, to $-c$, the Summ will be o, to wit, nothing. For supposing $-c$, or $-1c$ to be a Debt of one Crown that I owe; and $+c$, or $+1c$ to be one Crown in my purse, it is evident that one Crown in ready money will discharge or strike off a Debt of one Crown; and so that Debt and Credit being added or compared together, the Summ amounts to o.

$$\text{Add, } \begin{cases} -c \\ +c \end{cases} \qquad \text{Summ, } \quad 0$$

In like manner, if it be desired to add $-6\,l.$ to $+6\,l.$ the Summ will be o; for if my whole Estate be worth but 6 pounds, and I owe a Debt of 6 pounds, it is manifest that my cleer Estate is worth or amounts to just nothing.

$$\text{Add, } \begin{cases} +6\,l. \\ -6\,l. \end{cases} \qquad \text{Summ, } \quad 0$$

More Examples of the Rule of Addition in the preceding Sect. IV.

To be added,	$+3a$ $-3a$	$-5abc$ $+5abc$	$+7ddd$ $-7ddd$
The Summ,	o	o	o

V. When two Simple quantities propos'd to be added together be like, but their Signs unlike, and the prefixed numbers unequal between themselves; first subtract the lesser number prefixed from the greater; then to the Remainder annex the letter or letters by which either of the Quantities proposed is exprest; lastly, before the said Remainder set the Sign which stands before the greater number prefixt, so shall this new Quantity be the Sum desired.

As, for Example, if it be desired to add $-2a$ to $+3a$, the summ will be a. For first subtracting 2 from 3 the remainder is 1, to which annexing a, and prefixing $+$ (because $+$ belongs to that Quantity which hath the greater number prefixt) there ariseth $+1a$, or $+a$ for the Summ sought.

$$\text{Add, } \begin{cases} +3a \\ -2a \end{cases} \qquad \text{Summ, } \quad +1a, \text{ or, } +a$$

Again, to add $+b$ to $-3b$, I subtract 1 the lesser number prefixt, from 3 the greater, and to the Remainder 2 annexing b and prefixing $-$, (because $-$ belongs to $3b$ whose prefixt number 3 is greater than that of $+b$ or $+1b$) I find $-2b$ for the Summ desired.

$$\text{Add, } \begin{cases} -3b \\ +b \end{cases} \qquad \text{Summ, } \quad -2b$$

Thus you see that this last Rule of Addition is performed by Subtraction, and may easily be understood under the notion of discharging or paying off a Debt, or at least part of a Debt by so much ready Money or Credit, and then observing what Debt remains unpaid,

or what Money or Credit remains as an overplus: So in the first of the two last Examples; you may conceive $+3a$ to be three Pounds in ready Cash, and $-2a$ to be a Debt of two Pounds; then comparing the said ready Money and Debt together, you will find by Subtraction that the clear Money remaining after the Debt is payd, will be one Pound, to wit, $+1a$ or a which is the Summ of the quantities $+3a$ and $-2a$. Likewise in the latter Example, if $-3b$ be conceived to represent a Debt of three Pounds, and $+b$ or $+1b$ one Pound in ready Money; 'tis evident that this will strike off one Pound of that Debt, and so the Debt remaining will be two Pounds, to wit, $-2b$, which is the Summ of $-3b$ and $+b$.

More Examples of the Rule of Addition in the preceding Sect. V.

To be added,	$+5aa$ $-7aa$	$+6abcd$ $-4abcd$	$-8f^4$ $+3f^4$
The Summ,	$-2aa$	$+2abcd$	$-5f^4$

VI. When three or more simple Quantities propos'd to be added be like, but have unlike Signs; First, (by the Rule in *Sect.* III. of this *Chap.*) collect the Affirmative quantities into one Summ, and the Negative quantities into another; then (by *Sect.* IV. or V.) add those two Summs into one, so this last Summ shall be that which is sought.

As, for example, If the Summ of these four Quantities, $7a$, $2a$, $-3a$, $-5a$ be desired; First, (by *Sect.* III.) the summ of $7a$ and $2a$ is $+9a$; also the summ of $-3a$ and $-5a$ is $-8a$; lastly (by *Sect.* V.) $+9a$ added to $-8a$ makes $+a$, that is, a, which is the Summ desired.

More Examples of the Rule of Addition in Sect. VI.

To be added,	$+5a$ $+3a$ $-8a$	$-2bc$ $+3bc$ $-4bc$	$+4d^5$ $+3d^5$ $-5d^5$
The Summ,	0	$-3bc$	$+2d^5$

To be added,	$+5ee$ $+2ee$ $-ee$ $-4ee$	$-4fff$ $-3fff$ $-2fff$ $+8fff$	$+4ggbb$ $-3ggbb$ $+2ggbb$ $-ggbb$
The Summ,	$+2ee$	$-fff$	$+2ggbb$

VII. When two or more Simple quantities given to be added be unlike, write them down one after another without altering their Signs; as, if the number (or line) a be to be added to the number (or line) b; I write $a+b$, or, $b+a$ for the Summ.

In like manner the Summ of these Quantities $a, b, c,$ may be written thus, $a+b+c$; or thus, $a+c+b$; or thus, $b+a+c$.

More Examples of the Rule of Addition in Sect. VII.

To be added,	$+3a$ $+2d$	$+aa$ $-bb$
The Summ,	$3a+2d$	$+aa-bb$

To

Again,

To be added,	$+ab$ $-ac$ $+ad$	$+5ddd$ $-3dd$ $-4d$
The Summ,	$+ab-ac+ad$	$+5ddd-3dd-4d$

Addition of Compound Algebraical Integers.

VIII. The Addition of Compound whole Quantities may easily be dispatcht by the help of the Rules in the preceding *Sections* of this *Chap.* as will appear by the following Examples.

First then, If this Compound quantity $a+b$ be to be added to $a+2b$, their Summ is $a+b+a+2b$, that is $2a+3b$; for $a+a$ makes $2a$, and $+b+2b$ makes $+3b$.

Again, The Summ of these two Compound quantities $3b+5a$ and $2b-2a$ is $3b+5a+2b-2a$, that is, $5b+3a$; for $3b+2b$ makes $5b$; and (by *Sect.* V.) $+5a-2a$ makes $+3a$.

Likewise, The Summ of these two Compound quantities $5ee+3f-8$ and $3ee-2f+6$ will be found $8ee+f-2$: For $5ee$ added to $3ee$ makes $8ee$; also $+3f$ added to $-2f$ gives $+f$, and -8 added to $+6$ makes -2.

After the same manner, $3a-8$ added to $10-a$ makes $2a+2$; (for $+3a$ added to $-a$ makes $+2a$, and -8 added to $+10$ gives $+2$.)

Again, The Summ of these two Compound quantities $a+b$ and $c-d$ is $a+b+c-d$, which Summ admits of no Contraction, in regard all the Simple quantities are unlike.

More Examples of the Addition of Compound whole Quantities.

To be added,	$a+b$ $a-b$	$aa+2a-3$ $aa+a-6$
The Summ,	$2a$	$2aa+3a-9$

To be added,	$aa-2ab$ $aa+ab$	$4c-d+3$ $-4c+2d-2$
The Summ,	$2aa-ab$	$d+1$

To be added,	$2ee+3ef-ff$ $-3ee+5ef$	$a^3-abc+6$ $+3abc-6$
The Summ,	$-ee+8ef-ff$	a^3+2abc

To be added,	$-aaa+2bba$ $8aaa+4bba$ $6aaa-6bba$	$aa-5a+24$ $aa+a-17$ $-2aa+2a+12$
The Summ,	$13aaa$	$-2a+19$

To be added,	$a+b$ $c-d$ $e+f$	$5b^3+24$ $-2b^3+40$ $6b^3-64$
The Summ,	$a+b+c-d+e+f$	$9b^3$, or, $9bbb$

Chap. III.

Subtraction in Algebraick Integers.

I. ALgebraical Subtraction takes one Quantity, whether it be expreſt by a letter or letters, or partly by letters and partly by number, out of, or from another, in ſuch manner that if the Remainder be added (according to the Rules of Algebraick Addition) to the Quantity ſubtracted, the Summ will be alwayes equal to the ſaid other Quantity.

II. A general Rule to find out the Remainder in all caſes of Algebraical Subtraction is this; Firſt joyn both the given quantities together, by writing one after the other; but with this caution, that every Sign of the quantity given to be ſubtracted, be ever changed into the contrary Sign, *viz.* $+$ into $-$ and $-$ into $+$; then ſhall the Summ of both quantities ſo connected be the Remainder ſought, which is to be contracted (when it may be done) into the feweſt and ſmalleſt Terms, by the Rules of Algebraical Addition.

As, for Example, If from $5a$ it be deſired to ſubtract $3a$, firſt, I write down $5a$, then next after the ſame I write $-3a$; (where obſerve, that according to the Rule above given, I change $+$, the Sign belonging to $3a$ the quantity given to be ſubtracted, into $-$,) ſo there ariſeth $5a-3a$, which being contracted (by the Rule of Addition in *Sect.* V. *Chap.* II.) makes $2a$ the Remainder ſought.

Out of	$5a$
Subtract	$3a$
Remainder,	$5a-3a$
Remainder contracted,	$2a$

Likewiſe, If from $3b$ it be deſired to ſubtract $-2b$, I firſt write down $3b$, and next after the ſame I write $+2b$; ſo $3b+2b$, that is, $5b$ is the Remainder ſought; where obſerve (as before) that I change the Sign $-$, which belongs to $2b$ the quantity propos'd, to be taken out of $3b$, into the contrary ſign $+$. But that the ſaid $5b$ is a true Remainder, we may prove by Addition; for $+5b$ added to $-2b$ the quantity ſubtracted, makes $+3b$, which is the Quantity out of which the ſaid $-2b$ was ſubtracted.

Out of	$3b$
Subtract	$-2b$
Remainder,	$3b+2b$
Remainder contracted,	$5b$

Moreover, If a be to be ſubtracted from a, the Remainder will be $a-a$, that is, 0 or nothing. And if from $2b$ there be ſubtracted $-4b$, the Remainder will be $2b+4b$, that is, $6b$.

Likewiſe, If from $-2m$ it be required to ſubtract $-m$, the Remainder will be found $-2m+m$, that is, $-m$. In every one of which Examples you may obſerve that the ſign of the Quantity propos'd to be ſubtracted is changed into the contrary ſign.

Again, If from $2bc$, it be deſired to ſubtract $2ab$, the Remainder will be $2bc-2ab$, which, becauſe it conſiſts of unlike Quantities, cannot be contracted into fewer or leſſer Terms, by any of the Rules of Algebraical Addition. But according to the definition of Subtraction, the ſaid $2bc-2ab$ is a true Remainder, for if it be added to $2ab$ the quantity ſubtracted, the Summ is $2bc$, which is the quantity out of which the ſaid $2ab$ was ſubtracted.

Out of	$2bc$
Subtract	$2ab$
Remainder,	$2bc-2ab$

More Examples of Subtraction in Simple Algebraick Integers.

Out of	$2b$	$+3c$	$-2n$
Subſtract	b	$-c$	$-n$
Remainder,	$2b-b$	$+3c+c$	$-2n+n$
Remainder contracted,	b	$+4c$	$-n$

Again,

Again,

Out of	$3a$	$-8d$	$-a$
Subtract	$5a$	$-10d$	$+a$
Remainder,	$3a-5a$	$-8d+10d$	$-a-a$
Remainder contracted,	$-2a$	$+2d$	$-2a$

Out of	$-bcd$	$-4rs$	$+4abc$
Subtract	$-bcd$	$+9rs$	$-abc$
Remainder,	$-bcd+bcd$	$-4rs-9rs$	$+4abc+abc$
Remainder contracted,	0	$-13rs$	$+5abc$

From	d	$-2b$	$+a^3$
Subtract	e	$-3a$	$-3a$
Remainder,	$d-e$	$-2b+3a$	$+a^3+3a$

From	$8bbd$	$+3abcd$
Subtract	$7bbb$	$-7aa$
Remainder,	$8bbd-7bbb$	$+3abcd+7aa$

Nor will the Operation be otherwise in the Subtraction of Compound Algebraick Integers; as for Example, if from this Compound quantity $3a+2b$, it be desired to subtract $a+3b$. First I write down $3a+2b$, then next after the same I write $-a-3b$, where observe, that the sign $+$ which belongs to a, and also to $3b$, in the Quantity propos'd to be subtracted, is changed into the contrary sign $-$ (according to the Rule of Subtraction before given;) so the Remainder sought is $3a+2b-a-3b$, that is, $2a-b$, (by *Sect.* V. *Chap.* II.)

From	$3a+2b$
Subtract	$a+3b$
Remainder,	$3a+2b-a-3b$
Remainder contracted,	$2a-b$

Again, If from $2a+b$, it be desired to subtract $5a-6b$, the Remainder will be $2a+b-5a+6b$, that is, $7b-3a$, for (according to the Rule of Algebraical Subtraction) I joyn together the two given Quantities, changing only the Signs of $+5a-6b$ (the quantity to be subtracted) into the contrary Signs, so there ariseth $2a+b-5a+6b$, which contracted (by the Rules of Addition in *Sect.* III. and V. of *Chap.* II.) make $7b-3a$; which is the Remainder sought, as will easily appear by the Proof.

Out of	$2a+b$
Subtract	$5a-6b$
Remainder,	$2a+b-5a+6b$
Remainder contracted,	$7b-3a$

Likewise, to subtract $c-d$ from $a+b$, I change the Signs of $c-d$ into the contrary Signs; *viz.* instead of $c-d$, I take $-c+d$, which added to $a+b$ makes $a+b-c+d$, which because it consists altogether of unlike Quantities, cannot be contracted into fewer Terms, and therefore the said $a+b-c+d$ is the Remainder sought, to wit, that which ariseth by subtracting $c-d$ from $a+b$.

From	$a+b$
Subtract	$c-d$
Remainder,	$a+b-c+d$

After the same manner, $cd+36$ subtracted from $3aa+bc+24$ leaves $3aa+bc+24-cd-36$, that is, $3aa+bc-cd-12$.

More

More Examples of Subtraction in Compound Algebraick Integers.

Out of	$a+b$	$3c-8$
Subtract	$a-b$	$c+5$
Remainder,	$a+b-a+b$	$3c-8-c-5$
Remainder contracted,	$+2b$	$2c-13$

Out of	$5a-4b$	$29e$
Subtract	$3a-3b$	$-3e+7$
Remainder,	$5a-4b-3a+3b$	$29e+3e-7$
Remainder contracted,	$2a-b$	$32e-7$

Out of	$aa+2ba+bb$	$-2cd+6$
Subtract	$+4ba$	$+cd-2$
Remainder,	$aa+2ba+bb-4ba$	$-2cd+6-cd+2$
Remainder contracted,	$aa-2ba+bb$	$-3cd+8$

Out of	$5a^3+27$	$3aa+6$
Subtract	$-8+3a^3$	$-3dd$
Remainder,	$5a^3+27+8-3a^3$	$3aa+6+3dd$
Remainder contracted,	$2a^3+35$	

From	$a+b$	$aa-bb$
Subtract	$c-d$	$-cc+dd$
Remainder,	$a+b-c+d$	$aa-bb+cc-dd$

III. The reaſon of changing the ſigns of the Quantity to be ſubtracted into their contraries, to wit $+$ into $-$, and $-$ into $+$ (according to the Rule before given) will be manifeſt from a ſerious conſideration of the definition of Subtraction, which requires that the Summ of the quantity ſubtracted and the Remainder be equal to the quantity from which the ſubtraction is made: For firſt, (according to the ſaid Rule) the Remainder is alwayes compos'd of both the quantities propos'd for Subtraction, with this caution, that the ſigns $+$ and $-$ in the quantity to be ſubtracted be changed into the contrary ſigns; Secondly, (according to Algebraical Addition) the quantity to be ſubtracted with its own ſigns being added to it ſelf with contrary ſigns, will deſtroy or extinguiſh it ſelf, therefore the Summ of the Remainder and the Quantity to be Subtracted will neceſſarily be equal to the Quantity from which the Subtraction was made: And therefore the certainty of the ſaid Rule of Algebraical Subtraction, and the reaſon of changing the ſigns of the quantity to be ſubtracted into their contraries, to wit, $+$ into $-$, and $-$ into $+$, is manifeſt: So if from $a+b$ there be ſubtracted $a-b$, the Remainder (according to the Rule of Algebraical Subtraction before given) will be $a+b-a+b$, to which if $a-b$ (the quantity ſubtracted) be added, it is evident that $a-b$ will deſtroy $-a+b$, and ſo the Summ will be $a+b$, to wit, the quantity from which $a-b$ was ſubtracted.

CHAP.

Chap. IV.

Multiplication in Algebraick Integers.

I. A*Lgebraical Multiplication* doth by two Quantities; whether they be expreſt by letters wholly, or partly by letters and partly by numbers, find out a third Quantity, which is called the Product, the Fact, or the Rectangle.

The Quantities given to be multiplyed one by the other are called Factors; or (as in vulgar Arithmetick) either of them may be called the Multiplicand, and the other the Multiplicator or Multiplyer.

II. When two Simple (or ſingle) Quantities expreſt by letters, whether like or unlike, be to be multiplyed by one another, and have no numbers prefixt to them, joyn the letters of both Quantities together, like letters in a word, it matters not in what order they be written; then the new Quantity repreſented by the letters ſo ſet together is the Product ſought.

As, for example, If the number or line a be to be multiplyed by it ſelf, to wit, by a, I write aa for the Product: ſo alſo to multiply a by b, I write ab or ba for the Product; in like manner if I would multiply abc by bc, I write $abcbc$, or $abbcc$, or $accbb$, &c. for the Product.

And if a, b, and c be to be multiplyed one into another, firſt a multiplyed by b produceth ab, then ab multiplyed by c produceth abc, or bac, or bca, to wit, the Product made by the continual Multiplication of the three Quantities a, b, and c.

Again, if aa be to be multiplyed by ba, the Product will be $aaab$; which may alſo be written thus, a^3b; where the Learner muſt diligently note that the figure 3 which ſtands next after but a little higher than a, muſt not be taken as a number prefixt to b, but as an Index to ſhew the number of Dimenſions in a^3, or aaa, (as before hath been ſaid in *Sect.* XVI. and XVII. *Chap.* I.)

Likewiſe, if aaa be to be multiplyed by aaa, or a^3 by a^3, the Product will be $aaaaaa$, or a^6, in which latter way of expreſſing the Product, the Index 6 ſtanding at the head of a is the Summ of 3 and 3 the Indices of the Quantities a^3 and a^3 propos'd to to be multiplyed.

So the Product made by the multiplication of $bbbb$ by bbb or b^4 by b^3 will be $bbbbbbb$, or b^7 (7 being the ſumm of the Indices 4 and 3.)

Likewiſe if theſe three Quantities be to be multiplyed continually, to wit, $aaaaa$, $bbbb$ and ccc, the Product may be expreſt thus, $aaaaabbbbccc$; or compendiouſly thus, $a^5b^4c^3$: and ſo of others.

More examples of Multiplication in ſimple Algebraick Integers, according to the preceding Sect. II.

Multiplicand,	b	d	ac	ccc
Multiplicator,	c	d	d	cc
Product,	bc	dd	acd	$ccccc$

Multiplicand,	$aabc$	def	$aabbcc$
Multiplicator,	bca	abc	$aabbcc$
Product,	$aaabbcc$	$abcdef$	$a^4b^4c^4$

III. If two ſimple Quantities, whether like or unlike, having numbers prefixt before them, be to be multiplyed one by the other; firſt multiply the numbers prefixt, one into the other, then to this Product annex the letters of both Quantities, by ſetting them immediate-

immediately one after another, (as before in *Sect.* II.) so this new Quantity shall be the Product sought.

As, for Example, if it be desired to multiply $2a$ by $3b$; first I multiply 2 by 3, and the Product is 6; to which annexing ab, (to wit, the letters found in both Quantities given to be multiplyed) there ariseth $6ab$ the Product sought; which shews that six times the Product of the Multiplication of any two numbers, or right-lines, a and b, is equal to the Product made by the Multiplication of the double of a by the triple of b.

Multiply	$2a$
by	$3b$
Product,	$6ab$

In like manner, if $2b$ be multiplyed by c the Product will be $2bc$, or $2cb$; for 2 which is prefixt to b in the Multiplicand, being multiplyed by 1 which is suppos'd to be prefixt to the Multiplyer c, makes 2, to which annexing bc, there is found $2bc$ for the Product sought.

Multiply	$2b$
by	c
Product,	$2bc$

More Examples of Multiplication in Simple Algebraick Integers, according to Sect. III.

Multiply	$4b$	$12ac$	$5ddfg$
by	$2a$	$3d$	dgh
Product,	$8ab$	$36acd$	$5d^3fggh$

Multiply	aaa	$3a^3$	$16aab$
by	$3bbb$	b^3	4
Product,	$3aaabbb$	$3a^3b^3$	$64aab$

IV. The Multiplication of Compound Quantities depends upon the precedent Rules of multiplying Simple Quantities; for when a Compound quantity is to be multiplyed by a Simple (or single) quantity, every member of that must be multiplyed by this; also, when two Compound quantities are to be mutually multiplyed, every member of the one must be multiplyed into every member of the other. It matters not whether you begin to multiply at the right hand or the left, nor in what order the particular Products be set; (for Quantities exprest by Letters retain their peculiar and unaltered values wheresoever they stand;) but due regard must be had to the Signs $+$ and $-$, one of which alwayes belongs to every particular Product, and may be discovered by this Rule, *viz.* $+$ multiplyed by $+$, or $-$ by $-$, makes $+$ in the Product; but $+$ multiplyed by $-$, or $-$ by $+$, makes $-$ in the Product; lastly, all the particular Products added together (according to the Rules in the preceding *Chap.* 2.) make the total Product sought: All which will be made manifest by the following Examples.

First, if a Compound quantity, as $a+b$, be to be multiplyed by a Simple quantity, as c, I begin at the left hand, and multiplying $+a$ by $+c$ the Product is $+ac$, (for $+$ multiplyed by $+$ gives $+$;) likewise $+b$ multiplyed by $+c$ produceth $+bc$; which two Products added together make $ac+bc$, which is the Product of the multiplication of $a+b$ by c.

Multiply	$a+b$
by	c
Product,	$ac+bc$

So if $a-b$ be to be multiplyed by c, the Product will be $ac-bc$. For $+a$ multiplyed by $+c$ produceth $+ac$; and $-b$ multiplyed by $+c$ produceth $-bc$; (for according to the Rule, $-$ multiplyed by $+$ gives $-$:) Therefore $+ac-bc$ or $ac-bc$ is the Product sought.

Multiply	$a-b$
by	c
Product,	$ac-bc$

After

After the same manner, if it be desired to multiply $a+b$ by $c+d$, the Product will be found $ac+bc+ad+bd$. For, first $a+b$ being multiplyed by c, (as in the first Example) produceth $+ac+bc$; likewise $a+b$ again multiplyed by d, produceth $+ad+bd$; then adding those Products together, the Summ is $ac+bc+ad+bd$, which is the required Product of $a+b$ multiplyed by $c+d$.

Multiply	$a+b$
by	$c+d$
	$+ac+bc$
	$+ad+bd$
Product,	$+ac+bc+ad+bd$

Again, if $a-b$ be multiplyed by $c-d$ the Product will be $ac-bc-ad+bd$: For first, $a-b$ multiplyed by c produceth $ac-bc$, (as in the last Example but one;) then $a-b$ again multiplyed by $-d$ produceth $-ad+bd$; (for, according to the Rule, $+a$ multiplyed by $-d$ produceth $-ad$, and $-b$ by $-d$ produceth $+bd$.) Lastly, those particular Products added together make $ac-bc-ad+bd$, which is the Product of $a-b$ multiplyed by $c-d$.

Multiply	$a-b$
by	$c-d$
Product,	$ac-bc-ad+bd$

Likewise, if $a+b$ be multiplyed by $a-b$, the Product will be $aa-bb$: For first, $a+b$ multiplyed by a produceth $aa+ba$; then $a+b$ multiplyed by $-b$ produceth $-ba-bb$; lastly, the said Products $aa+ba$ and $-ba-bb$ added together make $aa-bb$; (for $+ba$ and $-ba$ by Addition do quite vanish;) Therefore $aa-bb$ is the Product of $a+b$ multiplyed by $a-b$.

Multiply	$a+b$
by	$a-b$
	$aa+ba$
	$-ba-bb$
Product,	$aa \quad -bb$

Moreover, If $aa-ab+bb$ be multiplyed by $a+b$, the Product will be only $aaa+bbb$; for the rest of the particular Products will vanish by Addition.

And if $a+b$ be multiplyed by it self, to wit, by $a+b$, the Product will be $aa+2ab+bb$, which is the Square of $a+b$.

Likewise the Square of $a-b$ will be found $aa-2ab+bb$.

Nor will the Operation be otherwise when Numbers are prefixed to compound Quantities proposed to be multiplyed, respect being had to the third *Sect.* of this *Chap.* as, for Example, to multiply $3a-2e$ by $3a-2e$; First $3a-2e$ multiplyed by $3a$ produceth $9aa-6ae$, and $3a-2e$ again multiplyed by $-2e$ produceth $-6ae+4ee$; which particular Products added together make $9aa-12ae+4ee$, which is the Square of $3a-2e$.

Multiply	$3a-2e$
by	$3a-2e$
	$+9aa-6ae$
	$-6ae+4ee$
Product,	$9aa-12ae+4ee$

When Absolute numbers are members of Quantities to be multiplyed, the Rules of Multiplication in Vulgar Arithmetick and those before given must be mixtly observed; as,

If it be desired to multiply . . . . $3a+6$
by the Absolute number 5

The Product will be . . . . $15a+30$

For five times $3a$ makes $15a$, and five times 6 makes 30.

Likewise, if $2aa-3$ be multiplyed by $a-6$, the Product will be $2aaa-12aa-3a+18$, and the work will stand as here you see;

Multiplicand,	$2aa-3$
Multiplicator,	$a-6$
	$+2aaa-3a$
	$-12aa+18$
Product,	$2aaa-12aa-3a+18$

For further illustration of the Multiplication of Algebraick Integers, the Learner may peruse the following Examples; in every one of which, as also in those afore-going, I begin to multiply at the left hand, because in Algebraical Multiplication it being a thing

 indifferent

indifferent to begin the work either at the right hand or the left, it will be easier to write forward than backward. And as to the placing of the particular Products, there is no necessity of observing any Order therein; for whether they be written upon one, two, or more Lines, they retain the same values, and must by Algebraical Addition be collected into one Summ to make the total Product: And therefore you may either write the particular Products all upon one line when there is room, or else upon so many several lines as there be particular Multiplyers, setting like Products (when they happen) under one another to facilitate their Addition; or otherwise, as you shall find it most convenient.

More Examples of Multiplication in Compound Algebraick Integers, according to Sect. IV.

Multiplicand,	$a + e$	$2b - 3d$	$5g - 8$
Multiplicator,	d	f	6
Product,	$da + de$	$2bf - 3fd$	$30g - 48$

Multiplicand,	$5a + 3c$	$2b + 3$
Multiplicator,	$3a - 2c$	$4b - 6$
	$+ 15aa + 9ca$	$8bb + 12b$
	$- 10ca - 6cc$	$- 12b - 18$
Product,	$15aa + 9ca - 10ca - 6cc$	$8bb + 12b - 12b - 18$
Product contracted,	$15aa - ca - 6cc$	$8bb - 18$

Multiplicand,	$3dd + 4de + ee$
Multiplicator,	$3dd - ee$
	$+ 9dddd + 12ddde + 3ddee$
	$- 3ddee - 4deee - eeee$
Product,	$9dddd + 12ddde + 3ddee - 3ddee - 4deee - eeee$
Product contracted,	$9d^4 + 12d^3e - 4de^3 - e^4$

Multiplicand,	$a + e$	$a + e$
Multiplicator,	$a + e$	$a - e$
	$aa + ae$	$aa + ae$
	$+ ae + ee$	$- ae - ee$
Product,	$aa + 2ae + ee$	$aa - ee$

Multiplicand,	$4aaa + 3aa - 2a + 1$
Multiplicator,	$aa - 5a + 6$
	$4aaaaa + 3aaaa - 2aaa + aa$
	$- 20aaaa - 15aaa + 10aa - 5a$
	$+ 24aaa + 18aa - 12a + 6$
Product,	$4aaaaa - 17aaaa + 7aaa + 29aa - 17a + 6$

Again,

Again,

$$\begin{array}{ll} \text{Multiplicand,} & 2aa + 3ba - bc \\ \text{Multiplicator,} & \underline{3aa - 2ba - cc} \\ & 6aaaa + 9baaa - 3bcaa \\ & \quad - 4baaa - 6bbaa + 2bbca \\ & \quad\quad \underline{- 2ccaa - 3bcca + bccc} \\ \text{Prod.} & 6aaaa + 5baaa \left.\begin{array}{l} -3bc \\ -6bb \\ -2cc \end{array}\right\} aa \left.\begin{array}{l} +2bbc \\ -3bcc \end{array}\right\} a + bccc \end{array}$$

V. Sometimes when Compound quantities be to be multiplyed one by the other, it will be very commodious to omit the Operation, and to set only the word *into*, or × (the sign of Multiplication) between the Quantities to be multiplyed, to signifie the Product of their Multiplication: But in such case, to avoid mistake, it will be convenient to draw a Line over each Compound quantity, to shew that every member of the one is to be multiplyed by every member of the other.

As to multiply $4aaa + 3aa - 2a + 1$ by $aa - 5a + 6$, I write

$$\overline{4aaa + 3aa - 2a + 1} \quad \text{into} \quad \overline{aa - 5a + 6}$$

$$\text{Or,} \quad \overline{4aaa + 3aa - 2a + 1} \quad \times \quad \overline{aa - 5a + 6}$$

But that + multiplyed by —, or — by + makes —; also, that — multiplyed by — makes + in the Multiplication of Compound quantities, I shall hereafter make manifest in the last *Section* of *Chap.* X I.

Chap. V.

Division in Algebraick Integers.

I. A*Lgebraical Division* doth by two Quantities, (whether they be exprest wholly by letters, or partly by letters and partly by numbers,) whereof one is called the Dividend, and the other the Divisor, find out a third called the Quotient; to wit, such a Quantity, that if it be multiplyed by the Divisor, the Product will be equal to the Dividend.

II. The nature of Division is to resolve or undo that which is composed or done by Multiplication; For the Dividend alwayes represents the Fact or Product in Multiplication, the Divisor one of the two Factors or Multiplyers, and the Quotient the other. As, if 12 be to be divided by 2, the Dividend 12 represents the Fact or Product made by the multiplication of two numbers, one of which is the Divisor 2, and the other is the Quotient sought, to wit, 6.

III. Every Fraction is equal to the Quotient of the Numerator divided by the Denominator: So $\frac{3}{4}$ is the Quotient of 3 divided by 4; for, according to the Proof of Division, If the Quotient $\frac{3}{4}$ be multiplyed by the Divisor 4, the Product will be equal to the Dividend 3. Uupon this ground, Division in Algebraick Integers, whether Simple or Compound is most commonly performed; *viz.* by setting the Dividend as the Numerator of a Fraction, and the Divisor as a Denominator; for this Fraction is equal to the Quotient sought.

As; for Example, to divide the Quantity a by b, I write $\frac{a}{b}$, which signifies that that a is divided by b; or $\frac{a}{b}$ is equal to the Quotient of the quantity a divided by the Quantity b.

In like manner, if b be propos'd to be divided by ac, I write $\frac{b}{ac}$ to represent the Quotient; also, if ac be to be divided by b, I write $\frac{ac}{b}$ to signifie the Quotient.

Again, If $2ab$ be given to be divided by $3cd$, the Quotient will be $\frac{2ab}{3cd}$; and if a be to be divided by 5, I write for the Quotient $\frac{a}{5}$; also, to divide 1 by a, I write $\frac{1}{a}$ to signifie the Quotient.

So also, If $a+b$ be given to be divided by c, the Quotient may be represented by $\frac{a+b}{c}$; and if $3a$ be to be divided by $2b-c$, the Quotient is $\frac{3a}{2b-c}$.

More Examples of Division in Algebraick Integers, according to the foregoing Sect. III.

Dividend,	bb	$2de$	$3abc$	a^4b
Divisor,	a	fg	$2dd$	$2d^3$
Quotient,	$\frac{bb}{a}$	$\frac{2de}{fg}$	$\frac{3abc}{2dd}$	$\frac{a^4b}{2d^3}$

Dividend,	$aa+bb$	$2ab-3bd$	aaa
Divisor,	c	$d+e$	$a+b-c$
Quotient,	$\frac{aa+bb}{c}$	$\frac{2ab-3bd}{d+e}$	$\frac{aaa}{a+b-c}$

Dividend,	$4aa$	$2cc+5dd$
Divisor,	3	3
Quotient,	$\frac{4aa}{3}$, or $\frac{4}{3}aa$	$\frac{2cc+5dd}{3}$, or, $\frac{2}{3}cc+\frac{5}{3}dd$.

IV. When the Dividend is equal to the Divisor, the Quotient is 1; for every Quantity contains it self once, and therefore being divided by it self gives 1 in the Quotient: As to divide 4 by 4 the Quotient is 1; likewise, a divided by a gives 1 for the Quotient; also, if $a+b$ be divided by $a+b$ the Quotient is 1; and if $3a+2cd$ be divided by $3a+2cd$ the Quotient is 1. The like is to be understood of others.

V. When the Quotient is expressed Fraction-wise, (according to *Sect.* III.) if the same letter or letters be found equally repeated in every member of the Numerator and Denominator, cast away those letters, so the remaining Quantities shall signifie the Quotient.

As, for Example, If ab be to be divided by a, the Quotient exprest Fraction-wise will be $\frac{ab}{a}$; But because the letter a is found in the Numerator and Denominator, I cast away a out of both, so b only is left, which is the Quotient of ab divided by a.

Likewise, If aa be divided by a the Quotient is $\frac{aa}{a}$, that is, a; (by casting away a out of the Numerator and Denominator.)

Again, If aaa be to be divided by aa, the Quotient will be $\frac{aaa}{aa}$, that is, a; by casting away aa out of the Numerator and Denominator. And if abc be to be divided by ab, the Quotient exprest Fraction-wise will be $\frac{abc}{ab}$, that is, c, after ab is cast out of the Numerator and Denominator.

After the same manner, If a^5 be propos'd to be divided by a^3, (that is, $aaaaa$ by aaa) the Quotient will be a^2, or aa, by expunging a^3 (or aaa) out of the Dividend and Divisor.

This

This Contraction of Division is like to the reducing of a Fraction exprest by large numbers to more simple Terms, by dividing the Numerator and also the Denominator by a common Divisor.

Again, If $ab + ac$ be to be divided by $ad - af$, the Quotient exprest Fraction-wise according to the preceding *Sect.* III. will stand thus, $\frac{ab+ac}{ad-af}$, where because the letter a is found in every member of the Numerator and Denominator, it may be quite struck out, and then the new Quotient will be $\frac{b+c}{d-f}$, which Fraction is equal to the former, and exprest by more simple Terms.

Dividend,	$ab + ac$
Divisor,	$ad - af$
Quotient,	$\frac{ab+ac}{ad-af}$
Quotient contracted,	$\frac{b+c}{d-f}$

Likewise, If $ab + a$ be divided by a, the Quotient (according to *Sect.* III.) will be $\frac{ab+a}{a}$, that is, $b + 1$; for by casting away a, there will remain $\frac{b+1}{1}$, that is, $b + 1$; (for $\frac{b}{1}$ is but b; and $\frac{1}{1}$ is 1;) but that $b + 1$ is the true Quotient it will appear by the proof of Division, for $b + 1$ multiplyed by the Divisor a will produce the Dividend $ab + a$.

So also to divide $3bc - 2b$ by $2bb + b$, I write $\frac{3c-2}{2b+1}$ for the Quotient; where observe, that although the letter b be cast out of every member of the given Dividend and Divisor, yet the number prefixt to the letter cast out must stand still in the new Quotient.

But note diligently, That in this kind of Division of Compound Algebraick Integers, a letter cannot be cancell'd or cast away, unless it be found in every member of the Dividend and Divisor; and therefore this Quotient $\frac{bc+cd}{c+f}$ cannot be contracted by casting away any letter.

More Examples of Contractions in Algebraick Division, according to the preceding Sect. V.

Dividend,	aab	$ddef$	abc	a^7
Divisor,	aa	ef	b	a^3
Quotient,	$\frac{aab}{aa}$	$\frac{ddef}{ef}$	$\frac{abc}{b}$	$\frac{a^7}{a^3}$
Quotient contracted,	b	dd	ac	a^4

Dividend,	$ab + ac - a$	$ab - 2a$
Divisor,	a	$3a$
Quotient,	$\frac{ab+ac-a}{a}$	$\frac{ab-2a}{3a}$
Quotient contracted,	$b + c - 1$	$\frac{b-2}{3}$, or, $\frac{1}{3}b - \frac{2}{3}$.

Dividend,	$2abd + 3bd$	$2ba^3 + caa - 3aa$
Divisor,	$3bb - b$	$baa - daa + aa$
Quotient,	$\frac{2ad+3d}{3b-1}$	$\frac{2ba+c-3}{b-d+1}$

VI. If

VI. If an Algebraick Integer, whether Simple or Compound, be to be divided by a simple Quantity, and there be such numbers prefixt to the letters in the Dividend and Divisor as may all be severally divided by some number as a common Divisor without leaving a Remainder, set the Quotients arising by the Division of those numbers by their common Divisor, before the letters respectively, instead of the numbers that were first prefixt: As, for Example, if $8a$ be to be divided by $6b$; First, the Quotient exprest Fraction-wise (according to *Section* III. of this *Chap.*) will be $\frac{8a}{6b}$, then dividing the prefixed numbers 8 and 6 by their common Divisor 2, I set the Quotients 4 and 3 instead of 8 and 6 before a and b; so the Quotient sought is $\frac{4a}{3b}$.

In like manner, $6abc - 3dbc$ divided by $9fbc$ gives the Quotient $\frac{2a-d}{3f}$;

Dividend,	$6abc - 3dbc$
Divisor,	$9fbc$
Quotient,	$\frac{6abc - 3dbc}{9fbc}$
Quotient contracted,	$\frac{2a-d}{3f}$

For first, the Dividend and Divisor being set Fraction-wise will stand thus, $\frac{6abc - 3dbc}{9fbc}$; then, (according to *Sect.* V.) bc is to be cast out of the Numerator and Denominator; lastly, the prefixed numbers 6, 3, and 9 being divided by their common Divisor 3, give 2, 1, and 3, which being set before the remaining letters a, d and f respectively, give the contracted Quotient $\frac{2a - 1d}{3f}$ or $\frac{2a-d}{3f}$.

More Examples of Contractions in Division, according to Sect. V. *and* VI.

Dividend,	$4cd$	$27ab$	$16gh$
Divisor,	$2c$	$9ad$	$8gh$
Quotient,	$\frac{4cd}{2c}$	$\frac{27ab}{9ad}$	$\frac{16gh}{8gh}$
Quotient contracted,	$2d$	$\frac{3b}{d}$	2

Dividend,	$18aaaa$	$30b^5c^4dd$
Divisor,	$6aa$	$5bbccd$
Quotient,	$\frac{18aaaa}{6aa}$	$\frac{30b^5c^4dd}{5bbccd}$
Quotient contracted,	$3aa$	$6b^3ccd$

Dividend,	$28bbc + 16bbd$
Divisor,	$20bb$
Quotient,	$\frac{28bbc + 16bbd}{20bb}$
Quotient contracted,	$\frac{7c + 4d}{5}$, or, $\frac{7}{5}c + \frac{4}{5}d$.

VII. If every member of a Compound quantity be multiplyed by one and the same Simple quantity, it is evident from the nature of Multiplication and Division, that if the Product of that Multiplication be divided by the said Compound quantity, the Quotient will be the Simple quantity.

As, for Example, If $b + c$ be multiplyed by a the Product will be $ba + ca$, and therefore $ba + ca$ divided by the Factor $b + c$ will give the other Factor a. And for

for the ſame reaſon, $2bca + a$, that is $2bca + 1a$, divided by $2bc + 1$ will give the Quotient a.

Likewiſe, If $6a + 5a - a$ (that is, $10a$) be divided by $6 + 5 - 1$ (that is, 10,) the Quotient will be a.

Again, If $2ba + 2ca + 2da$ be divided by $b + c + d$, the Quotient will be $2a$; and if $2baa + caa - daa - aa$ be divided by $2b + c - d - 1$, the Quotient will be aa.

More Examples of Contractions in Diviſion, according to the preceding Sect. VII.

Dividend,	$2da + 3ca$	$23b + 18b + 1b$
Diviſor,	$2d + 3c$	$23 + 18 + 1$
Quotient,	a	b

Dividend,	$2baa - 3caa$	$2af - 2bf + 2cf - 6f$
Diviſor,	$2b - 3c$	$a - b + c - 3$
Quotient,	aa	$2f$

VIII. When the Dividend and Diviſor are Compound whole Quantities, the precedent Rules of Algebraical Diviſion will not alwayes give the Quotient in the leaſt Terms; but the ſimpleſt Quotient may be found out by one of theſe two wayes, *viz.*

1. When the Dividend and Diviſor are Algebraick Integers, and there is a poſſibility of expreſſing the Quotient by an Algebraical Integer, it may be found out by the general method of Diviſion handled in the next following *Section*, which way is like that of dividing whole numbers in vulgar Arithmetick; but if the Learner find it difficult, he may wave it until he hath proceeded as far as the 8. *Chapter* of the 2. *Book*.

2. The Quotient, whether it happen to be an Algebraick Integer, or a Fraction, may be found out in its leaſt Terms by the method hereafter delivered in *Sect.* 7. *Chap.* 8. of the Second Book; where the manner of finding out all the *Aliquot parts* or juſt Diviſors, every one of which will divide the Dividend and Diviſor propos'd without any Remainder is exhibited.

IX. In this *Section* a general method of Diviſion in Algebraical Integers is handled. As to the order of the work, it agrees with that form of Diviſion in whole numbers which I have explained in Mr. *Wingate*'s Arithmetick, but the work it ſelf depends upon the preceding Rules of Algebraical *Diviſion*, *Multiplication*, and *Subtraction*, as alſo upon this Rule for diſcovering the due Sign belonging to every particular Quotient, *viz.* $+$ divided by $+$, or $-$ by $-$, gives $+$ in the Quotient; but $+$ divided by $-$, or $-$ by $+$, gives $-$ in the Quotient. Whether the Operation be begun at the right hand or the left, it mattets not; but becauſe 'tis eaſier to Write forwards than backwards, I ſhall (as in Vulgar Arithmetick) begin to Divide at the left hand, and proceed towards the right.

Example 1. Let it be required to divide $ac + ad + bc + bd$ by $c + d$.

Having placed the Dividend and Diviſor in ſuch order as you ſee in the next Page, firſt I divide $+ac$ by $+c$, according to *Sect.* 5. of this *Chap.*) and there ariſeth $+a$, ($+a$, becauſe $+$ divided by $+$ gives $+$,) therefore I write $+a$ or a in the Quotient; then multiplying the whole Diviſor $c + d$ by the ſaid Quotient a, I write the Product $ac + ad$ under the two firſt members of the Dividend towards the left hand, to wit, under $ac + ad$; that done, drawing a line under the ſaid Product $ac + ad$, I ſubtract the ſame from $ac + ad$, (the two firſt members of the Dividend) and there remains 0, which I ſet under the line, as you may ſee in the Page following.

Diviſor.

$$
\begin{array}{rl}
\text{Diviſor.} & \text{Dividend.} \qquad\qquad \text{Quotient.} \\
c + d \,\Big) & ac + ad + bc + bd \;\Big(\; a + b \\
 & ac + ad \\
\hline
 & o \quad o + bc + bd \\
 & \qquad\;\; + bc + bd \\
\hline
 & \qquad\quad\; o \quad o
\end{array}
$$

Then there remains to be divided $+bc+bd$ which I bring down to the Remainder o, and renew the work, *viz.* I divide $+bc$ by $+c$, and there ariſeth $+b$ which I write in the Quotient next after a; then multiplying the whole Diviſor $c+d$ by the ſaid Quotient b, the Product is $bc+bd$, which being ſubſcribed, and ſubtracted from that which remained to be divided, there remains o. So the Diviſion is finiſhed, and the Quotient is found $a+b$; but that it is a true Quotient the Proof will make manifeſt; for $a+b$ multiplyed by the Diviſor $c+d$ produceth the Dividend $ac+ad+bc+bd$.

Example 2. In like manner, if $aa-bb$ be to be divided by $a+b$ the Quotient will be found $a-b$; For firſt, aa divided by a gives a in the Quotient, by which multiplying the whole Diviſor $a+b$ the Product is $aa+ab$, which ſubtracted from the Dividend $aa-bb$, there remains to be divided $-bb-ab$. Now I renew the work, and divide $-bb$ by its correſpondent Diviſor $+b$, (not by a, becauſe the Quotient will be a Fraction, which is to be avoided when there is a poſſibility) and there ariſeth $-b$ to be written next after a in the Quotient, I ſay $-b$, not $+b$; for according to the Rule before given, $-$ divided by $+$ gives $-$ in the Quotient; then multiplying the whole Diviſor $a+b$ by $-b$ (laſt ſet in the Quotient) the Product is $-ab-bb$, or $-bb-ab$, which ſubtracted from $-bb-ab$ that remained to be divided, there remains o; ſo the Diviſion is finiſh'd and the Quotient is found $a-b$, to wit, ſuch a Quantity that if it be multiplyed by the Diviſor $a+b$, it will produce the Dividend $aa-bb$.

$$
\begin{array}{rl}
a + b \,\Big) & aa - bb \;\Big(\; a - b \\
 & aa + ab \\
\hline
 & \quad - bb - ab \\
 & \quad - bb - ab \\
\hline
 & \qquad o \quad o
\end{array}
$$

Example 3. Again, If it be deſired to divide $aaa+bbb$ by $aa-ba+bb$, the Quotient will be found $a+b$, and the work will ſtand thus:

$$
\begin{array}{rl}
aa - ba + bb \,\Big) & aaa + bbb \;\ldots\ldots\ldots\; \Big(\; a + b \\
 & aaa - baa + bba \\
\hline
 & \quad + bbb + baa - bba \\
 & \quad + bbb + baa - bba \\
\hline
 & \qquad\; o \quad o \quad o
\end{array}
$$

In which Example, firſt (as before) I begin at the firſt Term of the Dividend towards the left hand, and dividing aaa by aa, (not by $-ba$ nor by $+bb$, becauſe each of theſe will give a Fraction in the Quotient) there ariſeth a, which I ſet in the Quotient; then multiplying the whole Diviſor $aa-ba+bb$ by the ſaid Quotient a, the Product is $aaa-baa+bba$, which I ſubtract from the Dividend $aaa+bbb$; ſo there remains to be yet divided $+bbb+baa-bba$.

Now I renew the work, and divide $+bbb$ by its correſpondent Diviſor $+bb$, (not by $+aa$, nor by $-ba$, becauſe each of theſe gives a Fraction) and there ariſeth $+b$, which I write next after a in the Quotient; then multiplying the whole Diviſor $aa-ba+bb$ by the ſaid Quotient $+b$; the Product is $bbb+baa-bba$, which I ſet under, and ſubtract from the Quantity that remained to be divided, ſo there remains o, and the Quotient ſought is $a+b$: But that it is a true Quotient the proof will diſcover; for if the Diviſor $aa-ba+bb$ be multiplyed by the Quotient $a+b$, it will produce the Dividend $aaa+bbb$.

Exam-

Example 4. In like manner, if $aaa - bbb$ be divided by $aa + ba + bb$, the Quotient will be $a - b$, and the work will ſtand thus;

$$
\begin{array}{lll}
\text{Diviſor.} & \text{Dividend.} & \text{Quotient.} \\
aa + ba + bb \;) & aaa - bbb \ldots\ldots\ldots & (\; a - b \\
 & \underline{aaa + baa + bba} & \\
 & \quad - bbb - baa - bba & \\
 & \quad \underline{- bbb - baa - bba} & \\
 & \quad\quad 0 \quad 0 \quad 0 &
\end{array}
$$

Example 5. Again, If $9dddd + 12ddde - 4deee - eeee$ be to be divided by $3dd - ee$, the Quotient will be found $3dd + 4de + ee$, as will be manifeſt by the ſubſequent Operation.

$$
\begin{array}{lll}
3dd - ee \;) & 9dddd + 12ddde - 4deee - eeee & (\; 3dd + 4de + ee \\
 & \underline{9dddd - 3ddde} & \\
 & \quad + 12ddde + 3ddee - 4deee & \\
 & \quad \underline{+ 12ddde \qquad\quad - 4deee} & \\
 & \qquad\quad + 3ddee - eeee & \\
 & \qquad\quad \underline{+ 3ddee - eeee} & \\
 & \qquad\qquad 0 \quad\; 0 &
\end{array}
$$

In which Example, firſt I divide $9dddd$ by $3dd$, and it gives $3dd$, which I write in the Quotient; then multiplying the whole Diviſor $3dd - ee$ by the ſaid Quotient $3dd$, the Product is $9dddd - 3ddee$, which I write under the two firſt members of the Dividend, and ſubtract the ſame from the ſaid two members, ſo there remains $+ 12ddde + 3ddee$; to which I bring down $- 4deee$ (the next member of the Dividend) and it makes $+ 12ddde + 3ddee - 4deee$ which comes now to be divided; therefore I renew the work, and dividing $+ 12ddde$ by $+ 3dd$, it gives $+ 4de$, which I ſet in the Quotient next after $3dd$, then multiplying the whole Diviſor $3dd - ee$ by the ſaid Quotient $+ 4de$, the Product is $+ 12ddde - 4deee$, which I write under $+ 12ddde + 3ddee - 4deee$ (the Quantity laſt ſet apart to be divided;) and having drawn a line under the ſaid Product I ſubtract it from the ſaid particular Dividend, ſo there remains $+ 3ddee$ which I write underneath the line; that done, to the ſaid Remainder $+ 3ddee$ I bring down $- eeee$, (the laſt member of the total Dividend) and it makes $+ 3ddee - eeee$ which is yet to be divided: Therefore I renew the work, and dividing $+ 3ddee$ by $+ 3dd$, it gives $+ ee$ which I ſet in the Quotient next after $+ 4de$; (or I might here divide $+ 3ddee$ by $- ee$ in regard it will give an Algebraical Integer in the Quotient, as I ſhall ſhew in the next Example:) then multiplying the Diviſor $3dd - ee$ by $+ ee$, (laſt ſet in the Quotient,) and ſubtracting the Product $+ 3ddee - eeee$ from the quantity that remained to be divided, there now remains 0. So the Diviſion is finiſhed without any Quantity remaining, and the entire Quotient is $+ 3dd + 4de + ee$.

Note. By this General Method of Diviſion the Quotient may oftentimes be found out and expreſt various wayes, both as to the Order and Multitude of members in the Quotient, but yet the entire Quotient in each form will have one and the ſame value, as will appear by the following manner of Dividing the two quantities propos'd in the laſt Example.

Let it therefore be again propos'd to divide $9dddd + 12ddde - 4deee - eeee$ by $3dd - ee$.

Firſt, I work as before in the laſt Example to find out the two firſt members in the Quotient, to wit, $3dd + 4de$, and then there remains to be divided $+ 3ddee - eeee$ which you ſee ſtands at this mark ✱ in the following Operation: Now becauſe $+ 3ddee$ divided by $- ee$ gives an Algebraick Integer for the Quotient, to wit, $- 3dd$, therefore I write $- 3dd$ in the Quotient; then multiplying the whole Diviſor $3dd - ee$ by $- 3dd$ (laſt ſet in the Quotient) I ſubtract the Product $+ 3ddee - 9dddd$ from $+ 3ddee - eeee$ which remained to be divided; ſo there remains to be yet divided $- eeee + 9dddd$

$$\begin{array}{l|llll|l}
3dd - ee & 9dddd + 12ddde - 4deee - eeee & & & & 3dd + 4de \\
 & 9dddd - 3ddee & & & & -3dd + ee + 3dd \\
\hline
 & +12ddde + 3ddee - 4deee & & & & \\
 & +12ddde \qquad - 4deee & & & & \\
\hline
 & * \quad +3ddee - eeee & & & & \\
 & \quad\;\; +3ddee - 9dddd & & & & \\
\hline
 & \qquad -eeee + 9dddd & & & & \\
 & \qquad -eeee + 3ddee & & & & \\
\hline
 & \qquad\quad +9dddd - 3ddee & & & & \\
 & \qquad\quad +9dddd - 3ddee & & & & \\
\hline
 & \qquad\qquad 0 \quad 0 & & & &
\end{array}$$

Then I divide $-eeee$ (which ſtands immediately under the third black line) by its correſpondent Diviſor $-ee$, (for it cannot be divided by $3dd$ ſo as to give an Integer in the Quotient,) and there ariſeth $+ee$, which I ſet in the Quotient; then multiplying the whole Diviſor $3dd-ee$ by the ſaid Quotient $+ee$ the Product is $-eeee+3ddee$, which ſubtracted from $-eeee+9dddd$ (to wit, the quantity that remained to be divided) there remains to be yet divided $+9dddd-3ddee$, (which ſtands immediately under the laſt black line but one;) Therefore I divide $+9dddd$ by $+3dd$ and it gives $+3dd$ to be ſet in the Quotient; then multiplying the whole Diviſor $3dd-ee$ by the ſaid $+3dd$, it makes $+9dddd-3ddee$, which ſubtracted from $+9dddd-3ddee$ (the quantity that remained to be divided) leaves 0; ſo the Diviſion is finiſhed without any quantity remaining, and the Quotient is found $3dd+4de-3dd+ee+3dd$, that is, $3dd+4de+ee$: So that the Quotient found out by the latter Operation, after it is contracted by Algebraical Addition, is the ſame found out by the former way of dividing the Quantities given in the fifth Example.

Example 6. Again, If $yyyyyy-8yyyy-124yy-64$ be divided by $yy-16$, the Quotient will be found $yyyy+8yy+4$, and the work will ſtand thus:

$$\begin{array}{r|l|l}
\text{Diviſor.} & \text{Dividend.} & \text{Quotient.} \\
yy-16 & yyyyyy - 8yyyy - 124yy - 64 & yyyy + 8yy + 4 \\
 & yyyyyy - 16yyyy & \\
\hline
 & \quad +8yyyy - 124yy & \\
 & \quad +8yyyy - 128yy & \\
\hline
 & \qquad +4yy - 64 & \\
 & \qquad +4yy - 64 & \\
\hline
 & \qquad\quad 0 \quad 0 &
\end{array}$$

If the Powers of the Root y in the laſt Example be expreſſed according to *Carteſius* his way, the work will ſtand thus:

$$\begin{array}{r|l|l}
yy-16 & y^6 - 8y^4 - 124yy - 64 & y^4 + 8yy + 4 \\
 & y^6 - 16y^4 & \\
\hline
 & \quad +8y^4 - 124yy & \\
 & \quad +8y^4 - 128yy & \\
\hline
 & \qquad +4yy - 64 & \\
 & \qquad +4yy - 64 & \\
\hline
 & \qquad\quad 0 \quad 0 &
\end{array}$$

But

But *Cartesius* in dividing the Quantities propos'd in the last Example works backwards, *viz.* from the right hand of the Dividend towards the left, as you here see in the following Operation.

$$
\begin{array}{l}
yy - 16 \,)\, y^6 - 8y^4 - 124yy - 64 \,(\, 4 + 8yy + y^4 \\
\qquad\qquad\qquad\qquad + \; 4yy - 64 \\
\hline
\qquad\qquad\quad - 8y^4 - 128yy \\
\qquad\qquad\quad + 8y^4 - 128yy \\
\hline
\qquad\qquad\quad y^6 - 16y^4 \\
\qquad\qquad\quad y^6 - 16y^4 \\
\hline
\qquad\qquad\quad 0 \quad 0
\end{array}
$$

More Examples are here added for the fuller exercise and illustration of Division in Compound Algebraick Integers, according to the general method in Sect. IX. *of this* Chapter.

$$
\begin{array}{l}
\text{Divisor.} \qquad \text{Dividend.} \qquad\qquad \text{Quotient.} \\
2c - 3d \,)\, 6ca - 9da - 8bc + 12db \,(\, 3a - 4b \\
\qquad\quad 6ca - 9da \\
\hline
\qquad\quad 0 \quad 0 - 8bc + 12db \\
\qquad\qquad\quad - 8bc + 12db \\
\hline
\qquad\qquad\quad 0 \quad 0
\end{array}
$$

$$
\begin{array}{l}
\text{Divisor.} \qquad \text{Dividend.} \qquad\qquad \text{Quotient.} \\
a - b \,)\, aaa - 3aab + 3abb - bbb \,(\, aa - 2ab + bb \\
\qquad aaa - aab \\
\hline
\qquad\quad - 2aab + 3abb \\
\qquad\quad - 2aab + 2abb \\
\hline
\qquad\qquad + abb - bbb \\
\qquad\qquad + abb - bbb \\
\hline
\qquad\qquad 0
\end{array}
$$

$$
\begin{array}{l}
\text{Divisor.} \qquad \text{Dividend.} \qquad\qquad \text{Quotient.} \\
2aa + 3bb \,)\, 4aaaa + 12aabb + 9bbbb \,(\, 2aa + 3bb \\
\qquad\qquad 4aaaa + 6aabb \\
\hline
\qquad\qquad\quad + 6aabb + 9bbbb \\
\qquad\qquad\quad + 6aabb + 9bbbb \\
\hline
\qquad\qquad\quad 0 \quad 0
\end{array}
$$

$$
\begin{array}{l}
a + b \,)\, aaa - abb \,(\, aa - bb - ab + bb, \text{ that is, } aa - ab \\
\qquad aaa + aab \\
\hline
\qquad - abb - aab \\
\qquad - abb - bbb \\
\hline
\qquad\quad - aab + bbb \\
\qquad\quad - aab - abb \\
\hline
\qquad\qquad + bbb + abb \\
\qquad\qquad + bbb + abb \\
\hline
\qquad\qquad 0 \quad 0
\end{array}
$$

Again,

Again,

$$
\begin{array}{lll}
\text{Divisor.} & \text{Dividend.} & \text{Quotient.} \\
ab - aa \,\Big) & aab^3 + a^4b - 2a^5 & \Big(\, abb + a^3 + aab + a^3 \\
 & aab^3 - a^3bb & \\
\hline
 & + a^4b + a^3bb - 2a^5 & \\
 & + a^4b \quad\quad - a^5 & \\
\hline
 & + a^3bb - a^5 & \\
 & + a^3bb - a^4b & \\
\hline
 & - a^5 + a^4b & \\
 & - a^5 + a^4b & \\
\hline
 & 0 \quad 0 &
\end{array}
$$

$$
\begin{array}{lll}
\text{Divisor.} & \text{Dividend.} & \text{Quotient.} \\
\frac{2}{3}ab - \frac{1}{2}aa \,\Big) & \frac{4}{9}aab^3 + \frac{13}{12}a^4b - a^5. & \Big(\, \frac{2}{3}abb + \frac{13}{8}a^3 + \frac{1}{2}aab + \frac{3}{8}a^3 \\
 & \frac{4}{9}aab^3 - \frac{1}{3}a^3bb & \Big(\; viz.\ \frac{2}{3}abb + \frac{1}{2}aab + 2a^3 \\
\hline
 & + \frac{13}{12}a^4b + \frac{1}{3}a^3bb - a^5 & \\
 & + \frac{13}{12}a^4b \quad\quad - \frac{13}{16}a^5 & \\
\hline
 & + \frac{1}{3}a^3bb - \frac{3}{16}a^5 & \\
 & + \frac{1}{3}a^3bb - \frac{1}{4}a^4b & \\
\hline
 & - \frac{3}{16}a^5 + \frac{1}{4}a^4b & \\
 & - \frac{3}{16}a^5 + \frac{1}{4}a^4b & \\
\hline
 & 0 \quad 0 &
\end{array}
$$

If Algebraical Division according to this general Method will not work off just without a Remainder, then you may write the Dividend and Divisor fraction-wise, according to *Sect.* III. of this *Chapt.* Or sometimes the Quotient may be exprest partly by Integers, and partly by a Fraction; As if $bb + bd + cc$ be to be divided by $b + d$, the Quotient may be exprest either thus $\frac{bb + bd + cc}{b + d}$; or else thus, $b + \frac{cc}{b + d}$, which latter Quotient is found out by the help of the said general Method; for after you have thereby discovered as many Integers as can arise in the Quotient, you may set the remainder of the Dividend as a Numerator over the Divisor as a Denominator, so this Fraction together with the said Integer or Integers shall be equal to the Quotient sought; as in this following Example.

$$
\begin{array}{lll}
\text{Divisor.} & \text{Dividend.} & \text{Quotient.} \\
a - b \,\Big) & 2aac + 3aaa - 2abc - 3aab + 2cc & \Big(\, 2ac + 3aa + \frac{2cc}{a - b}. \\
 & 2aac \quad\quad - 2abc & \\
\hline
 & + 3aaa \quad\quad - 3aab & \\
 & + 3aaa \quad\quad - 3aab & \\
\hline
 & 0 \quad\quad 0 + 2cc &
\end{array}
$$

CHAP.

CHAP. VI.

Containing the Arithmetick of Algebraical Fractions.

Of the rise of Algebraick Fractions, and the manner of expressing Integers and mixed quantities Fraction-wise.

I. THe Operations about Algebraick Fractions are wrought like those of vulgar Fractions, by the help of the Rules of Algebraick Integers before delivered, as will appear by the following Rules of this *Chapter.*

II. From the manner of dividing quantities according to *Sect.* 3. of the preceding *Chap.* 5. Algebraick Fractions arise; as, If a be to be divided by b, the Quotient is represented by the Fraction $\frac{a}{b}$: Likewise $\frac{a+b}{c-d}$, which imports as much as the Quotient of $a+b$ divided by $c-d$; also $\frac{2aa+3cd}{bb}$, and such like, are called Algebraical Fractions.

III. If the Numerator be equal to the Denominator, that Fraction (or Quotient exprest fraction-wise) is equal to 1, (to wit, Unity;) as before hath been said in *Sect.* 4. *Chap.* 5.

$$\text{So } \frac{aa}{aa} = 1. \qquad \text{And } \frac{abc+dd}{abc+dd} = 1.$$

IV. When an Algebraick Integer is to be exprest fraction-wise, make it a Numerator, and set 1 for the Denominator; as if these quantities ab and $aa-bb$ be to be set in the form of Fractions they will stand thus;

$$\frac{ab}{1}. \quad \text{And} \quad \frac{aa-bb}{1}.$$

V. If an Algebraick Integer, as a, be to be set in the form of a Fraction that shall have for its Denominator some Algebraical Integer prescribed, as d, multiply a by the Denominator d, and write the Product ad as a Numerator over the Denominator d, thus, $\frac{ad}{d}$; which Fraction is equal to the Integer a first proposed, and hath for its Denominator the prescribed quantity d.

Likewise the quantity a reduced to the form of a Fraction whose Denominator is prescribed $b+c$ will stand thus, $\frac{ab+ac}{b+c}$.

Moreover, If $a+\frac{aa}{d}$ be to be reduced to the Form of a Fraction that shall have d for a Denominator; let a be multiplyed by the Denominator d, and to the Product ad add the Numerator aa; then set that Summ, to wit, $ad+aa$ over the Denominator d, so there will be $\frac{ad+aa}{d}$ for the Fraction desired. More Examples of this Rule are these following.

$\frac{bc}{c} = b.$	$\frac{aa+ab}{a+b} = a.$	$\frac{dda}{a} = dd.$
$\frac{bc+bb}{c} = b+\frac{bb}{c}$	$\frac{ab-ac+dd}{b-c} = a+\frac{dd}{b-c}.$	

How

How to reduce Algebraick Fractions to others of the same value in more simple Terms.

VI. When the same letter or letters be found in the Numerator and Denominator, let them be cast out of both; and if the numbers prefixt can be abbreviated by some common Divisor set the Quotients in the places of those numbers prefixt, so shall the new Fraction be of the same value with that first proposed: So this Fraction $\frac{abc}{abd}$ will be reduced to $\frac{c}{d}$; and this $\frac{12ab + 8ac}{16ad}$ will be reduced to $\frac{3b + 2c}{4d}$. This Rule hath already been explained in *Sect.* 5. and 6. of *Chap.* 5. and may be further illustrated by these following Examples.

$\frac{ad}{ac} = \frac{d}{c}$.	$\frac{12add}{4abc} = \frac{3dd}{bc}$.
$a + \frac{bcd}{cd} = a + b$.	$\frac{36aa}{4ba + 16da} = \frac{9a}{b + 4d}$.

VII. The searching out of the greatest common Divisor, for reducing an Algebraick Fraction to the smallest Terms, after the manner used in Vulgar Arithmetick, is for the most part a tedious and intricate work, especially when the Numerator and Denominator are Compound quantities consisting of many members; and therefore instead of that way of finding out a Common measure (or Divisor,) I shall by a clear Method in *Chap.* 8. of the Second *Book*, shew how to find out all such Divisors as will divide the Numerator and Denominator precisely without leaving a Remainder. But in the mean time I shall recommend to the Learners exercise the following Examples of Fractions abbreviated by Division, according to the general method in *Sect.* 9. *Chap.* 5. of this Book; which Examples, together with the Rule above-delivered in the 6. *Sect.* will be great helps to reduce Algebraical Fractions to lower terms, when there is a possibility.

Examples of Fractions reduced to their smallest Terms.

$\frac{aa + ab}{a + b} = a$	$\frac{aa - ab}{a - b} = a$.
$\frac{aac - aad}{c - d} = aa$	$\frac{aa + 2ba + bb}{a + b} = a + b$.
$\frac{a^4 + 2b^2a^2 + b^4}{aa + bb} = aa + bb$	$\frac{aa - 2ba + bb}{a - b} = a - b$.
$\frac{a^4 - 2b^2a^2 + b^4}{aa - bb} = aa - bb$	$\frac{aa - bb}{a + b} = a - b$.
$\frac{aaaa - bbbb}{aa + bb} = aa - bb$	$\frac{aa - bb}{a - b} = a + b$.
$\frac{aaaa - bbbb}{aa - bb} = aa + bb$	$\frac{aaa + bbb}{aa - ba + bb} = a + b$.

aaa +

$\frac{aaa+bbb}{a+b} = aa-ba+bb$	$\frac{aaa-bbb}{aa+ba+bb} = a-b.$
$\frac{aaa-bbb}{a-b} = aa+ba+bb$	$\frac{aaa-abb}{aa-ab} = a+b.$
$\frac{aaa-abb}{aa+ab} = a-b$	$\frac{aaaa-bbbb}{aaa-aab+abb-bbb} = a+b.$

More Examples of Fractions abbreviated.

$\frac{aa+ab}{ad+bd} = \frac{a}{d}.$ (By the common Divisor $a+b$)

$\frac{aa-ab}{ac-bc} = \frac{a}{c}.$ (By the common Divisor $a-b$)

$\frac{aac-aad}{cd-dd} = \frac{aa}{d}.$ (By the common Divisor $c-d$.)

$\frac{aaa-abb}{aa+2ab+bb} = \frac{aa-ab}{a+b}.$ (By $a+b$.)

$\frac{aaa-bbb}{aa-bb} = \frac{aa+ba+bb}{a+b}.$ (By $a+b$.)

$\frac{a^4-b^4}{aa+ab} = \frac{aaa-aab+abb-bbb.}{a}$ (By $a+b$.)

How to find out the smallest quantity that can be divided by two or more given quantities severally without a Remainder.

VIII. Two or more Algebraick quantities whether Simple or Compound being proposed, the smallest quantity that can be divided by every one of those given, without a Remainder, may be found out by the following Operation, (which is grounded upon 36. *prop.* 7. *Elem.* Euclid.) and the use thereof will hereafter appear.

As, for Example, If it be desired to find the smallest quantity that can be divided by aac and cd; set them in the form of a Fraction thus, $\frac{aac}{cd}$, and reduce the Fraction to its primitive or equivalent Fraction in the smallest Terms $\frac{aa}{d}$, which being set near the former, multiply cross-wise, *viz.* aac by d, or aa by cd, and there will arise one and the same Product, to wit $aacd$ the Quantity sought; which is the smallest quantity that can be divided severally by aac and cd without leaving any Remainder.

$$\frac{aac}{cd} \times \frac{aa}{d}$$
$$\overline{\quad aacd \quad}$$

In

In like manner to find the ſmalleſt quantity that can be divided by $ab+ac$ and $ad-af$ ſeverally, I ſet them Fraction-wiſe thus, $\frac{ab+ac}{ad-af}$, this reduced to its loweſt Terms gives $\frac{b+c}{d-f}$; then I multiply croſs-wiſe (as before) *viz.* $ab+ac$ by $d-f$ or $ad-af$ by $b+c$, and there ariſeth $abd+acd-fab-fac$, which is the ſmalleſt quantity that can be divided by $ab+ac$ and $ad-af$, ſo as to leave no Remainder.

$$\frac{ab+ac}{ad-af} \times \frac{b+c}{d-f}$$
$$abd+acd-fab-fac$$

IX. But if the given Quantities cannot be reduced to lower Terms, then multiply them one into another, and their Product is the quantity deſired: So to find the ſmalleſt quantity that can be divided by $bb+cc$ and $dd+ff$ ſeverally without leaving a Remainder; becauſe $\frac{bb+cc}{dd+ff}$ cannot be reduced to more ſimple Terms, I multiply $bb+cc$ by $dd+ff$, and there is produced $bbdd+ccdd+bbff+ccff$ the Quantity ſought.

$$\frac{bb+cc}{dd+ff} \times \frac{bb+cc}{dd+ff}$$
$$bbdd+ccdd+bbff+ccff$$

X. When three or more quantities are given, the ſmalleſt quantity that can be divided by them ſeverally without leaving a Remainder may be found out in this manner; *viz.* To find out the leaſt quantity that can be divided by $aaa-abb$, $aa+2ab+bb$ and $aa-bb$; I firſt ſeek (after the manner of the ſecond Example in *Sect.* 8.) the ſmalleſt quantity that can be divided by $aaa-abb$, and $aa+2ab+bb$, ſo I find $aaaa-aabb+aaab-abbb$; and becauſe this quantity may be alſo divided by $aa-bb$ (the third quantity propoſed) it is manifeſt that $aaaa-aabb+aaab-abbb$ is the quantity ſought.

$$\frac{aaa-abb}{aa+2ab+bb} \times \frac{aa-ab}{a+b}$$
$$aaaa-aabb+aaab-abbb$$

In like manner if there be given theſe four quantities, $aaaa-bbbb$; $aa+ab$; $aaaa+aabb$; and $a+b$; Firſt, I find out (as before) the ſmalleſt quantity $aaaaa-abbbb$ that can be divided by the firſt and ſecond quantities $aaaa-bbbb$ and $aa+ab$;

$$\frac{aaaa-bbbb}{aa+ab} \times \frac{aaa-aab+abb-bbb}{a}$$
$$aaaaa-abbbb$$

Then becauſe the ſaid $aaaaa-abbbb$ cannot be divided by the third quantity $aaaa+aabb$, I ſeek the ſmalleſt quantity that can be divided by $aaaaa-abbbb$ and $aaaa+aabb$, ſo I find (in like manner as before) $aaaaaa-aabbbb$, which, becauſe it is diviſible by the fourth quantity propoſed, to wit, by $a+b$ ſhall be the quantity ſought; *viz.* a^6-aab^4 is the ſmalleſt quantity that can be divided by every one of theſe four quantities, a^4-b^4; $aa+ab$; a^4+aabb; and $a+b$. And ſo of others.

$$\frac{aaaaa-abbbb}{aaaa+aabb} \times \frac{aa-bb}{a}$$
$$aaaaaa-aabbbb$$

How to reduce Algebraical Fractions which have different Denominators, into other Fractions of the ſame value that may have a Common Denominator.

XI. When two Fractions having different Denominators are to be reduced into two other Fractions of the ſame value that ſhall have a Common Denominator; multiply the Numerator of the firſt Fraction by the Denominator of the ſecond, and the Product ſhall be a new Numerator correſpondent to the Numerator of that firſt Fraction; Alſo, multiply

multiply the Numerator of the ſecond Fraction by the Denominator of the firſt, and the Product is a new Numerator correſpondent to the Numerator of the ſecond Fraction; laſtly, multiply the Denominators one by the other, and the Product ſhall be a common Denominator to both the new Numerators.

As, for Example, to reduce $\frac{ab}{c}$ and $\frac{bd}{a}$ (whoſe Denominators c and a are unlike) into two other Fractions that may be of the ſame value with thoſe given, and have a common Denominator; Firſt, I multiply croſs-wiſe, *viz.* the Numerator ab by the Denominator a, and the Product is aab for a new Numerator inſtead of ab; likewiſe I multiply the Numerator bd by the Denominator c, and the Product is bdc, for a new Numerator inſtead of bd; laſtly, the Denominators c and a multiplyed one by the other produce ac for a Denominator to each of thoſe new Numerators aab and bdc: So the Fractions $\frac{aab}{ac}$ and $\frac{bdc}{ac}$ are found out which have a common Denominator ac, and are equal in value to the Fractions firſt given, *viz.* $\frac{aab}{ac}$ is equal to $\frac{ab}{c}$, and $\frac{bdc}{ac}$ is equal to $\frac{bd}{a}$, as was required.

$$\frac{ab}{c} \times \frac{bd}{a}$$

$$\frac{aab}{ac}\,,\quad \frac{bdc}{ac}$$

In like manner $\frac{ad}{7bc}$ and $\frac{2bb}{5d}$ (which have unlike Denominators) will be reduced into $\frac{5daa}{35bcd}$ and $\frac{14bbbc}{35bcd}$ which have a common Denominator.

Alſo, $\frac{12}{a}$ and $\frac{b}{5}$ will be reduced into theſe, $\frac{60}{5a}$ and $\frac{ba}{5a}$.

Again, to reduce $\frac{aa+2bb}{c+d}$ and $\frac{3cc-dd}{ff}$ to a common Denominator, I multiply croſs-wiſe (as before,) *viz.* $aa+2bb$ by ff, and $3cc-dd$ by $c+d$; ſo the Products are $aaff+2bbff$, and $3ccc-cdd+3ccd-ddd$ for new Numerators; then multiplying the Denominators $c+d$ and ff one into the other, the Product is $cff+dff$ for a common Denominator, and the Fractions ſought are $\frac{aaff+2bbff}{cff+dff}$ and $\frac{3ccc-cdd+3ccd-ddd}{cff+dff}$.

XII. When three or more Fractions having unlike Denominators are to be reduced into as many other Fractions that may be of the ſame value, and have a common Denominator; Multiply the Numerator of each Fraction into all the Denominators except its own, ſo the Products made by that continual Multiplication ſhall be new Numerators; multiply alſo all the Denominators one into another, and the Product ſhall be a Denominator to every one of the new Numerators.

As, for Example, To reduce theſe three Fractions $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{2ef}{g}$ into three others that may be of the ſame value and have a common Denominator; I multiply the Numerator a into the Denominators d and g, ſo the Product adg is a new Numerator inſtead of a; again, I multiply the Numerator c into the Denominators b and g, and the Product cbg is a Numerator inſtead of c; likewiſe, multiplying the Numerator $2ef$ into the Denominators b and d, the Product $2bdef$ is a Numerator inſtead of $2ef$; laſtly, the Denominators b, d and g multiplyed one into another produce bdg for a common Denominator to thoſe three new Numerators, and the three Fractions ſought are $\frac{adg}{bdg}$, $\frac{cbg}{bdg}$ and $\frac{2bdef}{bdg}$.

$$\frac{a}{b}\,,\quad \frac{c}{d}\,,\quad \frac{2ef}{g}$$

$$\frac{adg}{bdg}\,,\quad \frac{cbg}{bdg}\,,\quad \frac{2bdef}{bdg}$$

In like manner these three Fractions $\frac{aa+8}{bb}$, $\frac{9}{aa-8}$, and $\frac{dd}{7}$ will be reduced to to these three, to wit, $\frac{7aaaa-448}{7aabb-56bb}$, $\frac{63bb}{7aabb-56bb}$, and $\frac{aaddbb-8ddbb}{7aabb-56bb}$, which have for a common Denominator $7aabb-56bb$.

XIII. But if the Denominators of the given Fractions can be reduced to lower Terms, then those Fractions may oftentimes be reduced more compendiously than by the Rules in the two last preceding *Sections*, into others in the smallest Terms that have a common Denominator, in this manner; *viz.* Seek (by the Rules in *Sect.* 8. and 10. of this *Chap.*) the smallest quantity that can be divided by every one of the Denominators without a Remainder, which quantity reserve for a common Denominator; then for the Numerators divide the common Denominator by the Denominator of the first Fraction, and multiply the Quotient by the Numerator of the first Fraction, so shall the Product be a new Numerator instead of that first Numerator; work in like manner to find out the other Numerators, and set every one of them over the common Denominator before found out.

As, for Example, to reduce these Fractions $\frac{bbbd}{aac}$ and $\frac{aaa}{cd}$ to a common Denominator; I seek first of all the smallest quantity that can be divided by the Denominators aac and cd, and I find that quantity to be $aacd$, which shall be the common Denominator; then I divide the said $aacd$ by each of the given Denominators aac and cd, and multiply the Quotients d and aa by the given Numerators $bbbd$ and aaa, so the Products $bbbdd$ and $aaaaa$ shall be the new Numerators, which being severally set over the common Denominator $aacd$, there will arise $\frac{bbbdd}{aacd}$ and $\frac{aaaaa}{aacd}$ for the Fractions sought.

Likewise, to reduce $\frac{bbbb}{aac-aad}$ and $\frac{aaa+bbb}{cd-dd}$ to a common Denominator, having first found the common Denominator $aacd-aadd$, to wit, the least quantity that can be divided by the given Denominators $aac-aad$ and $cd-dd$, I divide the said common Denominator by the said given Denominators severally, and the Quotients d and aa I multiply by the Numerators $bbbb$ and $aaa+bbb$, and then setting the Products severally over the common Denominator, the Fractions sought will be $\frac{bbbbd}{aacd-aadd}$ and $\frac{aaaaa+aabbb}{aacd-aadd}$.

Again, to reduce these three Fractions, to wit, $\frac{a-b}{aaa-abb}$, $\frac{bb}{aa+2ab+bb}$ and $\frac{aa-ab}{aa-bb}$ to a common Denominator; First (as in the first Example in *Sect.* 10. of this *Chap.*) I seek the smallest quantity that can be just divided by every one of the the three given Denominators, and I find $aaaa+aaab-aabb-abbb$, for a common Denominator; then dividing this quantity found by every one of the three given Denominators (according to the general Method in *Sect.* 9. *Chap.* 5.) the Quotients will be $a+b$, $aa-ab$ and $aa+ab$; that done, I multiply the first of those Quotients by the Numerator of the first Fraction; also the second Quotient by the second Numerator, and the third Quotient by the third Numerator; so the Products $aa-bb$, $aabb-abbb$ and $aaaa-aabb$ shall be new Numerators, which being severally set over the common Denominator first found, will give the Fractions sought, to wit, these:

$$\frac{aa-bb}{aaaa+aaab-aabb-abbb},$$

$$\frac{aabb-abbb}{aaaa+aaab-aabb-abbb},$$

$$\frac{aaaa-aabb}{aaaa+aaab-aabb-abbb},$$

Nor

Nor will the Operation be otherwise to reduce these four Fractions, to wit, $\frac{a^5}{a^4 - b^4}$, $\frac{a^3 - a^2b}{a^2 + ab}$, $\frac{a^5 - b^5}{a^4 + a^2b^2}$ and $\frac{a^2 + ab + b^2}{a + b}$, into these four following Fractions having a common Denominator.

1. $\frac{a^7}{a^6 - a^2b^4}$,
2. $\frac{a^7 - 2a^6b + 2a^5b^2 - 2a^4b^3 + a^3b^4}{a^6 - a^2b^4}$,
3. $\frac{a^7 - a^5b^2 - a^2b^5 + b^7}{a^6 - a^2b^4}$,
4. $\frac{a^7 + a^5b^2 - a^4b^3 - a^2b^5}{a^6 - a^2b^4}$.

For first by the help of the given Denominators, the smallest common Denominator $a^6 - aab^4$ is found out by the operation in the last Example of the preceding *Sect.* 10. of this *Chap.*) then the said common Denominator being divided severally by the given Denominators $a^4 - b^4$, $aa + ab$, $a^4 + aabb$, and $a + b$; the Quotients are aa, $a^4 - a^3b + aabb - ab^3$, $aa - bb$, and $a^5 - a^4b + a^3bb - aab^3$; which multiplyed respectively by the given Numerators a^5, $a^3 - aab$, $a^5 - b^5$, and $aa + ab + bb$, will produce those new Numerators which are before set over the common Denominator $a^6 - aab^4$.

Addition of Algebraical Fractions.

XIV. If two or more Fractions to be added have one common Denominator, add the Numerators together, and set their Summ as a new Numerator over the common Denominator, so shall this new Fraction be the summ of the Fractions given to be added.

As, for Example, to add $\frac{aa}{c}$ to $\frac{bb}{c}$, the Summ will be $\frac{aa + bb}{c}$.

So also, $\frac{2ab}{c + d}$ added to $\frac{3bb}{c + d}$ makes $\frac{2ab + 3bb}{c + d}$.

Likewise the Summ of $\frac{5a - 3b}{c + d}$ and $\frac{2b - 3a}{c + d}$ will be found $\frac{2a - b}{c + d}$; (For the given Numerators $5a - 3b$ and $2b - 3a$ added together make $2a - b$.)

Again, the Summ of $\frac{a - b + 24}{c + 5}$, $\frac{a + b - 24}{c + 5}$ and $\frac{4a}{c + 5}$ will be found $\frac{6a}{c + 5}$.

And if these be added, to wit, $\frac{3ab}{b + c + d}$, $a + \frac{3ac}{b + c + d}$, and $\frac{3ad}{b + c + d}$, the Summ will be $a + \frac{3ab + 3ac + 3ad}{b + c + d}$; that is, $4a$. (For by Division, $\frac{3ab + 3ac + 3ad}{b + c + d} = 3a$.)

XV. But if the Fractions propos'd to be added together have unlike Denominators, first reduce them to a common Denominator, and then add them as before; as to add $\frac{ab}{c}$ to $\frac{bd}{a}$, first I reduce them to $\frac{aab}{ac}$ and $\frac{bdc}{ac}$ which have the same Denominator ac; then setting the summ of the Numerators aab and bdc over the common Denominator ac, there will be $\frac{aab + bdc}{ac}$ for the Summ required.

So alſo to add $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{2ef}{g}$, their Summ will be found $\frac{adg + cbg + 2bdef}{bdg}$.

Likewiſe, to add theſe three Fractions $\frac{a - b}{aaa - abb}$, $\frac{bb}{aa + 2ab + bb}$ and $\frac{aa - ab}{aa - bb}$; firſt I reduce them to three others of the ſame value under a common Denominator, (as in the third Example of the preceding 13. *Sect.*) and then ſetting the Summ of the three new Numerators over the common Denominator, I find the ſumm of the given Fractions to be $\frac{aaaa + aa - abbb - bb}{aaaa + aaab - aabb - abbb}$.

XVI. When Mixed quantities are to be added together, collect the Fractions into one ſumm, and the Integers into another, then thoſe two ſumms added together give the ſumm deſired; as, for Example:

To add theſe mixed quantities . . . . $\frac{aa}{b} - a$ and $\frac{dd}{c} + d$.

The ſumm of the Fractions, after they are reduced to a common Denominator, is . . . . $\frac{caa + bdd}{bc}$

To which ſumm adding the Integers in the mixed quantities propoſed, the ſumm deſired will be $\frac{caa + bdd}{bc} - a + d$.

Or, when mixed quantities are to be added together, you may reduce them to improper Fractions, (by *Sect.* 5. of this *Chap.*) and then add theſe together as in the preceding Examples; as,

To add thoſe mixed quantities in the laſt Example, to wit, ... $\frac{aa}{b} - a$ and $\frac{dd}{c} + d$;

I firſt reduce them to theſe Fractions . . . . $\frac{aa - ba}{b}$ and $\frac{dd + cd}{c}$;

Which reduced to a common Denominator produce theſe . . . . $\frac{caa - cba}{bc}$ and $\frac{bdd + bcd}{bc}$

Which two laſt Fractions added together give the ſumm required, to wit, ... $\frac{caa - cba + bdd + bcd}{bc}$

Which is equal to the Summ before found, to wit, ... $\frac{caa + bdd}{bc} - a + d$.

Subtraction of Algebraical Fractions.

XVII. If the two Fractions given have the ſame Denominator, ſubtract the Numerator of the Fraction preſcribed to be ſubtracted, from the other Numerator, and ſet the Remainder as a new Numerator over the common Denominator, ſo ſhall this new Fraction be the Remainder ſought.

As, for Example, If from $\frac{aa}{c}$ you deſire to ſubtract $\frac{bb}{c}$, take bb from aa, and ſet the Remainder $aa - bb$ as a Numerator over the common Denominator c; ſo $\frac{aa - bb}{c}$ ſhall be the Remainder ſought.

In like manner, If from $\frac{2ab}{b - c}$ you would ſubtract $\frac{2ac}{b - c}$, the Remainder will be $\frac{2ab - 2ac}{b - c}$, that is, (by Diviſion) $2a$.

Again, If from $\frac{8aa - 7b + 6}{a + b}$ it be deſired to ſubtract $\frac{3aa + 12b - 18}{a + b}$, the Remainder

Remainder will be found $\frac{5aa - 19b + 24}{a+b}$. (For $3aa + 12b - 18$ subtracted from $8aa - 7b + 6$, leaves $5aa - 19b + 24$.)

So also, from $d + \frac{bb}{b+d}$ subtracting $\frac{bd}{b+d}$, there remains $\frac{dd+bb}{b+d}$. For, (by *Sect.* 5. of this *Chap.*) $d + \frac{bb}{b+d}$ will be reduced to $\frac{db + dd + bb}{b+d}$; from which subtracting $\frac{bd}{b+d}$, the Remainder is $\frac{dd+bb}{b+d}$.

XVIII. But if the two Fractions given have different Denominators, first reduce them to a common Denominator, and then subtract as before; so if from $\frac{dd}{c}$ it be desired to subtract $\frac{aa}{b}$, I reduce them to $\frac{ddb}{cb}$ and $\frac{aac}{cb}$, which have the same Denominator cb; then from $\frac{ddb}{cb}$ subtracting $\frac{aac}{cb}$, there remains $\frac{ddb - aac}{cb}$, which is the Remainder sought.

After the same manner, If from $\frac{aa+d}{b-c}$ you would take away $\frac{aa}{b}$, there will remain $\frac{db + aac}{bb - bc}$.

Likewise from $\frac{aaa + bbb}{cd - dd}$ to take away $\frac{bbbb}{aac - aad}$, I first reduce these given Fractions to a common Denominator, (as in the second Example of *Sect.* 13. of this *Chap.*) and so I find $\frac{aaaaa + aabbb}{aacd - aadd}$ and $\frac{bbbbd}{aacd - aadd}$, which latter Fraction subtracted from the former there remains $\frac{aaaaa + aabbb - bbbbd}{aacd - aadd}$.

Again, If from a it be desired to subtract $\frac{aa - ab}{a+b}$, I reduce a into the form of a Fraction whose Denominator shall be $a+b$, and so instead of a, I find $\frac{aa+ab}{a+b}$, from which subtracting $\frac{aa-ab}{a+b}$, there remains $\frac{2ab}{a+b}$.

Multiplication of Algebraical Fractions.

XIX. When two Algebraick Fractions are given to be multiplyed one by the other, multiply their Numerators one into the other, and take the Product for a new Numerator; likewise multiplying the Denominators one into the other, this Product shall be a new Denominator, and the new Fraction is the Product sought.

As, for Example, to multiply $\frac{2a}{c}$ by $\frac{b}{3d}$, I multiply (as in vulgar Fractions) the Numerator $2a$ by the Numerator b, and the Product $2ab$ is a new Numerator; likewise I multiply the Denominators $3d$ and c one into the other, and the Product $3dc$ shall be a new Denominator; so $\frac{2ab}{3dc}$ is the Product sought.

In like manner, $\frac{aa - bb}{c}$ multiplyed by $\frac{2ab}{b+c}$ gives the Product $\frac{2aaab - 2abbb}{bc + cc}$.

XX. When either or both the given Terms are mixed Quantities, reduce the mixt Quantity to the form of a Fraction (by the Rule in *Sect.* 5. of this *Chap.*) and and then multiply as before: So to multiply $c + \frac{bb}{d}$ by $a + \frac{ad}{c-d}$, I first reduce those

those mixt quantities to these Fractions, $\frac{cd + bb}{d}$ and $\frac{ac}{c - d}$, then multiplying the Numerator $cd + bb$ by the Numerator ac, the Product is $accd + acbb$ for a new Numerator; also multiplying the Denominators d and $c - d$ one by the other, the Product is $dc - dd$ for a new Denominator, and the Product sought is $\frac{accd + acbb}{dc - dd}$.

XXI. When an Integer is to be multiplyed by a Fraction, express the Integer fraction-wise by giving it unity, (to wit, 1) for a Denominator, (according to *Sect.* 4. of this *Chap.*) and then multiply as in the preceding Examples.

As, to multiply a by $\frac{b}{c}$, that is, $\frac{a}{1}$ by $\frac{b}{c}$, the Product will be $\frac{ab}{c}$. Likewise to multiply $aa - bb$ by $\frac{aa + bb}{cd + fg}$, I reduce $aa - bb$ to $\frac{aa - bb}{1}$, then multiplying the Numerator $aa + bb$ by the Numerator $aa - bb$, the Product $aaaa - bbbb$ shall be a new Numerator; Likewise the Denominator $cd + fg$ multiplyed by the Denominator 1 gives $cd + fg$ for a new Denominator, and the new Fraction $\frac{aaaa - bbbb}{cd + fg}$ is the Product sought.

XXII. But oftentimes there may be this useful Contraction in the Multiplication of Fractions, *viz.* When the Numerator of the one and the Denominator of the other may be severally divided by some common Divisor without a Remainder, take the Quotients instead of the said Numerator and Denominator, and then multiply as in the preceding Examples.

As, for Example, to multiply $\frac{aa + 2ab + bb}{cd - dd}$ by $\frac{dd}{a + b}$:
Forasmuch as the Numerator of the first Fraction and the Denominator of the latter may be divided severally by $a + b$ without a Remainder, I set the Quotients $a + b$ and 1 in the places of $aa + 2ab + bb$ and $a + b$; and by that exchange these Fractions will arise, to wit;

$$\frac{a + b}{cd - dd} \text{ and } \frac{dd}{1}.$$

In like manner, because $cd - dd$ the Denominator of the first of the two Fractions last above-written, and dd the Numerator of the latter Fraction, may be severally divided by d without a Remainder, I set the Quotients $c - d$ and d in the places of $cd - dd$, and dd, and so these new Fractions arise, to wit;

$$\frac{a + b}{c - d} \text{ and } \frac{d}{1},$$

Then I multiply (as before) the Numerators $a + b$ and d, one by the other, and the Product $da + db$ is a new Numerator: Also multiplying the Denominator $c - d$ by the Denominator 1, the Product $c - d$ is a new Denominator, and the new Fraction $\frac{da + db}{c - d}$ is the Product sought; being equal to that which would be made by the mutual multiplication of $\frac{aa + 2ab + bb}{cd - dd}$ and $\frac{dd}{a + b}$ the Fractions first proposed to be multplyed.

So also, If it be desired to multiply $a + \frac{bb}{a - b}$ by $a - 2b + \frac{bb}{a}$, that is, $\frac{aa - ab + bb}{a - b}$ by $\frac{aa - 2ab + bb}{a}$; Forasmuch as the Numerator $aa - 2ab + bb$ of the latter Fraction, and the Denominator $a - b$ of the former, being severally divided by their common Divisor $a - b$ will give the Quotients $a - b$ and 1; therefore I set these in the places of $aa - 2ab + bb$ and $a - b$, whence these Fractions will arise, to wit;

$$\frac{aa - ab + bb}{1} \text{ and } \frac{a - b}{a}:$$

Which

Which being multiplyed one by the other will give $\frac{aaa - 2aab + 2abb - bbb}{a}$, or $aa - 2ab + 2bb - \frac{bbb}{a}$, the Product fought.

Again, this Fraction $\frac{aac - aad - bbc + bbd}{aa + 2ab + bb}$ multiplyed by $\frac{aaa - abb}{cd - dd}$, will produce $\frac{aaaa - aaab - aabb + abbb}{ad + bd}$; For the Numerator of the firft Fraction and the Denominator of the latter being feverally divided by their common Divifor $c - d$, the Quotients will be $aa - bb$ and d; Alfo, the Denominator of the firft Fraction and the Numerator of the fecond being feverally divided by their common Divifor $a + b$, the Quotients will be $a + b$ and $aa - ab$; then fetting the two former Quotients in the places of the two firft Dividends, and the two latter Quotients in the places of the two latter Dividends, thefe two Fractions will arife, to wit;

$$\frac{aa - bb}{a + b} \text{ and } \frac{aa - ab}{d}:$$

Laftly, multiplying the Numerators $aa - bb$ and $aa - ab$ one into the other; as alfo the Denominators $a + b$ and d, (as in former Examples,) you will find the Product fought, to wit;

$$\frac{aaaa - aaab - aabb + abbb}{ad + bd}.$$

XXIII. When a Fraction is to be multiplyed by fome Integer that happens to be the fame with the Denominator of the Fraction, take the Numerator for the Product required. As, for Example, to multiply $\frac{aa + ab + bb}{a + d}$ by $a + d$; I write $aa + ab + bb$ for the Product of their multiplication.

Likewife, If $\frac{b}{c}$ be to be multiplyed by the Denominator c; I write the Numerator b for the Product. The reafon of this Contraction is evident; for if $\frac{b}{c}$ be multiplyed by c, or $\frac{c}{1}$, in the ordinary way, the Product will ftand thus, $\frac{bc}{c}$, which, by cafting away the common Factor c out of the Numerator and Denominator, gives b for the Product; to wit, the Numerator of the given Fraction $\frac{b}{c}$.

Hence alfo, if an Algebraical Fraction be to be multiplyed by fome letter or letters that are found among others in every member of the Denominator, that multiplication needs no other work but the cafting away fuch letter or letters out of the Denominator: As to multiply $\frac{ab}{cd}$ by c, the Product is $\frac{ab}{d}$; where obferve, that becaufe the multiplyer c is found in the given Denominator cd, I ftrike it quite out.

Likewife, to multiply $\frac{ab}{cd}$ by d, I write $\frac{ab}{c}$ for the Product: And to multiply $\frac{bbb - ccc}{3faa - 3gaa}$ by $3aa$, I cancel $3aa$ in the Denominator, and write $\frac{bbb - ccc}{f - g}$ for the Product required.

Note. The taking of $\frac{2}{3}$ parts of the Quantity a, imports the fame thing with the multiplying of a by $\frac{2}{3}$, and the Product may be expreft either thus, $\frac{2a}{3}$; or thus, $\frac{2}{3}a$.

Likewife $\frac{2}{5}$ of $b + c$, or the Product of $b + c$ multiplyed by $\frac{2}{5}$, may be expreft either thus, $\frac{2b + 2c}{5}$, or thus, $\frac{2}{5}b + \frac{2}{5}c$. And fo of others.

Divifion

Division in Algebraical Fractions.

XXIV. When the two given Fractions, to wit, the Dividend and Divisor, have a common Denominator, cast away the Denominator, and divide the Numerator of the Dividend by the Numerator of the Divisor; so that which ariseth shall be the Quotient sought. As, to divide $\frac{aab}{c}$ by $\frac{bb}{c}$; I cast away the common Denominator c, and divide aab by bb, so the Quotient sought is $\frac{aab}{bb}$, that is, $\frac{aa}{b}$.

In like manner, $\frac{aabb}{d}$ divided by $\frac{ab}{d}$ gives $\frac{aabb}{ab}$, that is, ab for the Quotient.

Again, If $\frac{aaa - abb}{c - d}$ be divided by $\frac{aa + 2ab + bb}{c - d}$, there will arise $\frac{aaa - abb}{aa + 2ab + bb}$, which abbreviated (by dividing the Numerator and Denominator severally by their common Divisor $a + b$) gives $\frac{aa - ab}{a + b}$ the Quotient sought.

XXV. If the given Fractions have not a common Denominator, then (as in Division of vulgar Fractions) multiply the Numerator of the Dividend by the Denominator of the Divisor, and the Product shall be a new Numerator; also, multiply the Denominator of the Dividend by the Numerator of the Divisor, and the Product shall be a new Denominator; so the new Fraction is the Quotient sought.

As, for Example, to divide $\frac{ab}{c}$ by $\frac{dd}{a}$, I multiply ab by a, and the Product is aab for a new Numerator; also, multiplying c by by dd, the Product is ddc for a new Denominator; so the Quotient sought is $\frac{aab}{ddc}$.

$$\frac{dd}{a} \Big) \frac{ab}{c} \Big(\frac{aab}{ddc}$$

Likewise, If $\frac{aa - bb}{c + d}$ be divided by $\frac{c - d}{aa + bb}$, the Quotient will be $\frac{aaaa - bbbb}{cc - dd}$; For $aa - bb$ the Numerator of the Dividend being multiplyed by $aa + bb$ the Denominator of the Divisor, the Product $aaaa - bbbb$ is the new Numerator; and $c + d$ the Denominator of the Dividend being multiplyed by $c - d$ the Numerator of the Divisor produceth $cc - dd$ for a new Denominator; whence the Quotient sought is $\frac{aaaa - bbbb}{cc - dd}$.

XXVI. But oftentimes there may be this useful Contraction in the Division of Fractions, *viz.* when either the two Numerators, or the two Denominators may be divided by some common Divisor without a Remainder, set the Quotients arising out of such Division (or imagine them to be set) in the places of the said Numerators or Denominators that were divided, and then divide as in the former Examples.

As, to divide $\frac{aa - ab}{cc}$ by $\frac{a - b}{cd}$; Forasmuch as the Numerators $aa - ab$ and $a - b$ may be reduced to more simple Terms, to wit, a and 1, (for $aa - ab$ and $a - b$ being severally divided by their common Measure $a - b$ give a and 1. And, because the Denominators cc and cd may likewise be reduced to more simple Terms c and d, (by dividing the said cc and cd by their common Divisor c,) therefore in the places of the two given Numerators $aa - ab$ and $a - b$ I set the two former Quotients a and 1, and in the places of the two given Denominators cc and cd I set the two latter Quotients c and d; so there will be $\frac{a}{c}$ and $\frac{1}{d}$ for a new Dividend and Divisor; then (as before) I multiply a by d, and the Product is ad or da for a new Numerator; Also, c multiplyed by 1 gives c for a new Denominator, and the new Fraction $\frac{da}{c}$ is

$$\frac{1}{d} \Big) \frac{a}{c} \Big(\frac{da}{c}$$

the

the Quotient sought; which is equal to that which would arise by dividing $\frac{aa-ab}{cc}$ by $\frac{a-b}{cd}$, to wit, the Fractions first proposed.

Again, If it be desired to divide $\frac{aaaa-bbbb}{aa-2ab+bb}$ by $\frac{aa+ab}{a-b}$; Forasmuch as the Numerators $aaaa-bbbb$ and $aa+ab$ may be reduced to $aaa-aab+abb-bbb$ and a by their common Divisor $a+b$; and the Denominators $aa-2ab+bb$ and $a-b$ may be reduced to $a-b$ and 1, by the common Divisor $a-b$; therefore instead of multiplying $aaaa-bbbb$ by $a-b$, I multiply the said $aaa-aab+abb-bbb$ by 1, and the Product is $aaa-aab+abb-bbb$ for a new Numerator; and instead of multiplying $aa-2ab+bb$ by $aa+ab$, I multiply $a-b$ by a; so the Product $aa-ab$ shall be a new Denominator, whence the Quotient sought is $\frac{aaa-aab+abb-bbb}{aa-ab}$.

In like manner, If $\frac{aaaa-625}{aa-10a+25}$ be divided by $\frac{aa+5a}{a-5}$, the Quotient will be $\frac{aaa-5aa+25a-125}{aa-5a}$; For $aaaa-625$ and $aa+5a$ may be reduced to $aaa-5aa+25a-125$, and a by the common Divisor $a+5$; Also, $aa-10a+25$ and $a-5$ may be reduced to $a-5$ and 1 by the common Divisor $a-5$ and 1; whence instead of the Fractions given we may divide

$$\frac{aaa-5aa+25a-125}{a-5} \text{ by } \frac{a}{1},$$

and the Quotient sought will be $\frac{aaa-5aa+25a-125}{aa-5a}$.

Again, to divide $aaa-2aab+abb$ by $\frac{aa-ab}{a+b}$, I set 1 for a Denominator under the Dividend $aaa-2aab+abb$, and it stands thus $\frac{aaa-2aab+abb}{1}$; then forasmuch as the Numerators $aaa-2aab+abb$ and $aa-ab$ may be reduced to $a-b$ and 1, (by the common Divisor $aa-ab$,) therefore instead of the given Dividend and Divisor we may take $\frac{a-b}{1}$ and $\frac{1}{a+b}$, whence the Quotient sought will be found $aa-bb$.

So also, If $aa+\frac{3abb}{a+4b}$ be to be divided by $a+b$, that is, $\frac{aaa+4aab+3abb}{a+4b}$ by $\frac{a+b}{1}$, the Quotient will be found $\frac{aa+3ab}{a+4b}$: And $\frac{xx+5x}{x-5}$ divided by $xx+5x$, gives the Quotient $\frac{1}{x-5}$: Lastly, $\frac{xx+5x}{x-5}$ divided by $x+5$ gives the Quotient $\frac{x}{x-5}$.

Chap. VII.

The Rule of Three in Quantities represented by Letters.

I. As in Vulgar Arithmetick ſo here in Algebraical, if three Quantities be given to find out a fourth in a direct Proportion, that is, when the nature of the Queſtion is ſuch; that as the firſt Term is in proportion to the ſecond, ſo is the third to the fourth ſought; then (reſpect being had to the preceding Rules of Algebraical Multiplication and Diviſion) multiply the ſecond and third Terms one into another, and divide the Product by the firſt Term; ſo the Quotient ſhall be the fourth Proportional ſought.

As, for example, If the Quantity a give b, what ſhall c give, in a direct Proportion? Or, to the ſame effect, find out a quantity which ſhall have the ſame proportion to c, as b hath to a; here I multiply b by c, and then dividing the Product bc by a, the Quotient $\frac{bc}{a}$ is the fourth Proportional ſought; as will appear by the Proof of the Rule of Three direct: For if the fourth Term $\frac{bc}{a}$ be multiplyed by the firſt Term a, the Product will be $\frac{abc}{a}$, which (by *Sect.* 5. *Chap.* 5.) is equal to bc, to wit, the Product of the ſecond Term multiplyed by the third.

$$a \,.\, b :: c \,.\, \frac{bc}{a}$$

The Proof,

$$\frac{abc}{a} = bc.$$

In like manner, If $a+b$ give d, what ſhall $c+d$ give in a Direct proportion? Anſwer, $\frac{dc+dd}{a+b}$.

Again, If 4 give 3, what ſhall $8aa$ give? Anſw. $\frac{24aa}{4}$, that is, $6aa$.

Moreover, If $aaa-aab+abb-bbb$ give $aa+bb$, what ſhall $aa-bb$ give? Anſw. $a+b$: For the ſecond and third Terms being multiplyed one by the other will produce $aaaa-bbbb$, which divided by the firſt Term $aaa-aab+abb-bbb$ (according to the general method of Diviſion in *Sect.* 9. *Chap.* 5.) gives $a+b$ the fourth Proportional ſought.

II. When any one of the three given Quantities is an Algebraick Fraction, ſet the other two if they be Integers, in the form of Fractions, by placing 1 as a Denominator under each Integer.

Alſo, when any one of the three given Quantities is compos'd of an Integer and a Fraction, let it be reduced into the form of a Fraction, (by *Sect.* 5. *Chap.* 6.) then if the Proportion be Direct, multiply and divide as before.

As, for example, If $a+\frac{bb}{c}$ give cd, what ſhall $\frac{ab}{f}$ give in a direct proportion? Anſw. $\frac{abccd}{acf+bbf}$: For firſt, $a+\frac{bb}{c}$ being reduced to the form of a fraction will ſtand thus $\frac{ac+bb}{c}$; alſo cd ſet fraction-wiſe is $\frac{cd}{1}$; then multiplying the third Term $\frac{ab}{f}$ by the ſecond Term $\frac{cd}{1}$, the Product is $\frac{abcd}{f}$, which divided by the firſt Term $\frac{ac+bb}{c}$ gives $\frac{abccd}{acf+bbf}$ for the fourth Proportional ſought.

In like manner, If $\frac{ab}{c}$ give d, then $\frac{bb}{d}$ will give $\frac{cdbb}{abd}$, that is, $\frac{cb}{a}$, (for $\frac{cdbb}{abd}$ being abbreviated according to *Sect.* 5. *Chap.* 5. gives $\frac{cb}{a}$.

Alſo,

Alſo, If $\frac{a+c}{d}$ give $\frac{aa}{bb}$; then $\frac{bb}{a-c}$ will give $\frac{daa}{aa-cc}$.

III. If after the three given Quantities are ordered or ſet in the Rule according to the uſual manner in Vulgar Arithmetick, the Proportion flows backwards, *viz.* if the nature of the Queſtion be ſuch, that as the third Term is in proportion to the ſecond, ſo is the firſt to the fourth Term ſought; then (as in the Inverſe or backward Rule of Three in Vulgar Arithmetick) multiply the firſt and ſecond Terms one by the other, and divide the Product by the third, ſo the Quotient ſhall be the fourth Proportional ſought. But I ſhall not need to give Examples of this Rule, nor to make application of Algebraical Arithmetick to the Double Rule of Three, Rules of Fellowſhip and Alligation; ſince he that underſtands the manner of working thoſe Rules in Vulgar Arithmetick, as alſo the Rules of Algebraical Arithmetick before delivered, cannot miſs of performing the like work Algebraically when there is occaſion.

CHAP. VIII.

An Introduction to the Extraction of ROOTS out of Algebraical Quantities.

I. IT is not my deſign in this Chapter to treat of the Extraction of *Roots* in general, (that Doctrine being hereafter handled in the third and fourth Chapter of the ſecond Book) but chiefly to ſhew how to extract the Roots or ſides of Simple Powers expreſt by Letters, as alſo of Squares formed from Rational Binomial Roots, in order to the explication of divers Equations in the following Chapters: For I would not willingly affright the Learner with tedious and intricate Operations until he hath had a conſiderable taſte of the practice of Algebra in the ſolving of Arithmetical Queſtions.

II. As in Vulgar Arithmetick, the extraction of the Square root of a given number imports nothing elſe but the finding out ſuch a number that being multiplyed by it ſelf will produce the given number; ſo the extracting of the Square root of the quantity aa implyes onely the finding out ſuch a quantity, which if it be multiplyed by it ſelf will produce aa; and ſince a multiplyed by a produceth aa, therefore a is the Root or ſide of the Square aa.

Likewiſe the ſquare Root of $4bb$ is $2b$; for $2b$ multiplyed by $2b$ produceth $4bb$: And for the ſame reaſon, the ſquare Root of $\frac{1}{4}aa$ (or $\frac{aa}{4}$) is $\frac{1}{2}a$; (or $\frac{a}{2}$.) Alſo, the ſquare Root of $bbaa$ is ba; and the ſquare Root of $aaaa$ is aa.

Moreover, Foraſmuch as aa, or the Square of the Root a, being multiplyed by the Root a produceth aaa, or the Cube of a, therefore the cubick Root of aaa being extracted there will come forth again the Root a. In like manner, the cubick Root of $8aaa$ is $2a$; for $2a$ multiplyed cubically, (that is, firſt by it ſelf and then again by the Product) produceth $8aaa$.

III. The like is to be underſtood in the extraction of the Root of a Compound Power; For, as the Binomial Root $a+b$, which may repreſent the Summ of the two parts into which ſome Number or Right line is divided, being ſquared or multiplyed by it ſelf, produceth the Square $aa+2ab+bb$; So the ſquare Root of $aa+2ab+bb$ being extracted, there will ariſe the Root $a+b$. Here the Learner may obſerve, That if a Number or Right-line be divided into any two parts, (a and b) the Square ($aa+2ab+bb$) which is made of $a+b$) the Summ of the parts, is compoſed of (aa and bb) the Squares of the parts, and of ($2ab$) the double Product made by the multiplication of the parts (a and b) one into the other.

$$\begin{array}{rl} a+b. & \text{The Root.} \\ a+b & \\ \hline aa+ab & \\ +ab+bb & \\ \hline aa+2ab+bb. & \text{The Square.} \end{array}$$

So the Square of 8, or of $5+3$, is equal to $25+9+30$, that is, 64.

Again, As the Binomial, or (as ſome call it) the Reſidual Root $a-b$, or $b-a$ being multiplyed by it ſelf produceth the Square $aa-2ab+bb$; So the ſquare Root of $aa-2ab+bb$ being extracted, there will come forth the Root $a-b$, or $b-a$; (for either of theſe Roots will produce the ſame Square.) Here alſo the Learner may obſerve, That if a Number or Right-line be divided into any two parts, (a and b) the Square ($aa-2ab+bb$) which is made by the multiplication of ($a-b$, or $b-a$) the difference of the parts into it ſelf, is equal to ($aa+bb$) the ſumm of the Squares of the parts, leſs by ($2ab$) the double Product of the Multiplication of the parts one into the other: So the Square of $5-3$, that is, of 2, is equal to $25+9-30$, that is, 4.

$$\begin{array}{rl} a-b. & \text{The Root.} \\ a-b & \\ \hline aa-ab & \\ -ab+bb & \\ \hline aa-2ab+bb. & \text{The Square.} \end{array}$$

IV. From what hath been ſaid in the laſt Section, this Theorem may be inferr'd, *viz.* If a Compound quantity conſiſts of three ſuch members or Simple quantities, that two of them are Squares, each of them having the ſign $+$ prefixt to it, and the third is the double Product made by the mutual multiplication of the Roots of thoſe ſimple Squares, the ſaid double Product alſo having the ſign $+$ prefixt to it; that Compound quantity ſhall be a Square whoſe Root is the ſumm of the two Roots of the ſaid two ſimple Squares: But if the ſaid double Product hath the ſign $-$ prefixt to it, then the difference of the ſaid Roots ſhall be the Root of the ſaid compound Square.

Hence $aa+6a+9$ will be found a Square, whoſe Root is $a+3$; for it is evident that aa and 9 are Squares, whoſe Roots are a and 3; and $6a$ is the double Product of the multiplication of thoſe Roots a and 3 one by the other.

Likewiſe, $9bb+6bc+cc$ is a Square, whoſe Root is $3b+c$; for $9bb$ and cc are Squares whoſe Roots are $3b$ and c, and $6bc$ is the double Product of the multiplication of the Roots $3b$ and c one into the other. Alſo, $aaaa+baa+\frac{1}{4}bb$ will be found a Square, whoſe Root is $aa+\frac{1}{2}b$.

Moreover, (agreeable to the latter Caſe in the Theorem) This Compound quantity $aa-10a+25$ will be diſcovered to be a Square whoſe Root is $a-5$, or $5-a$. And $bbaa-2bca+cc$ is a Square whoſe Root is $ba-c$, or $c-ba$; For from either of theſe Roots the ſame Square $bbaa-2bca+cc$ will be produced by Algebraical Multiplication.

If the Learner be well vers'd in this Theorem, he may oftentimes diſcern at firſt ſight whether a Compound quantity that conſiſts of three members or Single quantities be a Square or not; and if a Square, what its Root is.

V. If a quantity out of which a Root is to be extracted be ſuch, that the Root cannot any manner of way be exactly extracted; that Root is uſually deſign'd or repreſented by prefixing the Radical ſign before the Quantity propoſed. So to extract the ſquare Root of the quantity a, (whether it repreſents a Plane number or a Superficies) I write $\surd a$, or $\surd(2)a$, which ſignifies that the ſquare Root of a is extracted or to be extracted.

So alſo, $\surd:\overline{aa+bb}$: or, $\surd(2):\overline{aa+bb}$: denotes the Square Root of the ſumm of the Squares aa and bb.

Likewiſe, to extract the Cubick Root of b, I write $\surd(3)b$; as alſo $\surd(3)aab$, to ſignifie the Cubick Root of aab; which kind of Roots are called Surd or Irrational Quantities. (As hereafter in *Chap* 9. of the II. *Book* will be more fully declared.)

VI. When it is required to extract the Root of a Fraction, the Root of the Numerator and the Root of the Denominator ſhall give a new Fraction which is the Root ſought. As, for example, If the Square root of $\frac{aa}{bb}$ be deſired; foraſmuch as the ſquare Root of aa is a, and the ſquare Root of bb is b; I write $\frac{a}{b}$ for the Root ſought.

In

In like manner, the ſquare Root of $\frac{aabb}{dd}$ is $\frac{ab}{d}$; (for the ſquare Root of $aabb$ is ab, and the Root of dd is d.)

Again, the ſquare Root of $\frac{aa + 6a + 9}{bb}$ is $\frac{a + 3}{b}$; For (by the foregoing *Sect.* 4.) the ſquare Root of the Numerator $aa + 6a + 9$ is $a + 3$; and the ſquare Root of the Denominator bb is b. Alſo, the ſquare Root of $\frac{9bb + 6bc + cc}{\frac{1}{4}dd}$ is $\frac{3b + c}{\frac{1}{2}d}$; and the cubick Root of $\frac{27ddd}{64}$ is $\frac{3d}{4}$, or $\frac{3}{4}d$.

VII. But if the Root ſought cannot be extracted out of the Numerator and Denominator as before, the Radical ſign is to be ſet before the given Fraction; as to extract the ſquare Root of $\frac{aa}{b}$, I write $\sqrt{\frac{aa}{b}}$; or becauſe the ſquare Root of the Numerator is a, the ſquare Root of $\frac{aa}{b}$ may be expreſt thus $\frac{a}{\sqrt{b}}$; likewiſe the ſquare Root of $\frac{aa + bb}{aabb}$ may be written either thus, $\sqrt{\frac{aa + bb}{aabb}}$; or thus, $\frac{\sqrt{:aa + bb:}}{ab}$.

Chap. IX.

Which teacheth how by any two of the three Members of a Square formed from a Binomial Root, to find out the third Member.

I. From *Sect.* 3. of the precedent 8. *Chap.* it is evident that every Square formed from a Binomial Root, that is, a Root of two Names or Parts, conſiſts of three Members or diſtinct Quantities, to wit, two Affirmative Squares, and the double of the Product made by the mutual multiplication of the two Roots of thoſe Squares; which double Product is ſometimes Affirmative, and ſometimes Negative: So each of theſe compound Squares $9aa + 12a + 4$, and $9aa - 12a + 4$, whoſe Roots are $3a + 2$, and $3a - 2$, (or $2 - 3a$) conſiſts of two Squares, to wit, $9aa$ and 4, together with $12a$, the double Product of $3a$ multiplyed by 2; which $3a$ and 2 are the Roots of the ſaid Squares $9aa$ and 4: Now if any two of the three members of a Square formed from a Binomial root be given, we may find out the third member by one of theſe two following Rules.

II. When two Affirmative Squares are given as two of the three members or parts of a compound Square formed from a Binomial root to find out the third or mean member; extract the Square root out of each of thoſe given Squares, then the double of the Product made by the multiplication of thoſe Roots one into the other ſhall be the mean or middle member ſought, which if it be annexed to the two given Squares either by $+$ or $-$, will make a compleat Compound Square having a Binomial Root.

As, for example, If the Squares $9aa$ and 4 be given, firſt I extract their Roots which are $3a$ and 2, then multiplying theſe Roots one by the other the Product is $6a$, which doubled makes $12a$, the middle member ſought; this joyned by $+$ to the ſumm of the given Squares $9aa$ and 4 makes the compound Square $9aa + 4 + 12a$, or $9aa + 12a + 4$, whoſe Root is $3a + 2$: But if the ſaid double Product $12a$ be joyned to the ſumm of the Squares by $-$, there will ariſe the compound Square $9aa + 4 - 12a$, or $9aa - 12a + 4$; whoſe Root is $3a - 2$, or, $2 - 3a$.

In like manner, If $4aa$ and $9bb$ be propos'd as two of the three members of a compound Square that hath a Binomial Root, the third member will be found $12ab$; and the Square ſought will be either $4aa + 12ab + 9bb$, whoſe Root is $2a + 3b$; or elſe $4aa - 12ab + 9bb$, whoſe Root is $2a - 3b$, or $3b - 2a$.

III. When

III. When the double Product and either of the two Affirmative Squares aforesaid are given as two of the three members of a compound Square having a Binomial Root, to find out the other Square or third member; divide half the said double Product by the Root of the given Square, and the square of the Quotient shall be the third member sought, which added by + to the two given members will compleat the Compound Square.

As, for example, If $9aa + 12a$ be proposed; the half of $12a$ is $6a$; this divided by $3a$ (the square Root of $9aa$) gives 2 whose Square is 4, which added by + to $9aa + 12a$ makes $9aa + 12a + 4$, which is a compleat Compound Square, whose Root is $3a + 2$.

In like manner, If $12a + 4$ be given; the half of $12a$ is $6a$, which divided by 2, (the square Root of 4) gives $3a$, whose Square is $9aa$, which added by + to $12a + 4$, makes the compound Square $12a + 4 + 9aa$, that is, $9aa + 12a + 4$, whose Root is $3a + 2$.

Again, If $aa - 2ba$ be given; the half of $2ba$ is ba, which divided by a, (the square Root of aa) gives the Quotient b, whose Square is bb; which added to $aa - 2ba$ makes the Square $aa - 2ba + bb$, whose Root, because — is prefixt to $2ba$, shall be $a - b$, or, $b - a$; But if + had been prefixt to $2ba$, then the Root would have been $a + b$, or $b + a$.

Note. If the said Affirmative Square given be exprest by letters, and hath only 1 (to wit, Unity) prefixt to it, then instead of the Rule above delivered in this *Sect.* 3. there may be this *Compendium*, *viz.* The Square of half that quantity which in the double Product given is drawn into the Root of the given Square shall be the third Member sought to compleat the compound Square: As in the last Example, where $aa - 2ba$ was given, because 1 is prefixt (or must be imagined to be prefixt) to aa; I take the half of $2b$ to wit, b, which multiplyed by it self gives bb, which added by + to $aa - 2ba$, will make (as before) the compleat Compound Square $aa - 2ba + bb$. So also to make $aa + 6da$ a Compleat Square, I take the half of $6d$ which is $3d$, whose Square $9dd$ added by + to $aa + 6da$ makes the compound Square $aa + 6da + 9dd$, whose Root is $a + 3d$. This will be further illustrated in the next *Section*.

IV. If a Compound quantity consists of two such quantities that one of them is an Affirmative Square exprest by letters, before which 1 is prefixt, (or suppos'd to be prefixt) and the other is the Product made by the multiplication of the Root of that Square by some quantity, which is usually called the Coefficient; that Compound quantity may be made a compleat Square thus, *viz.* Add by the sign + the Square of half the Coefficient to the Compound quantity given, so shall the summ be a Square, whose Root, when + is prefixt to the said Product, is the summ of the Roots of the Square given and the Square added: But when — is prefixt to the said Product, then the Root of the Compound Square found shall be the difference of those two Roots.

As, for example, If the Compound quantity $aa + ca$ be proposed, I take the half of the Coefficient c, to wit, $\frac{1}{2}c$; then the Square of $\frac{1}{2}c$ is $\frac{1}{4}cc$, which added to $aa + ca$ makes $aa + ca + \frac{1}{4}cc$; which is a Square whose Root or Side is $a + \frac{1}{2}c$, to wit, the summ of the Roots of the Squares aa and $\frac{1}{4}cc$; But if the said $\frac{1}{4}cc$ be added to $aa - ca$, then there will arise the Square $aa - ca + \frac{1}{4}cc$, whose Root is $a - \frac{1}{2}c$, or $\frac{1}{2}c - a$.

In like manner, To make $aa + 5ba$ a compleat Square, and to discover its Root; I take the half of $5b$, to wit, $\frac{5}{2}b$, the Square whereof is $\frac{25}{4}bb$, which added to the given Compound quantity $aa + 5ba$ makes $aa + 5ba + \frac{25}{4}bb$, which is a Square whose Root is $a + \frac{5}{2}b$, as will easily appear by multiplying the said Root into it self.

So also, To make $aa - 12a$ a perfect Square, I add 36 (the Square of half the Coefficient 12) to $aa - 12a$, and it makes the compound Square $aa - 12a + 36$, whose Root is $a - 6$, or $6 - a$.

Again, To find what Quantity must be added to $aaaa + aa$, or $aaaa + 1aa$, to make a compleat Square; I take $\frac{1}{2}$, to wit, half the Coefficient 1 which is prefixt to aa, (the Square root of $aaaa$) and then the Square of the said $\frac{1}{2}$ is $\frac{1}{4}$; this added to $aaaa + 1aa$ makes the Square $aaaa + 1aa + \frac{1}{4}$, or, $aaaa + aa + \frac{1}{4}$, whose Root is $aa + \frac{1}{2}$, to wit, the summ of the Roots of the Squares $aaaa$ and $\frac{1}{4}$.

After

After the ſame manner, To make this Compound Quantity a compleat Square, ...	$aa + \frac{2b+3c}{d}a.$
I take the half of the Coefficient $\frac{2b+3c}{d}$, to wit, ...	$\frac{2b+3c}{2d}$,
Then the Square of that half Coefficient is	$\frac{4bb+12bc+9cc}{4dd}$;
Which Square added to the Compound quantity propoſed, makes	$aa+\frac{2b+3c}{d}a+\frac{4bb+12bc+9cc}{4dd}$,
Which laſt Compound quantity is a Square, whoſe Root is	$a+\frac{2b+3c}{2d}$.

Likewiſe, If it be deſired to make this Compound quantity a compleat Square, to wit, $aaaaaa + baaa$, I add to it the Square of half the Coefficient b, to wit, $\frac{1}{4}bb$; ſo there will be $aaaaaa + baaa + \frac{1}{4}bb$ the Square deſired, whoſe Root is $aaa + \frac{1}{2}b$.

CHAP. X.

A Collection of eaſie Queſtions to exerciſe the Rules hitherto delivered.

I. THere are two Quantities whereof the greater is a (or, 3,) the leſſer is e, (or 2,) What is their Summ? What is their Difference? What is the Product of their Multiplication? What is the Quotient of the greater divided by the leſſer? What is the Quotient of the leſſer divided by the greater? What is the Summ of their Squares? What is the Difference of their Squares? What is the ſumm of the Summ and Difference of the two Quantities firſt propoſed? What is the difference of their Summ and Difference? What is the Product made by the multiplication of the Summ by the Difference? What is the Square of the Summ? What is the Square of the Difference? What is the Summ of the Squares of the Summ and Difference? What is the Difference between the Square of the Summ, and the Square of the Difference? What is the Square of the Product of the multiplication of the ſaid two Quantities?

Anſwers	*by Letters,*	*by Numbers.*
1. The Summ of the two Quantities propoſed is	$a+e$	5
2. Their Difference, or the exceſs of the greater above the the leſs, is	$a-e$	1
3. The Product of their Multiplication is	ae	6
4. The Quotient of the greater divided by the leſs is . .	$\frac{a}{e}$	$\frac{3}{2}$
5. The Quotient of the leſſer divided by the greater is	$\frac{e}{a}$	$\frac{2}{3}$
6. The Summ of their Squares is	$aa+ee$	13
7. The Difference of their Squares is	$aa-ee$	5
8. The ſumm of the Summ and Difference of the two Quantities firſt propoſed is	$2a$	6
9. The difference of their Summ and Difference is . . .	$2e$	4
10. The Product of the Multiplication of the Summ by the Difference is	$aa-ee$	5
11. The Square of the Summ is	$aa+2ae+ee$	25
12. The Square of the Difference is	$aa-2ae+ee$	1

13. The

13. The Summ of the Squares of the Summ and Difference is	$2aa + 2ce$	26
14. The difference between the Square of the Summ and the Square of the Difference is	$4ac$	24
15. The Square of the Product of the multiplication of the two Quantities is	$aace$	36

In like manner, If the greater of two Quantities be c, (or 4,) and the lesser be $\frac{b-d}{c}$; (which we may suppose to represent $\frac{20-12}{4}$, that is, 2; by putting b for 20, and d for 12;) then

1. The Summ of those two Quantities will be	$c + \frac{b-d}{c}$	6
2. Their Difference is	$c - \frac{b-d}{c}$	2
3. The Product of their Multiplication is	$b - d$	8
4. The Quotient of the greater divided by the less is . .	$\frac{cc}{b-d}$	2
5. The Quotient of the lesser divided by the greater is	$\frac{b-d}{cc}$	$\frac{1}{2}$
6. The summ of their Squares is	$cc + \frac{bb - 2bd + dd}{cc}$	20
7. The difference of their Squares is	$cc - \frac{bb - 2bd + dd}{cc}$	12
8. The summ of the Summ and Difference of the two quantities is	$2c$	8
9. The difference between the Summ and Difference is	$\frac{2b - 2d}{c}$	4
10. The Product of the Summ multiplyed by the Difference is	$cc - \frac{bb - 2bd + dd}{cc}$	12

II. There are two Quantities whose Summ is b, (or 20,) and the greater of them is put a, (or 12;) What is the Lesser? What is their Difference? What is the Product of their multiplication? What is the Summ of their Squares? What is the Difference of their Squares?

1. If from the Summ of two quantities the greater be subtracted, the Remainder shall be the lesser; therefore the lesser quauntity sought is	$b - a$	8
2. If from the greater quantity a, the lesser $b-a$ be subtracted, the Remainder or Difference will be . . .	$2a - b$	4
3. The Product of the multiplication of the two quantities is	$ba - aa$	96
4. The Summ of their Squares is	$2aa + bb - 2ba$	208
5. The Difference of their Squares is	$2ba - bb$	80

1. But if the Summ of two quantities be represented by	b	20
2. And for the lesser of them there be put	e	8
3. The Greater quantity shall be	$b - e$	12
4. Their Difference shall be	$b - 2e$	4
5. The Product of their Multiplication	$be - ee$	96
6. The Summ of their Squares	$2ee + bb - 2be$	208
7. The Difference of their Squares	$bb - 2be$	80

III. There

III. There are two Quantities whose Difference is d, (or 4,) and if for the Greater quantity there be put a, (or 12 ;) What is the Lesser? What is their Summ? What is the Product of their Multiplication? What is the Summ of their Squrres? What is the Difference of their Squares?

1. By subtracting the Difference from the Greater quantity, the Lesser will be	$a - d$	8
2. The Summ of the two quantities is	$2a - d$	20
3. The Product of their Multiplication is	$aa - da$	96
4. The Summ of their Squares is	$2aa + dd - 2da$	208
5. The Difference of their Squares is	$2da - dd$	80
1. But if the Difference of two quantities be	d	4
2. And for the Lesser quantity you put	e	8
3. The Greater shall be the summ of the Difference and the Lesser, to wit,	$d + e$	12
4. The Summ of the two Quantities is	$d + 2e$	20
5. The Product of their Multiplication is	$de + ee$	96
6. The Summ of their Squares is	$dd + 2de + 2ee$	208
7. The Difference of their Squares is	$dd + 2de$	80

IV. There are two Quantities, whereof the Greater hath such proportion to the Lesser as r (3) to s, (2,) now if for the Greater quantity there be put a, (15,) What is the Lesser? What is their Summ? What is their Difference? What is the Product of their Multiplication? What is the Summ of their Squares? What is the Difference of their Squares?

1. First, say by the Rule of Three, If r give s, what will a give? *Answ.* $\frac{sa}{r}$, which is the Lesser quantity sought	$\frac{sa}{r}$	10
2. Then the Summ of the two quantities will be	$a + \frac{sa}{a}$	25
3. Their Difference is	$a - \frac{sa}{r}$	5
4. The Product of their Multiplication is	$\frac{saa}{r}$	150
5. The Summ of their Squares is	$aa + \frac{ssaa}{rr}$	325
6. The Difference of their Squares is	$aa - \frac{ssaa}{rr}$	125

But if the Lesser of two quantities be e (10,) and hath such proportion to the Greater as s (2,) to r (3 ;) Then

1. The Greater quantity will by the Rule of Three be found	$\frac{re}{s}$	15
2. And the Summ of the two quantities will be	$\frac{re}{s} + e$	25
3. Their Difference is	$\frac{re}{s} - e$	5
4. The Product of their Multiplication is	$\frac{ree}{s}$	150
5. The Summ of their Squares is	$\frac{rree}{ss} + ee$	325
6. The Difference of their Squares is	$\frac{rree}{ss} - ee$	125

V. There are two Quantities, the Product of whose multiplication is b (20;) and if for the Greater quantity there be put a (5,) What is the Lesser? What is their Summ? What is their Difference? What is the Summ of their Squares? What is the Difference of their Squares?

1. The Product b divided by the Greater quantity a, gives the Lesser, to wit, ...	$\frac{b}{a}$	4
2. Then the Summ of the two quantities is	$a + \frac{b}{a}$	9
3. Their Difference is	$a - \frac{b}{a}$	1
4. The Summ of their Squares is	$aa + \frac{bb}{aa}$	41
5. The Difference of their Squares is	$aa - \frac{bb}{aa}$	9

But if the Product of the multiplication of two quantities be b (20,) and for the Lesser there be put e (4.)

1. The greater quantity will be	$\frac{b}{e}$	5
2. The Summ of the two quantities is	$\frac{b}{e} + e$	9
3. The Difference is	$\frac{b}{e} - e$	1
4. The summ of their Squares is	$\frac{bb}{ee} + ee$	41
5. The difference of their Squares is	$\frac{bb}{ee} - ee$	9

VI. The extraction of Roots may be exercised by these following Questions, respect being had to *Sect.* 28. *Chap.* 1. as also *Chap.* 8.

1. What is the square Root of $144aa$? *Answ.* $12a$.
2. What is the square Root of $\frac{169}{16}aabb$? *Answ.* $\frac{13}{4}ab$.
3. What is the square Root of $9aa - 6ab + bb$? *Answ.* $3a - b$, or, $b - 3a$.
4. What is the square Root of $\frac{4aa + 16ab + 16bb}{9cc}$? *Answ.* $\frac{2a + 4b}{3c}$.
5. What is the Cubick Root of $125aaabbb$? *Answ.* $5ab$.
6. If b be put for 65, and c for 8. what number is signified by $\sqrt{:b + \frac{1}{4}cc:} - \frac{1}{2}c$? *Answ.* 5.
7. The same things being put as in the last Question, what number is signified by $\sqrt{:b + \frac{1}{4}cc:} + \frac{1}{2}c$? *Answ.* 13.
8. If d be put for 8, and f for 48, what number is signified by $\sqrt{:\sqrt{f + \frac{1}{4}dd} - \frac{1}{2}d:}$ *Answ.* 2.
9. But the same things being put as in the last Question, this quantity $\sqrt{:\sqrt{f + \frac{1}{4}dd} + \frac{1}{2}d:}$ signifies $\sqrt{12}$, or, 3.464, &c. that is, $3\frac{464}{1000}$, &c.
10. If g be put for 4, and h for 837, what number is signified by $\sqrt{(3):\sqrt{h + \frac{1}{4}gg} - \frac{1}{2}g:}$ *Answ.* 3.
11. But the same things being put as in the last Question, this quantity $\sqrt{(3):\sqrt{h + \frac{1}{4}gg} + \frac{1}{2}g:}$ signifies $\sqrt{(3)}\,31$, or, 3.141, &c.

VII. The Rules of the ninth *Chap.* may be exercised by these following Questions.

1. What Quantity is that which if it be added to $aa + 25$, will make the summ a Square? *Answ.* The Quantity to be added may be either $+ 10a$, or $- 10a$; and

and the Square sought is either $aa + 10a + 25$, whose Root or side is $a + 5$; or else the Square is $aa - 10a + 25$, whose Root is $a - 5$, or $5 - a$.

2. What Quantity is that which if it be added to $\frac{9}{16}aa + \frac{4}{9}bb$, will make the summ a Square? *Answ.* The Quantity to be added may be either $+ ab$, or $- ab$; and the Square is either $\frac{9}{16}aa + ab + \frac{4}{9}bb$, whose Root is $\frac{3}{4}a + \frac{2}{3}b$: Or else the Square is $\frac{9}{16}aa - ab + \frac{4}{9}bb$, whose Root is $\frac{3}{4}a - \frac{2}{3}b$; or $\frac{2}{3}b - \frac{3}{4}a$.

3. What Quantity is that which if it be added to $aa + 3a$ will make the summ a Square? *Answ.* The Quantity to be added is $\frac{9}{4}$; and the Square is $aa + 3a + \frac{9}{4}$, whose Root is $a + \frac{3}{2}$.

4. What Quantity is that which together with $aaaa - 2bbaa$ will make a perfect Square? *Answ.* The Quantity to be added is $bbbb$; and the Square is $aaaa - 2bbaa + bbbb$, whose Root is $aa - bb$, or $bb - aa$.

5. What Quantity is that which if it be added to $aa + \frac{bb}{c}a$, will make the summ a Square? *Answ.* The Quantity to be added is $\frac{bbbb}{4cc}$; and the Square is $aa + \frac{bb}{c}a + \frac{bbbb}{4cc}$, whose Root is $a + \frac{bb}{2c}$.

6. What Quantity is that which together with $aaaaaa - aaa$ will make a compleat Square? *Answ.* The Quantity to be added is $\frac{1}{4}$; and the Square sought is $aaaaaa - aaa + \frac{1}{4}$, whose Root is $aaa - \frac{1}{2}$, or $\frac{1}{2} - aaa$.

CHAP. XI.

Concerning an Equation, and the Reduction of Equations.

I. AN *Equation* in the Algebraical Art is a mutual Comparing of two Equal quantities or things of different Denominations: as, If the value of three shillings be compared to thirty six pence of *English* money, that comparison imports an Equation, which may be Symbolically exprest thus, $3s = 36d$, that is, three shillings are equal to thirty six pence. Likewise, forasmuch as nine Crowns are of equal value with the summ of two Pounds and five Shillings of *English* money; the comparing of these two summs to one another is nothing else but an Equation which may be briefly exprest thus, $9c = 2l + 5s$. In each of which Equations the Moneys compared are of different kinds; for Equations between equal things of one and the same name, as $2s = 2s$, or $5 = 5$, and such like, are fruitless.

After the same manner, this Equation $a = b + c$ may signifie that some number or line represented by a is equal to two other numbers or lines b and c taken together as one; or, if the number or line a be divided into two parts b and c, then also $a = b + c$; for the whole is equal to all its parts.

II. Every Equation consists of two Parts, which are usually separated one from another by this Character $=$; so in the first Equation in the precedent *Sect.* $3s$ is the first Part, and $36d$ the latter; also in the second Equation, $9c$ is the first Part, and $2l + 5s$ is the latter; likewise in the last Equation of the same *Section*, a is the first Part, and $b + c$ the latter.

III. The single Quantities or things, whereof each part of an Equation is composed, are called the Terms of an Equation; as in this Equation, $a = b + c$, the Terms are a, b and c.

IV. How Equations are found out, the Resolution of Questions will hereafter shew; but when known quantities are intermingled with unknown in an Equation, the first scope is to clear the Equation from all superfluous quantities, and to separate the known quantities from the unknown, that at length an Equation may remain in the fewest and simplest

Terms, so disposed, that the unknown quantity or quantities may possess one part of the Equation, and the known the other; this work is called *Reduction*, and how 'tis perform'd the Examples in the following *Sections* will make manifest.

Reduction by Addition.

V. Reduction by Addition is grounded upon this Axiom, (or common Notion) *viz.* If equal quantities, or one and the same quantity, be added to equal quantities, the whole shall be equal. As, for Example;

If the letter a represent some number unknown, and it be granted or found out that	$a - 3 = 12$
Then by adding $+ 3$ to each part of that Equation, this ariseth, to wit, . . .	$a - 3 + 3 = 12 + 3$
That is, (because $- 3$ and $+ 3$ added together make 0,)	$a = 15$
In like manner, to reduce this Equation	$3a - 4 = 6 - a$
I add $+ 4$ to each part, and there ariseth . .	$3a - 4 + 4 = 6 - a + 4$
Which Equation contracted makes	$3a = 10 - a$
Then by adding $+ a$ to each part of the last Equation, this ariseth, ...	$3a + a = 10 - a + a$
That is, after each part is contracted, ...	$4a = 10$
Again, If this Equation be propos'd to be reduced	$aa - b = d + b$
By adding $+ b$ to each part, this Equation ariseth, ...	$aa - b + b = d + b + b$
Which last Equation, after due contraction gives	$aa = d + 2b$
So also, If	$a - b = 0$
By adding $+ b$ to each part; there ariseth	$a = b$
Likewise, If	$b - a = 0$
By adding a to each part there ariseth	$b = a$
Moreover, If	$aa - bb - cc = dd$
Then by adding $bb + cc$ to each part this Equation comes forth, ...	$aa = dd + bb + cc$
Lastly, If	$aa - bb = cc - da$
By adding $+ bb$ to each part, this Equation ariseth, ...	$aa = cc - da + bb$
And by adding $+ da$ to each part of the last Equation, this ariseth, to wit, . . .	$aa + da = cc + bb$

From the premises it is evident, That if in any Equation any Quantity which hath the sign $-$ prefixed to it, be transfer'd to the other part of the Equation with the sign $+$, that work effects the same thing as the adding of that Quantity to each part of the Equation, and is called *Transposition*.

Reduction

Reduction by Subtraction.

V. If from equal Quantities you take away equal Quantities, or one and the ſame Quantity, the Quantities remaining will be equal; therefore,

If it be taken for granted that . . . . $a + 3 = 12$
Then by ſubtracting $+ 3$ from each part, there ariſeth . . . . $a = 9$

In like manner, If . . . . $b + a = 4b$
I ſubtract $+ b$ from each part, and there ariſeth . . . . $a = 3b$

Again, If . . . . $bb + 2aa = aa + cc$
Firſt, I ſubtract bb from each part, and there remains . . . . $2aa = aa + cc - bb$
Then aa ſubtracted from each part of the laſt Equation leaves this, to wit, ... $aa = cc - bb$

So alſo, If . . . . $aa + b + c = 2ca + df$
By ſubtracting $+ b + c$ from each part, there ariſeth . . . . $aa = 2ca + df - b - c$
And by ſubtracting $2ca$ from each part of the laſt Equation, this ariſeth, to wit, ... $aa - 2ca = df - b - c$

Hence it is evident, That if in any Equation any Quantity which hath the ſign $+$ prefixed to it be transferr'd to the other part of the Equation with the ſign $-$, that work effects the ſame thing as the ſubtracting of that Quantity from each part of the Equation; and is alſo called *Tranſpoſition.*

Reduction by Multiplication.

VII. If equal Quantities be multiplyed by equal Quantities, or by one and the ſame Quantity, the Products ſhall be equal: Hence Equations expreſt by Algebraical Fractions are reduced to other Equations conſiſting altogether of Integers.

As, for Example, If . . . . $\frac{a}{5} = 6$
Then by multiplying each part by 5, this Equation is produced . . . . $a = 30$

Again, to reduce this Equation to another in Integers, *viz.* . . . . $a = \frac{dd}{a - b}$
I multiply each part by $a - b$ and there comes forth . . . . $aa - ab = dd$

Likewiſe, to reduce this Equation to another in Integers, ... $\frac{3aa}{c} = \frac{dd}{b}$
Firſt, I multiply each part by the Denominator b, and there will be produced . . . . . $\frac{3aab}{c} = dd$
Then multiplying each part of the laſt Equation by the Denominator c, I find this Equation . . . . $3aab = cdd$

Hence it is manifeſt, That an Equation whereof each part is a Fraction, may be reduced to another Equation in Integers, by multiplying croſs-wiſe, as in the reduction of

of Fractions to a common Denominator, and then omitting the common Denominator, a new Equation may be instituted between the new Numerators only.

When either part of an Equation is compos'd of Integers and Fractions, first reduce that part into a Fraction, (after the manner of the latter Example in *Sect.* 16. *Chap.* 6.) and then multiply as in the preceding Examples: as,

If this Equation be proposed, ... $\frac{aa}{b} + c + d = bc + \frac{dd}{a}$

First, I reduce that Equation to this, . . . $\frac{aa + bc + bd}{b} = \frac{bca + dd}{a}$

Which last Equation reduced by Multiplication as in the preceding Examples, gives $aaa + abc + abd = bbca + bdd$

But here is to be noted, that in reducing Equations which consist of Fractions into other Equations in Integers, the Operation may oftentimes be facilitated by the same compendium that hath before been shewn in the Division of Fractions (in *Sect.* 26. *Chap.* 6.) *viz.* When either the Numerators or Denominators can be reduced to more simple Terms by some common Divisor, set the Quotients in the places of those Numerators or Denominators; and then reduce these new Fractions into an Equation in Integers, by multiplying cross-wise as before: As, for example,

To reduce this Equation to another in Integers, ... $\frac{aaa}{aa - bb} = \frac{ba - bb}{a + b}$

First, after the Denominators $aa - bb$ and $a + b$ are reduced to $a - b$ and 1, by the common Divisor $a + b$, this new Equation ariseth, ... $\frac{aaa}{a - b} = \frac{ba - bb}{1}$

Whence, by multiplying cross-wise, (as in the preceding Examples) this Equation in Integers is produced, ... $aaa = baa - 2bba + bbb$

Again, to reduce this Equation to another in Integers, ... $\frac{bba - cca}{a + b} = \frac{bbb - bcc}{a}$

First, the Numerators reduced to a and b by the common Divisor, $bb - cc$ will give . . . . $\frac{a}{a + b} = \frac{b}{a}$

Whence by mutiplying cross-wise, this Equation is produced . . . . . . $aa = ba + bb$

In like manner, to reduce this Equation, . . $\frac{baa - caa}{cc - ca} = \frac{bb - bc}{c}$

First, I reduce the Numerators to aa and b, by the common Divisor $b - c$; also, the Denominators to $c - a$ and 1, by the common Divisor c; which new Numerators and Denominators constitute this Equation, ... $\frac{aa}{c - a} = \frac{b}{1}$

Whence by multiplying cross-wise, this Equation is produced . . . . . . $aa = bc - ba$

So also to reduce this Equation . . . . . $\frac{ba^3 - ca^3}{aa - ba + bb} = bc - cc$

First, I set 1 for a Denominator under the Integer $bc - cc$, so the Equation proposed will stand thus, ... $\frac{ba^3 - ca^3}{aa - ba + bb} = \frac{bc - cc}{1}$

Then

Then, after the Numerators $ba^3 - ca^3$ and $bc - cc$ are reduced to a^3 and c,) by the common Divisor $b - c$, this Equation ariseth, ...	$\frac{a^3}{aa - ba + bb} = \frac{c}{1}$
Which last Equation, by multiplying cross-wise, gives this in Integers, ...	$aaa = caa - cba + cbb$

When one part of an Equation is a Surd quantity, (that is, such which hath a Radical sign prefixt to it, as, $\surd$, or $\surd(3)$, &c.) and the other part is a Rational quantity; that Equation may be reduced to another which shall be free from any Surd quantity, by casting away the Radical sign, and multiplying the rational part of the given Equation either quadratickly or cubickly, &c. according to the import of the Radical sign; as,

If there be proposed	$\surd a = 6$
Forasmuch as the Squares of equal Roots or Sides are also equal, therefore by squaring each part of that Equation, this is produced, to wit, ...	$a = 36$
Likewise, If	$\surd a = bc$
By multiplying each part into it self, this Equation is produced, ...	$a = bbcc$
Again, If	$\surd a = \surd 5$
By squaring each part, there comes forth	$a = 5$
And, If	$\surd a = \surd : \overline{bcc - b} :$
By squaring each part, which is done by casting away $\surd$, there will arise . . .	$a = bcc - b.$
So also if this Equation be proposed, ...	$\surd ca = b - d$
By multiplying each part into it self, this Equation is produced, ...	$ca = bb - 2bd + dd$
And, If	$\surd(3)a = 8.$
By multiplying each part into it self cubically, there ariseth	$a = 512.$
Also, If	$\surd(3)a = \surd(3) : \overline{b + c} :$
By casting away $\surd(3)$ from each part it gives	$a = b + c.$

Reduction by Division.

VIII. If equal Quantities be divided by equal Quantities, or by one and the same Quantity, there will come forth equal Quotients. Hence Equations are reduced to others of lower Degrees: As, for example;

If it be granted or found out that	$aa = 5a$
Then by dividing each part by a, you will find	$a = 5$
Again, If	$aaa + baa = bba$
By dividing each part by a, this Equation ariseth, ...	$aa + ba = bb.$
Also, If	$5a = 15$
By dividing each part by 5, there ariseth	$a = 3$
Likewise, If	$ba = bc$
By dividing each part by b, this Equation ariseth, ...	$a = c$
Again, If	$ba - ca = cc$
By dividing each part by $b - c$, there ariseth . . .	$a = \frac{cc}{b - c}$
Also, If	$baa + caa = bd + cd$
By dividing each part by $b + c$, there ariseth	$aa = d$

More-

Moreover, If . . . . $3aa + 4a = 39$
By dividing each part by 3, there ariseth . . $aa + \frac{4}{3}a = 13$
Likewise, If . . . . $caa - ba = cdd$
By dividing each part by c, there ariseth . . . $aa - \frac{b}{c}a = dd$

Reduction by Extraction of ROOTS.

IX. Forasmuch as the Sides or Roots of equal Squares and Cubes, *&c.* are also equal between themselves; therefore,

If there be proposed . . . . $aa = 36$
By extracting the square Root of each part, there ariseth . . . . $a = 6$
In like manner, If . . . . $aa = bb + 2bc + cc$
By extracting the square Root of each part, there comes forth . . . . . . $a = b + c$
Again, If . . . . $aa = 29$
By extracting the square Root of each part, there will arise . . . . $a = \surd 29$
Likewise, If . . . . $aa = bb - dd$
Then, by extracting the square Root out of each part, there ariseth . . . . $a = \surd:\overline{bb - dd}:$
Again, If . . . . $aaa = 27$
Then, the cubick Root being extracted out of each part, there comes forth . . . . $a = 3$
Also, If . . . . $aaa = 12$
By extracting the cubick Root out of each part, this Equation will arise, ... $a = \surd(3)12.$
Likewise, If . . . . $aaa = bbc + cdd$
Then, the cubick Root extracted out of each part, gives . . . . $a = \surd(3):\overline{bbc + cdd}:$

X. By the help of some of the foregoing Reductions, I shall here shew, (after the manner of *Fran. van Scooten* in his *Principia Mathes. universal.*) the certainty of the Rule before given concerning + and — in the Algebraical Multiplication of Compound quantities: *viz.* That + multiplyed by —, or — by + makes —; also, That — multiplyed by — makes +.

First, let $a - b$ be to be multiplyed by c, then the Product according to Algebraical Multiplication is $ac - bc$: now it must be proved that $-b$ multiplyed by $+c$ makes $-bc$; to which end, let f be put equal to $a - b$, and then if it be proved that $ac - bc = fc$, it is evident that $ac - bc$ is the true Product sought; and consequently, $-b$ multiplyed by $+c$ makes $-bc$: But that $ac - bc = fc$ may be proved thus,

Forasmuch as by supposition, ... $a - b = f$
Therefore by adding b to each part, it makes . . . . . $a = f + b$
And by multiplying each part of the last Equation by c, there will be produced . . . . $ac = fc + bc$
Wherefore, by subtracting bc from each part of the last Equation there remains . . . $ac - bc = fc$
Which was to be proved.

After the same manner it may be proved that — multiplyed by — makes +: For, If $a - b$ be to be multiplyed by $c - d$, and there be put (as before) $f = a - b$, it may be shewn that $ac - bc - ad + bd$ is equal to $\overline{a - b} \times \overline{c - d}$ the Product sought; and therefore $-b$ multiplyed by $-d$ produceth $+bd$. For,

By

By supposition . . . . $f = a - b$

Therefore, by multiplying each part into $c - d$ . . $f \times \overline{c - d} = \overline{a - b} \times \overline{c - d}$

That is, ... $fc - fd = \overline{a - b} \times \overline{c - d}$

But it hath been proved in the former Example, that . . . . $ac - bc = fc$

Therefore instead of fc in the third Equation of this latter Example, taking $ac - bc$ (equal to fc) there ariseth . . . . $ac - bc - fd = \overline{a - b} \times \overline{c - d}$

Again, If each part of the first Equation be multiplyed by d, this will be produced, ... $fd = ad - bd$

Wherefore, If from $ac - bc$ in the fifth Equation there be subtracted $ad - bd$ instead of fd equal to $ad - bd$, there will remain according to the Rule of Algebraical Subtraction. . . . . . $ac - bc - ad + bd = \overline{a - b} \times \overline{c - d}$

Which was to be proved.

Chap. XII.

Which shews in what Order the Reductions in the foregoing Chap. 11. *are to be used to resolve Equations, or at least to prepare them for Resolution.*

I. BY the help of the precedent Reductions, either the value of the unknown Root or Quantity sought in an Equation will be found equal to some known Quantity or Quantities, and consequently the Quantity sought is then known also; or else a new Equation will be discovered, from whence the same Quantity sought may be made known by some other Rule or Rules hereafter delivered: But in the use of those Reductions, the work may oftentimes be facilitated by an orderly process, which is the scope of the five following *Sections*; where I assume the Vowel a to stand for the unknown Root or Quantity sought, and Consonants for known Quantities.

II. If in any Equation the Quantity sought, or any Power or Degree of it be found in a Fraction, reduce that Equation to another that may be exprest altogether by Integers, (by *Sect.* 7. *Chap.* 11.) As, for Example;

If this Equation be proposed, ... $\frac{b - a}{c} = d + f - g$

By multiplying each part thereof by the Denominator c, this Equation ariseth in Integers, ... $b - a = cd + cf - cg$

After the same manner, this Equation multiplyed by 4, ... $\frac{aa}{4} + 6 = 15.$

Will be reduced into . . . . $aa + 24 = 60.$

Likewise this Equation . . . . $\frac{aa + bb}{d} + b + c = a - c$

Will be reduced to . . . . $aa + bb + db + dc = da - dc.$

III. When Quantities given or known be intermingled with those that are sought in an Equation, let Quantities be transferr'd from one part of the Equation to the other under a contrary Sign, (according to *Sect.* 5. and 6. of *Chap.* 11.) until at length the

unknown

unknown Quantity may make one part of an Equation, and all the known Quantities the other: As, for example;

If there be proposed	$2a - 26 = 8$.
By transposition of -26 to the other part of the Equation, under the contrary sign $+$, there will arise	$2a = 8 + 26 = 34$.

In like manner, If	$aa + 24 = 60$
By transposition of $+24$, under the contrary sign $-$ it gives	$aa = 60 - 24$
That is,	$aa = 36$

Again; If	$6a - 4 = 20 - a$
First, by transposition of -4, this Equation ariseth,	$6a = 20 - a + 4$
Then by transposition of $-a$, I find	$6a + a = 20 + 4$
Which last Equation being contracted by Addition, gives	$7a = 24$

Likewise, If	$b - a = cd - cf$
After due Transposition, this Equation will arise,	$b + cf - cd = a$
Or,	$a = b + cf - cd$

IV. When some Power or Degree of the Quantity sought happens to be multiplyed into every Term or Member of an Equation, divide every Term by that Degree, so will that Degree or Power quite vanish, and consequently the Equation will be depressed, that is, reduced to lower Degrees or more simple Terms: As, for example;

If there be proposed	$aa + 3a = 20a$
Forasmuch as a is drawn into every Term of that Equation, I divide every Term by a, and there ariseth	$a + 3 = 20$
Whence by equal subtraction of 3 I find	$a = 17$

In like manner, If	$aaa = 3aa$
By casting away aa, that is, by dividing each part by aa, there will arise	$a = 3$.

Again, If	$aaaa + baaa = ddaa$
By expunging aa out of every Term, there ariseth	$aa + ba = dd$

V. When some known Quantity is multiplyed into the highest Power or Degree of the Quantity unknown or sought in an Equation; divide each part of the Equation by that known Quantity, to the end the said highest unknown Power may have no Coefficient or fellow-multiplyer but 1, (or unity;) As, for example,

If there be proposed	$5a = 60$
Because the unknown quantity a is multiplyed by 5, I divide each part of the Equation by 5, and there ariseth	$a = 12$

Again, If	$ca = cc + dd$
Because c is drawn into a the Root sought, I divide every Term of the Equation by c, and there ariseth	$a = c + \frac{dd}{c}$

Like-

Likewiſe, If . . . . $2ba + 3ca = 2ddb + 3cdd$

Becauſe $2b + 3c$ is drawn into the unknown Root a, I divide each part by $2b + 3c$, and there ariſeth . . . $a = dd$

So alſo, If . . . . $4aa = 60$

By dividing each part by 4 which is drawn into aa, there ariſeth . . . . . . $aa = 15$

Again, If . . . . $3aa - 5a = 24$

Becauſe 3 is drawn into aa which is the higheſt unknown Power in the Equation, I divide every Term by 3, and there ariſeth . . $aa - \frac{5}{3}a = 8$

Likewiſe, If . . . . $2ccaa - 4dda = 5bbcc$

Becauſe $2cc$ is drawn into aa which is the higheſt unknown Power in the Equation, I divide every Term by $2cc$, and there ariſeth . . . . $aa - \frac{2dd}{cc}a = \frac{5}{2}bb.$

Again, If . . . . $2bbaa + 3cdaa - dda = ccdd$

Becauſe $2bb + 3cd$ is drawn into aa the higheſt unknown Degree in the Equation, I divide each part by $2bb + 3cd$, and there ariſeth . . . . $aa - \frac{dd}{2bb + 3cd}a = \frac{ccdd}{2bb + 3cd}$

Alſo, If . . . . $3aaa + 24aa - 6a = 1200$

Becauſe 3 is drawn into aaa the higheſt unknown Power in the Equation, I divide each part by 3, and there ariſeth . . $aaa + 8aa - 2a = 400$

VI. If there be a Surd quantity in an Equation, that is, if a Radical ſign as $\sqrt{}$; or $\sqrt{(3)}$ be prefixed before ſome Quantity; firſt by Tranſpoſition (according to *Sect.* 5. or 6. of *Chap.* 11.) make the Surd quantity ſole poſſeſſor of one part of an Equation, then caſt away the Radical ſign, and exalt the other part of the Equation to the ſame degree or Power which is denoted by the Radical ſign, by multiplying Quadratically or Cubically, *&c.* ſo at length an Equation will be found expreſt altogether by Rational quantities: As, for example;

If this Equation be propoſed . . . . $\sqrt{a} = 3$

By ſquaring each part, there will be produced . . . $a = 9$

In like manner, If . . . . $\sqrt{ba} = 3bc$

By multiplying each part into it ſelf quadratickly, there comes forth . . . . $ba = 9bbcc$

Then dividing each part of the laſt Equation by b, there ariſeth . . . . $a = 9bcc$

Again, If . . . . $b + \sqrt{ba} = c$

Firſt by tranſpoſition of b there ariſeth . . . . . $\sqrt{ba} = c - b$

Then by ſquaring each part of the laſt Equation, there will be produced . . . . . . $ba = cc - 2cb + bb.$

Whence, by dividing each part by b, there ariſeth . . . . $a = \frac{cc}{b} - 2c + b$

Likewise, If	$-d+\sqrt{:ba+da} = b$
First by transposition of $-d$, this Equation ariseth, ...	$\sqrt{:ba+da} = b+d$
Then by squaring each part, there will be produced	$ba+da = bb+2bd+dd$
Lastly, by dividing each part of the last Equation by $b+d$, there ariseth . .	$a = b+d$

Again, If	$\sqrt{(3)}9a = 3$
By multiplying each part Cubically, there will be produced	$9a = 27$
And, by dividing each part of the last Equation by 9 there ariseth	$a = 3$

Likewise, If	$\sqrt{(3)}:\overline{ba-ca}:+c = b$
First, by transposition of $+c$ this Equation ariseth, ...	$\sqrt{(3)}:\overline{ba-ca}: = b-c$

Then multiplying each part of the last Equation cubically, this Equation will be produced, to wit,

$$ba - ca = bbb - 3bbc + 3bcc - ccc:$$

Whence, by dividing each part by $b-c$, the value of a will be discovered, *viz.*

$$a = bb - 2bc + cc.$$

VII. When after the using of all or any of the foregoing Rules of this Chapter an Equation ariseth between a perfect Square, Cube or other higher Power of the quantity sought, and some known quantity; then extract such a Root out of each part of the said Equation as the Index of the said unknown Power denoteth, so will the value of the unknown Root or Quantity sought be made known: As, for example;

If this Equation be proposed, to wit,	$\frac{6aa}{5}+8 = 128$
First by subtracting 8 from each part, this Equation ariseth, ...	$\frac{6aa}{5} = 120$
Then each part of the last Equation being multiplyed by 5, gives	$6aa = 600$
And by dividing each part of the last Equation by 6, this ariseth, ...	$aa = 100$
Lastly, the Square root of each part of the last Equation being extracted, the value of a will be discovered, to wit, . .	$a = 10$

Again, If	$\frac{3aaaa}{4} - 8a = 154a$
Then by transposition of $-8a$ there ariseth	$\frac{3aaaa}{4} = 162a$
And by multiplying each part of the last Equation by 4, this will be produced,	$3aaaa = 648a$
And by dividing each part of the last Equation by a this ariseth, to wit, ...	$3aaa = 648$
Likewise each part of the last Equation divided by 3 gives	$aaa = 216$
Lastly, by extracting the Cubick root out of each part of the last Equation, the value of a will be discovered, to wit,	$a = 6$

Like-

Likewise, If . . . . $aa + 2ba + bb = cc$

The Square root extracted out of each part, gives . . . . $a + b = c$

And then by transposition of b, the value of a is discovered, to wit, ... $a = c - b$

CHAP. XIII.

Which shews how to convert Analogies into Equations; and Equations into Analogies.

I. IF four right-lines or numbers be Proportionals, the Product made by the multiplication of the two extremes is equal to the Product of the two means. And if three right-lines or numbers be Proportionals, the Product of the extremes is equal to the Square of the mean, (by *Prop.* 16. and 17. of 6. *Elem.* and by 19. and 20. of 7. *Elem. Euclid.*) Hence Analogies may be converted into Equations, as in the following Examples; where for the greater evidence let a represent 2; b 6; c, 12; and d, 3; Then

1. Let there be four Proportionals, suppose these, ... $d \,.\, b :: d - a \,.\, a$ — $3 \,.\, 6 :: 1 \,.\, 2$

Then by the Theorem above exprest, this Equation will follow, ... $da = bd - ba$

Now to find the value of a in that Equation, first by transposition of $-ba$ this Equation ariseth, ... $da + ba = bd$

Then each part divided by $d + b$ gives . . . . . $a = \dfrac{bd}{d + b}$

2. If there be three Continual proportionals, suppose these, ... $4a \,,\, c \,,\, 9a \;\div$ — $8 \,,\, 12 \,,\, 18 \;\div$

That is, If . . . . $4a \,.\, c :: c \,.\, 9a$

Then, by the latter part of the said Theorem, this Equation will follow, ... $36aa = cc$

Now to find the value of a in that Equation, extract the Square root out of each part, and there ariseth . . . . $6a = c$

Lastly, each part of the last Equation divided by 6 gives . . . . . . $a = \dfrac{c}{6}$, or $\frac{1}{6}c$.

II. If the Product of the multiplication of two Quantities be found equal to the Product of two other Quantities, that Equation may be resolved into Proportionals; for as either of the Factors in either of the two equal Products is to a Factor of the same kind in the other Product, so is the remaining Factor in this latter Product to the other Factor in the former. Hence Equations may oftentimes be resolved into Proportionals; as,

If there be proposed . . . . $3ba = cd$

From that Equation this Analogy may be inferr'd, *viz.* As . . . . $3b \,.\, c :: d \,.\, a$

Again, If . . . . $bd = da + ba$

That Equation may be resolved into these Proportionals, *viz.* As . . . . $d + b \,.\, b :: d \,.\, a$

Likewise, If . . . . $6da = bb$

Then it shall be, As . . . . $6d \,.\, b :: b \,.\, a$

III. When

III. When there happens to be an Equation between an Algebraical Fraction and an Integer, and the Numerator of the Fraction can be resolved into two such quantities that being mutually multiplyed will produce the said Numerator, then that Equation may be resolved into Proportionals in this manner, *viz.* Let the Denominator of the Fraction, and the Integer to which the Fraction is equal, be made the extreme Terms of an Analogy; and let the two quantities which being mutually multiplyed will constitute the Numerator be made the mean Terms; but with this caution in Geometrical Questions, that the first and second Terms be of one and the same kind, that is, either both Lines, or both Planes, or both Solids. As, for example;

If this Equation be proposed, ... $\frac{cd}{3b} = a$

It may be resolved into these Proportionals, . $3b \,.\, c :: d \,.\, a$

But that they are Proportionals, I prove thus;

First, It is evident that these are Proportionals, (because the Product of the extremes is equal to the Product of the means) . . . . $3b \,.\, c :: d \,.\, \frac{cd}{3b}$

And by the Equation proposed, ... $a = \frac{cd}{3b}$

Therefore . . . . $3b \,.\, c :: d \,.\, a \,.\, \left(\frac{cd}{3b}\right)$

Again, If . . . . $\frac{bb}{b+c} = a$

That Equation may be resolved into these Proportionals, ... $b+c \,.\, b :: b \,.\, a$

Likewise this Equation . . . . $\frac{cc-bb}{5b+2c} = a$

may be resolved into this Analogy, . . $5b+2c \,.\, c+b :: c-b \,.\, a$

And this Equation . . . . $\frac{bb+2bc+cc}{54d} = a$

may be converted into these Proportionals, $54d \,.\, b+c :: b+c \,.\, a$

Also, this Equation . . . . $\frac{bbc}{36d} = aa$

may be resolved into these Proportionals, $36d \,.\, b :: bc \,.\, aa$

Or into these, ... $36d \,.\, c :: bb \,.\, aa$

But this Equation . . . . $\frac{b}{c} = a$

cannot be resolved into Proportionals any otherwise than thus, ... $c \,.\, \sqrt{b} :: \sqrt{b} \,.\, a$

Nor can this Equation . . . . $\frac{bb+cd}{g} = a$

be converted into Proportionals, unless thus, $g \,.\, \sqrt{:\overline{bb+cd}:} :: \sqrt{:\overline{bb+cd}:} \,.\, a$

CHAP.

Chap. XIV.

Various Arithmetical Questions Algebraically resolved; whereby most of the Rules hitherto delivered are exercis'd, in the Invention and Resolution of pure or simple Equations.

I. Equations may be divided into two kinds, *viz.* $\left\{\begin{array}{l}\text{1. Pure or Simple,}\\ \text{2. Adfected or Compounded.}\end{array}\right.$

II. A pure or simple Equation is of two kinds, *viz.* First, when the quantity sought is exprest by a simple Root only, as a; as in this Equation, $6a = 12$: secondly, when the quantity sought is exprest by a simple Power only, as aa, or aaa, &c. as in this Equation, $3aaa = 24$; likewise in this, $2aaaa = 32$, and such like.

III. An adfected or compounded Equation is that, wherein there are two or more different Degrees or Powers of the quantity sought; as in this Equation, $aa + 6a = 27$, where aa and a express two different Degrees or Powers of the quantity sought, the one signifying a Square, and the other its Root or side: also in this Equation, $aaa + 6aa - 2a = 28$, there are three unlike Powers or Degrees of the quantity sought, to wit, aaa, aa, and a.

IV. The Invention and Resolution of Pure or Simple Equations is copiously illustrated by Arithmetical Questions in this Chapter, as also in the second and third Books of my *Algebraical Elements*; and the Resolution of Adfected or Compound Equations in Numbers is handled in the 15, 16, and 17. Chapters of this Book, as also in the 10, and 11. Chapters of the second Book. But how Algebraical operations are applicable to the solving of Geometrical Problems, I shall shew in my fourth Book of *Algebraical Elements*.

V. When an Arithmetical Question is proposed, the number sought must first of all be assumed or supposed to be known; and you may represent it by the letter a, or any other Vowel: you may likewise represent the given numbers by Consonants, as, b, c, d, &c. *Renates des Cartes* puts for given Quantities the former letters of the Alphabet, as, a, b, c, d, &c. but for Quantities sought the latter letters, z, y, x, &c. Then with the letters representing the numbers given and sought, an orderly process must be made, by adding, subtracting, multiplying or dividing, *&c.* according to the import of the Question, until at length an Equation be found out between the number sought or some Power or Powers of it, and some number or numbers given: Lastly, when the Equation so found out is a Pure or Simple Equation, the number sought may be discovered by some of the Reductions in the foregoing 12, and 13. Chapters; but when the Equation is Adfected or Compounded, the Resolution thereof belongs either to the 15. Chapter of this first Book, or the 10, and 11. Chapters of the second Book.

VI. In the Resolution of every Question, I proceed from the beginning to the end by steps numbred in the Margin, by 1, 2, 3, 4, 5, *&c.* And because *Numeral Algebra* is more easie for Learners than the *Literal*, (though not so useful for the reasons before given in *Sect.* 8. *Chap.* 1.) I have in many Questions exprest the Operation belonging to every step in both kinds of *Algebra*, that the one may explain the other: So in the second step of the Resolution of the following first Question, the lesser number sought is exprest by *Numeral Algebra* thus, $26 - a$; but by *Literal Algebra* thus, $b - a$. Also, in the fourth step, the Equation by numeral Algebra is $2a - 26 = 8$; but by literal Algebra it is $2a - b = c$.

VII. When an Equation is found out in any of the following Questions, I take it for granted that the Reader knows how to reduce it, if need be, according to the Rules in the foregoing 11, 12, and 13. Chapters, that I may avoid tedious repetitions of what hath been already explain'd. These things premised, I proceed to the Questions themselves.

QUEST.

QUEST. 1.

There are two numbers whose Summ is 26, (or b,) and their difference, (to wit, the excess of the greater above the lesser) is 8, (or c;) What are the Numbers ?

RESOLUTION:	*Numeral*,	*Literal.*
1. For the greater number put	a	a
2. Then subtracting that number a from the given Summ, the Remainder will be the lesser number, to wit, ...	$26 - a$	$b - a$
3. And by subtracting the lesser number from the greater, the Remainder will be their difference, to wit, ...	$2a - 26$	$2a - b$
4. Which difference found out in the last step must be equal to the given difference 8, (or c) whence this Equation ariseth, ...	$2a - 26 = 8$	$2a - b = c$
5. From which Equation, after it is duly reduced according to *Sect.* 3. and 5. of *Chap.* 12. the greater number sought will be discovered, to wit, ...	$a = 17$	$a = \frac{1}{2}b + \frac{1}{2}c$
6. And consequently from the fifth and second steps the lesser number is also discovered, to wit, ...	9, that is,	$\frac{1}{2}b - \frac{1}{2}c.$

So the numbers sought are found 17 and 9, whose summ is 26, and their difference is 8, as was prescribed.

Moreover, If the two last steps of the literal Resolution be exprest by words, they will give this

THEOREM.

Half the difference of any two numbers added to half their Summ, gives the greater number: But half the difference of any two numbers subtracted from half their Summ, leaves the lesser number.

Therefore the Summ and difference of any two numbers being given severally, the numbers themselves are also given by the said Theorem ; but it presupposeth that the number given for the Difference must not exceed the number given for the Summ.

Note here once for all, That the numbers given in a Question cannot alwayes be chosen at pleasure, but sometimes they must be subject to one or more Determinations, which for the most part (though not alwayes) are discoverable by the Theorem or Canon that resulteth from the Resolution. But how limits or Determinations are discovered, I shall have occasion to shew hereafter in my second, third, and fourth Books of *Algebraical Elements.*

QUEST. 2.

There are two numbers whose Summ is 40, (or b,) and the greater number hath such proportion to the lesser as 3 to 2, or, as r to s ;) What are the Numbers ?

1. For the greater number sought put . . .	a	a
2. Then to find the lesser number, say by the Rule of Three, If 3 . . 2 :: a . $\frac{2a}{3}$ Or, r . . s :: a . $\frac{sa}{r}$ whence the lesser number is	$\frac{2a}{3}$	$\frac{sa}{r}$

3. There-

3. Therefore the Summ of the two numbers ſought is	$\frac{5a}{3}$	$a + \frac{sa}{r}$
4. Which Summ found out in the laſt ſtep muſt be equal to the given Summ 40, (or b,) whence this Equation	$\frac{5a}{3} = 40$	$a + \frac{sa}{r} = b.$
5. Which Equation, after due Reduction according to *Sect.* 2. and 5. of *Chap.* 12. gives the greater number	$a = 24$	$a = \frac{rb}{r+s}$
6. And from the fifth, firſt, and ſecond ſteps, the leſſer number is alſo diſcovered, to wit,	16, or	$\frac{sb}{r+s}$

So the numbers ſought are found 24 and 16, which will ſatisfie the conditions in the Queſtion; for their ſumm is 40, and the greater hath ſuch proportion to the leſs as 3 to 2, as was preſcribed.

Moreover, If the two laſt ſteps of the literal Reſolution be reſolved into Proportionals, according to *Sect.* 3. *Chap.* 13. there will ariſe this

THEOREM.

As the Summ of both the Terms which expreſs the Reaſon (or Proportion) of two numbers, is to the Summ of the ſame two numbers; ſo is the greater Term to the greater number; and ſo is the leſſer Term to the leſſer number.

Therefore the Summ of two numbers being given, as alſo their Reaſon, or Proportion; the numbers ſhall alſo be given ſeverally by the ſaid Theorem.

QUEST. 3.

There are two numbers whoſe difference is 8, (or d,) and the greater number hath ſuch proportion to the leſſer as 3 to 2, (or as r to s;) what are the Numbers?

1. For the greater number put	a	a
2. Then to find the leſſer number ſay by the Rule of Three, If $3 \,.\, 2 :: a \,.\, \frac{2a}{3}$ Or if $r \,.\, s :: a \,.\, \frac{sa}{r}$ whence the leſſer number is	$\frac{2a}{3}$	$\frac{sa}{r}$
3. Therefore by ſubtracting the leſſer number from the greater, the Remainder ſhall be their difference, to wit, ...	$\frac{a}{3}$	$a - \frac{sa}{r}$
4. Which difference muſt be equal to the given difference 8 (or d,) hence this Equation ariſeth	$\frac{a}{3} = 8$	$a - \frac{sa}{r} = d.$
5. Which Equation, after due Reduction, diſcovers the greater number ſought, to wit,	$a = 24$	$a = \frac{rd}{r-s}$
6. And from the fifth, firſt, and ſecond ſteps the leſſer number will be alſo made known, to wit,	$= 16$	$= \frac{sd}{r-s}$

So the Numbers ſought are found 24 and 16, which will ſolve the Queſtion; for their difference is 8, and they are in the proportion of 3 to 2, as was preſcribed.

Moreover, If the two laſt ſteps of the literal Reſolution be converted into Proportionals (according to *Sect.* 3. *Chap.* 13.) there will ariſe this

THEOREM.

As the difference of the two Terms which expreſs the Reaſon or Proportion of two numbers is to the difference of the ſame two numbers, ſo is the greater Term to the greater Number; and ſo is the leſſer Term to the leſſer Number.

Therefore the Difference and Reaſon of two numbers being ſeverally given, the numbers themſelves ſhall be alſo given by the ſaid Theorem.

QVEST. 4.

There are two numbers whoſe Summ is 7, (or b,) and the difference of their Squares is 21, (or d;) what are the numbers?

1. For the greater number ſought put	a	a
2. Then ſubtracting the greater number from the given Summ, the Remainder is the leſſer number, to wit, ...	$7 - a$	$b - a$
3. Therefore from the firſt ſtep the Square of the greater number is	aa	aa
4. And from the ſecond ſtep the Square of the leſſer number is	$aa - 14a + 49$	$aa - 2ba + bb$
5. Therefore the difference of the Squares of the two numbers ſought ſhall be	$14a - 49$	$2ba - bb$
6. Which difference muſt be equal to the given difference 21 (or d,) whence this Equation ariſeth, ...	$14a - 49 = 21$	$2ba - bb = d$
7. Which Equation, after due Reduction according to *Sect.* 3, and 5. of *Chap.* 12. diſcovers the greater number ſought, to wit,	$a = 5$	$a = \frac{bb + d}{2b}$
8. And from the ſeventh and ſecond ſteps, the leſſer number will be alſo made known, to wit, ...	$= 2$	$= \frac{bb - d}{2b}$

So the numbers ſought are found 5 and 2, which will ſolve the Queſtion; for their Summ is 7, and the difference of their Squares is 21, (to wit, 25 — 4;) as was preſcribed.

Moreover, If the two laſt ſteps of the literal Reſolution be expreſt by words, they will give this

THEOREM.

If to the Square of the ſumm of any two numbers the difference of their Squares be added, and the ſumm of that addition be divided by the double ſumm of the two numbers, the Quotient will be the greater number: But if from the Square of the ſumm of two numbers the difference of their Squares be ſubtracted, and the Remainder be divided by the double ſumm of the two numbers, the Quotient will give the leſſer number.

Therefore the Summ of two numbers being given, as alſo the difference of their Squares, the numbers themſelves ſhall be given ſeverally; but it preſuppoſeth the ſquare of the given ſumm to exceed the given difference.

QVEST. 5.

There are two numbers whoſe difference is 3, (or c,) and the difference of their Squares is 21, (or d;) what are the numbers?

1. For the leſſer number ſought put	a	a
2. To which adding the given difference 3, (or c,) the ſumm will make the greater number, to wit, ...	$a + 3$	$a + c$
3. Therefore the Square of the greater number is	$aa + 6a + 9$	$aa + 2ca + cc$
4. And the Square of the leſſer number is . . .	aa	aa
5. Therefore the difference of thoſe Squares is	$6a + 9$	$2ca + cc$
6. Which difference muſt be equal to the given difference of the Squares; whence this Equation ariſeth, to wit, ...	$6a + 9 = 21$	$2ca + cc = d$
7. Which Equation, after due Reduction (according to *Sect.* 3, and 5. of *Chap.* 12.) diſcovers the leſſer number, to wit, . . .	$a = 2$	$a = \frac{d - cc}{2c}$
8. And from the ſeventh and ſecond Equations, the greater number will be found	$= 5$	$= \frac{d + cc}{2c}$

So

So the numbers sought are 5 and 2, which will solve the Question; for their difference is 3, and the difference of their Squares is 21; as was prescribed.

Moreover, the two last steps of the literal Resolution afford this

THEOREM.

If to the difference of the Squares of any two numbers the Square of their difference be added; and the summ of that addition be divided by the double of the difference of those two numbers, the Quotient will give the greater number: But if from the difference of the Squares of two numbers the Square of their difference be subtracted, and the Remainder be divided by the double of the difference of those two numbers, the Quotient shall be the lesser number.

Therefore the difference of any two numbers being given, as also the difference of their Squares, the numbers themselves shall also be given severally by this Theorem; but it presupposeth the given difference of the Squares of the two numbers to exceed the Square of the given difference of the same two numbers.

QUEST. 6.

A certain person being asked what was the age of every one of his four Sons, answered; the eldest was four years (or b) elder than the second, the second was four years elder than the third, the third was four years elder than the fourth or youngest; and the double of the youngest Sons age was equal to the age of the eldest; what was the age of each Son?

1. For the age of the eldest Son put	a	a
2. Then from the age of the eldest Son subtracting 4 (or b) there will remain the second Sons age, to wit, ...	$a - 4$	$a - b$
3. Likewise from the second Sons age subtracting 4 (or b) the Remainder will be the third Sons age, to wit, ...	$a - 8$	$a - 2b$
4. Again, from the third Sons age subtracting 4 (or b) there will remain the fourth or youngest Sons age, to wit, ...	$a - 12$	$a - 3b$
5. But according to the Question, the double of the age in the fourth step must be equal to the age in the first step, whence this Equation will arise, ...	$2a - 24 = a$	$2a - 6b = a$
6. Which Equation duly reduced discovers the age of the eldest Son, to wit, ...	$a = 24$	$a = 6b$

Wherefore the ages of the four Sons were 24, 20, 16, and 12; for the first exceeds the second by 4, which is also the excess of the second above the third, the third above the fourth, and the double of the fourth is equal to the first, as was prescribed in the Question.

Moreover the last step of the literal Resolution shews, that if instead of 4, any other number be given for the common difference of the four Sons ages, then six times that common difference will give the eldest Sons age, which shall be equal to the double of the age of the youngest.

QUEST. 7.

A Merchant began to Trade with a certain number of pounds: By his first Voyage he doubled that Stock; by his second he lost 1200. pounds (or b;) by his third he doubled his remaining Stock; by his fourth he lost again 1200. pounds, and then had no money left. The question is, to find how many pounds the Merchant began to Trade with?

1. For the number of pounds which the Merchant began to trade with put . . .	a	a
2. Then the double of that number gives the number of pounds he had at the end of his firſt voyage, to wit, ...	$2a$	$2a$
3. From which laſt number ſubtracting 1200 (or b,) the Remainder ſhews the number of pounds that remained to the Merchant at the end of his ſecond voyage, to wit, . . .	$2a - 1200$	$2a - b$
4. Which remaining number being doubled gives the number of pounds which the Merchant had at the end of his third voyage, to wit, ...	$4a - 2400$	$4a - 2b$
5. From which laſt number ſubtracting again 1200 (or b) pounds loſt by the fourth voyage, the Remainder muſt be equal to nothing; hence this Equation, ...	$4a - 3600 = 0$	$4a - 3b = 0$
6. Which Equation, after due Reduction, gives	$a = 900$	$a = \frac{3}{4}b$.

Whence it is found that the Merchant began to trade with 900. pounds; which number will ſatisfie the conditions in the Queſtion.

Moreover the laſt ſtep of the literal Reſolution ſhews, that if inſtead of 1200. any other number were given, the Merchants ſtock at firſt would be three quarters of that given number.

QUEST. 8.

A Gentleman hired a Servant for a year, for 120. ſhillings (or c,) together with a livery Cloak valued at a certain number of ſhillings: But when $\frac{7}{12}$ (or d) parts of the year were expired, the Maſter falling at variance with his Servant puts him away, and gives him the Cloak with 50. ſhillings, (or f;) and ſo the Servant received full ſatisfaction for the time of his ſervice. The queſtion is, to find How many ſhillings the Cloak was valued at?

1. For the number of ſhillings which the Cloak was valued at put	a	a
2. Then to find what part of the value of the Cloak was due to the Servant when $\frac{7}{12}$ (or d) parts of the year were expired, ſay by the Rule of Three, If $1 \,.\, a :: \frac{7}{12} \,.\, (\frac{7a}{12}$ Or, if $1 \,.\, a :: d \,.\, (da$ whence the deſired part of the value of the Cloak is found	$\frac{7a}{12}$	da
3. Find likewiſe what part of the 120 (or c) ſhillings was due to the Servant when $\frac{7}{12}$ (or d) parts of the year were expired, and ſay, If $1 \,.\, 120 :: \frac{7}{12} \,.\, (70$ Or, $1 \,.\, c :: d \,.\, (cd$ whence the part deſired is found	70	cd

4. Now foraſmuch as the Cloak together with the 50. ſhillings the Servant received, ought to be equal to the part of the Cloak, together with the part of the 120. ſhillings that was due to him at the time he left his ſervice; therefore from the premiſes there ariſeth this Equation:

$$a + 50 = \frac{7a}{12} + 70; \qquad \text{Or,} \qquad a + f = da + cd.$$

5. Which

5. Which Equation after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. will give the desired value of the Cloak, to wit,

$$a = 48 = \frac{cd \backsim f}{1 \backsim d}:$$

Whence it is evident that the Cloak was valued at 48. shillings; and the last Equation discovers this

CANON.

Multiply the money which the Servant was to receive besides the Cloak for a years wages, by the time he served; then divide the difference between that Product and the money he received when he left his service by the difference between 1 (or unity) and the same time he served; so the Quotient gives the value of the Cloak.

By which Canon the value of the Cloak will be found to be 48. *s.* as above.

The Proof.

$$48 + 50 = 98.$$
$$\tfrac{7}{12} \text{ of } 48, + \tfrac{7}{12} \text{ of } 120 = 98.$$

QUEST. 9.

A certain man finding divers poor persons at his door, gave every one of them three pence (or b,) and had six pence (or c) left; but if he would have given them four pence (or f) a piece, he should have wanted two pence (or g.) How many poor persons were there?

1. For the number of poor persons put . . . . a
2. Then forasmuch as that number multiplyed by 3 (or b) and the Product increased with 6 (or c) makes the whole number of pence that the giver had: And, because if the same number of poor persons be multiplyed by 4 (or f,) the Product less by 2 (or g) must also make the same number of pence: hence this Equation;

$$3a + 6 = 4a - 2:$$
Or, $$ba + c = fa - g.$$

3. Which Equation after due Reduction according to *Sect.* 3, and 5. of *Chap.* 12. discovers the number of poor persons to be 8: *viz.*

$$8 = \frac{c + g}{f - b} = a.$$

QUEST. 10.

One being asked what a Clock it was, answered, That the time then past from noon was equal to $\frac{33}{40}$ (or, b) parts of the time remaining until midnight: What was the present Hour? supposing the time between noon and midnight to be divided into 12 (or c) equal Hours.

1. For the Hour sought after noon put	a	a
2. Which subtracted from 12 (or c) leaves the time remaining until midnight, to wit, . . .	$12 - a$	$c - a$
3. Then $\frac{33}{40}$ (or b) parts of the said remaining time will be . . . :	$\frac{396}{40} - \frac{33}{40}a$	$bc - ba$
4. Therefore from the first and third steps (according to the Question) this Equation ariseth, to wit, ...	$a = \frac{396}{40} - \frac{33}{40}a$	$a = bc - ba$
5. Which Equation after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. gives the Hour sought, to wit, ...	$a = 5\frac{31}{73}.$	$a = \frac{bc}{b+1}:$

So the time sought was $5\frac{31}{73}$ Hours after noon, and consequently the remaining time until midnight was $\frac{480}{73}$ Hours, whereof $\frac{33}{40}$ is equal to the said $5\frac{31}{73}$; as was prescribed in the Question.

QUEST.

QUEST. 11.

A General of an Army having ſet his Souldiers in a Square Battel, there happened to be 500 (or b) Souldiers to ſpare; but to increaſe the Square ſo as that its ſide might conſiſt of 1 (or c) Souldier more than the ſide of the former Square, there would be 29 (or d) Souldiers wanting. The queſtion is, to find How many Souldiers the General had in his Army.

1. For the number of Souldiers that made the ſide of the firſt Square, put	a	a
2. Then that ſide multiplyed by it ſelf gives the number of Souldiers in the firſt ſquare Battel, to wit, ...	aa	aa
3. Therefore the number of Souldiers in the whole Army was	$aa + 500$	$aa + b$
4. Then to the end the ſide of another Square may exceed the ſide of the former by 1 (or c,) let it be	$a + 1$	$a + c$
5. Which latter ſide multiplyed by it ſelf gives the number of Souldiers in the latter ſquare Battel, to wit, ...	$aa + 2a + 1$	$aa + 2ca + cc$

6. But the number of Souldiers in the laſt ſtep exceeded the number of Souldiers in the Generals Army by 29 (or d;) therefore ſubtracting 29 (or d) from the number in the laſt ſtep, the Remainder muſt be equal to the number in the third ſtep: hence this Equation ariſeth, to wit,

$$aa + 2a + 1 - 29 = aa + 500,$$

Or, $$aa + 2ca + cc - d = aa + b.$$

7. Which Equation after due Reduction (according to *Sect.* 3, and 5. of *Chap.* 12.) makes known the ſide of the firſt Square, *viz.*

$$a = 264 = \frac{b + d}{2c} - \tfrac{1}{2}c.$$

8. Laſtly, If the ſide or number found out in the laſt ſtep be multiplyed by it ſelf, and the Product be increaſed with 500 (or b,) there will come forth the number of Souldiers that were in the Generals Army, to wit,

$$70196 = \frac{bb + 2bd + dd}{4cc} + \tfrac{1}{4}cc + \tfrac{1}{2}b - \tfrac{1}{2}d.$$

Whence it is manifeſt that the General had 70196 Souldiers in his Army: Alſo, the Side of the firſt Square Battel conſiſted of 264 Souldiers; and the Side of the latter 265; this multiplyed by it ſelf produceth 70225, which exceeds the ſaid 70196 by 29: Moreover, the ſaid 70196 exceeds the Square of 264 by 500; as the queſtion requires.

QUEST. 12.

Two perſons, A and B, diſcourſe of their Money in this manner, *viz.* A ſaith, if B would give him a Crown (or c,) then A ſhould have as many Crowns as B had left; but B ſaith, if A would give him a Crown, then B ſhould have twice as many Crowns as A had left. How many Crowns had each perſon?

1. For the number of Crowns which A had, put	a
2. Then, according to the queſtion, if that number be increaſed with 1 Crown (or c,) the ſumm will be the number of Crowns that remained to B after he had given 1 Crown to A, to wit, . . .	$a + c$
3. And conſequently, by adding 1 Crown (or c) to the ſaid number of Crowns that remained to B after he had given 1 Crown to A, the ſumm will be the number of Crowns which B had at firſt, to wit, ...	$a + 2c$

4. Again,

4. Again, according to the question, if 1 Crown (or c) be added to the said $a + 2c$ in the last step, and subtracted from a in the first step, the summ must be equal to the double of the Remainder; hence this Equation, ...	$a + 3c = 2a - 2c$
5. Which Equation, after due Reduction, discovers the number of Crowns that *A* had at first, to wit, ...	$a = 5c$
6. And from the fifth and third steps, the number of Crowns which *B* had at first will also be made known, to wit, ...	$a + 2c = 7c$

So it is found that *A* had 5 Crowns, and *B* 7 Crowns, as will be evident by

The Proof.

$$5 + 1 = 7 - 1 = 6$$
$$7 + 1 = 4 + 4 = 8.$$

QUEST. 13.

A Vintner having two sorts of *French* Wines, to wit, one sort worth 10. *d.* (or b) the quart, and the other 6. *d.* (or c) *per* quart, would have a mixed quantity of both sorts to consist of 100. quarts (or m) that might be worth 7. *d.* (or f) *per* quart. The question is, to find What quantity of each sort of Wine must be taken to make that mixture?

1. For the number of quarts that must be taken of the better sort of wine to make the mixture put	a	a
2. Which number subtracted from 100 (or m) leaves the number of quarts of the worser sort of wine in the mixture, to wit, ...	$100 - a$	$m - a$
3. Then find the worth of the better sort of wine in the mixture at 10. *d.* (or b) *per* quart, and say by the Rule of Three, If 1 . 10 :: a . ($10a$, Or, if 1 . b :: a . (ba. So the quantity of the better sort of wine in the mixture is found worth	$10a$	ba
4. Find likewise the worth of the worser sort of wine in the mixture at 6. *d.* (or c) *per* quart; and say, If 1 . 6 :: $100 - a$. ($600 - 6a$, Or, 1 . c :: $m - a$. ($cm - ca$. So the quantity of the worser sort of wine in the mixture is found worth	$600 - 6a$	$cm - ca$
5. Therefore the Summ of the values of both the quantities mentioned in the two last steps is	$4a + 600$	$ba + cm - ca$

6. Which Summ must be equal to the Product made by the multiplication of 100 (or m) the total mixed quantity, by 7 (or f) the prescribed mean price; hence this Equation ariseth, to wit,

$$4a + 600 = 700,$$

Or, ... $ba + cm - ca = fm.$

7. Which Equation, after due Reduction, discovers the value of a, to wit, the number of quarts that must be taken of the better sort of wine to make the mixture, *viz.*

$$a = 25 = \frac{fm - cm}{b - c}.$$

8. And from the seventh and second steps the number of quarts that ought to be taken of the worser sort of wine to make the mixture will also be made known, *viz.*

$$75 = \frac{bm - fm}{b - c}.$$

9. From the two last steps it is evident, That 25 quarts of the better sort of wine, and 75 quarts of the worser sort, must be taken to make the prescribed mixture; for those

quantities

quantities at their respective prices will be worth in the whole 700 pence, which is also the just worth of 100 quarts at 7 pence *per* quart.

Moreover, If the latter parts of the two last Equations be resolved into Proportionals, (according to *Sect.* 3. *Chap.* 13.) and be exprest by words, they will give this following

THEOREM.

As the difference between the given prices of two sorts of Wines or other things whereof a mixture is desired, is to the total quantity required to be in the mixture; So is the excess by which some mean price prescribed for the total quantity mixed exceeds the lesser of the two given prices, to the quantity to be taken of the better sort of Wine: And so is the excess of the greater of the two given prices above the mean price, to the quantity that is to be taken of the worser sort of Wine.

This Theorem contains the substance of the Rule of Alligation-alternate in Vulgar Arithmetick. But how Questions of this nature, when three or more things are to be mixed, may be solved more generally than by that Rule, I shall hereafter shew in *Chap.* 13. of my second Book of *Algebraical Elements.*

QUEST. 14.

A Cistern in a certain Conduit is supplyed with water by two Pipes, of such capacities, that by both their Cocks A and B set open at once the Cistern will be filled in 12 (or b) hours; but by the Cock A alone in 20 (or c) hours: the question is, to find In what time the Cistern will be filled by the Cock B alone?

1. Suppose the time sought to be	a	a
2. Then find what part of the Cistern will be filled by the Cock B alone in 12 (or b) hours, and say by the Rule of Three, If a . 1 :: 12 . $\left(\frac{12}{a}\right.$, Or, if a . 1 :: b . $\left(\frac{b}{a}\right.$; whence the said part is found	$\frac{12}{a}$	$\frac{b}{a}$
3. Find likewise what part of the Cistern will be filled by the Cock A alone in 12 (or b) hours, and say, If 20 . 1 :: 12 . $\left(\frac{3}{5}\right.$, Or, if c . 1 :: b . $\left(\frac{b}{c}\right.$; whence the said part is found	$\frac{3}{5}$	$\frac{b}{c}$
4. But those parts found out in the second and third steps must be equal to the whole Cistern, to wit, 1; hence this Equation ariseth,	$\frac{12}{a} + \frac{3}{5} = 1.$	$\frac{b}{a} + \frac{b}{c} = 1.$
5. Which Equation, after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. discovers the value of a, to wit, the time sought, *viz.*	$a = 30$	$a = \frac{bc}{c-b}$

Whence it appears, that by the Cock B set open alone the Cistern would be filled in 30 hours: And, if the last Equation of the literal Resolution be resolved into Proportionals according to *Sect.* 3. *Chap.* 13. there will arise this following

CANON.

As the difference of the two numbers or spaces of Time given in the Question is to either of them, so is the other to the Time sought, *viz.*

$$\text{As } 8\ (20-12)\ .\ 12\ ::\ 20\ .\ 30,$$

$$\text{Or, as } \ldots\ c-b\ .\ b\ ::\ c\ .\ \frac{bc}{c-b}.$$

The

The Proof may be made by solving this Question, *viz.*

If a Cistern will be filled with water by a Cock A in 20 hours, and by another Cock B in 30 hours; in what time will the Cistern be filled by both Cocks set open at once? *Answ.* 12 hours.

First find what part or parts of the Cistern will be filled by each Cock in one and the same time; then it shall be, As the summ of those parts is to that common time, so is the whole Cistern (to wit, 1,) to the time wherein the whole Cistern will be filled by both Cocks set open at once; *viz.*

	ho.	*Cist.*	*ho.*	
First, if	30	1 ::	20	($\frac{2}{3}$ Cistern.
			add	1 Cistern.
			Summ,	$1\frac{2}{3}$ Cist.

So it is found that $1\frac{2}{3}$ Cistern will be filled in 20 hours by both Cocks A and B set open at once; then say again by the Rule of Three;

Cist.	*ho.*	*Cist.*	
$1\frac{2}{3}$	20 ::	1	(12 hours.

If the Operation of this latter Question be formed Algebraically by letters, it will afford this

CANON.

As the Summ of the two given numbers expressing spaces of time in the latter Question, is to either of them; So is the other to the Time sought.

QUEST. 15.

A Shepherd in the time of War driving a flock of Sheep, fell into the hands of three Companies of plundering Souldiers, who compell'd him to deliver the half of his flock with half a Sheep over and above, to the first Company; also half of his remaining flock with half a Sheep to the second Company; likewise the half of the rest of his flock with half a Sheep to the third Company: All which Divisions the Shepherd exactly perform'd without killing a Sheep, and then there remained only 20 (or b) Sheep for himself. The question is, to find How many Sheep the Shepherd had in his Flock at first?

1. Let the number of Sheep which the Shepherd had in his flock at first be represented by . . . . a
2. Then the half of that number is $\frac{1}{2}a$, to which adding $\frac{1}{2}$, (that is, half a Sheep,) the summ will be the number of Sheep delivered to the first Company of Souldiers, to wit, ... $\frac{1}{2}a + \frac{1}{2}$
3. And by subtracting the said $\frac{1}{2}a + \frac{1}{2}$ from a, the remainder will be the number of Sheep that were left to the Shepherd after he had satisfied the first Company of Souldiers, to wit, ... $\frac{1}{2}a - \frac{1}{2}$
4. Then the half of that remaining flock is $\frac{1}{4}a - \frac{1}{4}$, to which adding $\frac{1}{2}$, (that is, $\frac{1}{2}$ Sheep,) the summ will be the number of Sheep delivered to the second Company of Souldiers, to wit, . . . $\frac{1}{4}a + \frac{1}{4}$
5. Which $\frac{1}{4}a + \frac{1}{4}$ being subtracted from $\frac{1}{2}a - \frac{1}{2}$ in the third step, the remainder will be the number of Sheep that were left to the Shepherd after he had satisfied the second Company of Souldiers, to wit, ... $\frac{1}{4}a - \frac{3}{4}$
6. Then the half of the remaining flock in the last step is $\frac{1}{8}a - \frac{3}{8}$, to which adding $\frac{1}{2}$, (to wit, $\frac{1}{2}$ Sheep,) the summ will be the number of Sheep delivered to the third Company, to wit, ... $\frac{1}{8}a + \frac{1}{8}$
7. Which $\frac{1}{8}a + \frac{1}{8}$ being subtracted from $\frac{1}{4}a - \frac{3}{4}$ in the fifth step, the remainder will be the number of Sheep that were left to the Shepherd after he had satisfied all the three Companies, to wit, $\frac{1}{8}a - \frac{7}{8}$
8. But the remainder in the last step must be equal to 20 (or b) the number given in the Question; hence this Equation, . . . $\frac{1}{8}a - \frac{7}{8} = b = 20.$
9. Which Equation, after due Reduction, discovers the number sought; to wit, ... $a = 8b + 7 = 167$

So it appears that the Shepherd had 167 Sheep in his Flock at first.

The Proof.

1. The half of 167 is $83\frac{1}{2}$, to which adding $\frac{1}{2}$, the summ is 84, which was the number of Sheep delivered to the first Company of Souldiers; and then there remained 83 Sheep to the Shepherd.

2. Again, the half of 83 is $41\frac{1}{2}$, which increased with $\frac{1}{2}$ makes 42, the number of Sheep delivered to the second Company; and then there remained 41 Sheep to the Shepherd.

3. Lastly, the half of 41 is $20\frac{1}{2}$, which increased with $\frac{1}{2}$ makes 21, which was the number of Sheep delivered to the third Company; and so there remained 20 Sheep to the Shepherd, as the Question declareth.

Moreover, the Equation in the last step of the Resolution shews, That if any whole number instead of 20 be prescribed in the Question, that number multiplyed by 8, and the Product increased with 7 will give a number capable of the like Division as 167 that answered the Question: So if there had been but one Sheep left for the Shepherd, then his Flock at first was 15 Sheep; if 2 Sheep had been left, his Flock at first was 23; if 3 Sheep had been left, then he had 31 when he first met with the Souldiers; and so by a continual addition of 8, all the odd numbers capable of that Division the Question requires may be orderly found out. But to have nothing left after such Division is made, the number first to be divided is 7.

It is also evident, that by continuing the Resolution an odd number may be found out, that shall be capable of being divided according to the import of the Question, as many times as shall be desired.

QUEST. 16.

Two Merchants, *A* and *B*, were Copartners in traffick: the summ of their Stocks was 300 *l.* (or b;) the Stock of *A* continued in company 9 (or c) months, and the Stock of *B* 11 (or d) months; they gained a certain summ of Money which they divided equally. The question is, to find What each Merchants Stock was at first?

1. For the Stock of *A* when he entred Partnership, put	a	a
2. Then subtracting that stock from the joynt stock 300 *l.* (or b,) the Remainder will be the stock of *B*, to wit, ...	$300 - a$	$b - a$
3. The first stock multiplyed by the time it continued in Company produceth	$9a$	ca
4. And the other stock multiplyed by its time produceth	$3300 - 11a$	$db - da$

5. Now forasmuch as the Merchants divided the gain equally, therefore the Products in the third and fourth steps must be equal to one another, (according to the nature of the Rule of Fellowship with Time.) Hence this Equation ariseth;

$$9a = 3300 - 11a,$$

$$\text{Or, } \ldots \quad ca = db - da.$$

6. Which Equation, after due Reduction, according to *Sect.* 3, and 5. of *Chap.* 12. will discover the Stock which *A* put in, *viz.*

$$a = 165 = \frac{db}{c + d}.$$

7. And from the 6, and 2. steps the Stock which *B* put in will also be made known, to wit,

$$135 = \frac{cb}{c + d}.$$

So it is found that the Stock of *A* was 165 *l.* and that of *B*, 135 *l.* For, $165 \times 9 = 135 \times 11$.

Moreover, If the latter parts of the two Equations in the sixth and seventh steps be resolved into Proportionals, according to *Sect.* 2. *Chap.* 13. there will arise this

CANON.

As the summ of both spaces of time given in the Question, is to the given summ of the two particular Stocks sought; so is the greater time to the particular Stock belonging to the lesser time: and so is the lesser time to the Stock belonging to the greater time.

QUEST.

QUEST. 17.

A certain man being asked how many years old he was, answered, If $\frac{1}{20}$ (or b) part of the number of years he had lived, were multiplyed by $\frac{5}{8}$ (or c) parts of the same number, the Product would give his Age. What was his Age?

1. For the number of years sought put . . .	a	a
2. Then according to the Question, multiplying $\frac{1}{20}a$ by $\frac{5}{8}a$ (or ba by ca) the Product will be	$\frac{1}{32}aa$	$bcaa$
3. Which Product must be equal to the number of years sought, *viz.*	$\frac{1}{32}aa = a$	$bcaa = a$
4. Then, by reducing that Equation according to *Sect.* 4, and 5. of *Chap.* 12. the number of years sought will be discovered, *viz.* . . .	$a = 32$	$a = \frac{1}{bc}$

Whence it is manifest that the Respondent was 32 years of age; for if $1\frac{3}{5}$, that is, $\frac{1}{20}$ of 32, be multiplyed by 20, that is, $\frac{5}{8}$ of 32; the Product will be 32, to wit, the number of years sought. It is also evident by the last Equation in the literal Resolution, that if 1 (to wit Unity) be divided by the Product made by the multiplication of the two numbers given in the question, the Quotient will be the number sought.

QUEST. 18.

There are two numbers, the greater of which hath such proportion to the lesser as 3 to 2, (or as r to s;) and the summ of the said numbers hath such proportion to the summ of their Squares, as 1 to 13, (or as b to c.) What are the numbers?

1. For the greater number sought put . . .	a	a
2. Then (according to *Quest.* 2. in *Sect.* 4. *Chap.* 10.) the summ of the two numbers will be found	$\frac{5a}{3}$	$a + \frac{sa}{r}$
3. And (according to *Quest.* 5. in the said *Sect.* 4. *Chap.* 10.) the summ of the Squares of the two numbers sought will be . . .	$\frac{13aa}{9}$	$aa + \frac{ssaa}{rr}$
4. Again, by the help of the latter Proportion given in the Question, and of the summ found in the second step search out the summ of the Squares of the two numbers sought; *viz.* say by the Rule of Three, If $1 \,.\, 13 :: \frac{5a}{3} \,.\, \left(\frac{65a}{3}\right.$ Or, if $b \,.\, c :: a + \frac{sa}{r} \,.\, \frac{cra + csa}{br}$ whence the summ of the said Squares is found	$\frac{65a}{3}$	$\frac{cra + csa}{br}$

5. But the summ of the Squares found out in the third step must be equal to the summ in the fourth; hence this Equation, *viz.*

$$\frac{13aa}{9} = \frac{65a}{3}$$

$$\text{Or, ... } aa + \frac{ssaa}{rr} = \frac{cra + csa}{br}.$$

6. Which Equation, after due Reduction, will discover the greater of the two numbers sought, *viz.*

$$a = 15 = \frac{crr + crs}{brr + bss}.$$

7. Whence, by the help of the first proportion given in the Question, the lesser number sought will also be made known, *viz.*

$$10 = \frac{css + crs}{brr + bss}.$$

So the numbers ſought are 15 and 10; for they are in the given Reaſon of 3 to 2; and their ſumm 25 is to 325 the ſumm of their Squares, as 1 to 13; as was preſcribed.

Moreover, the letters in the latter parts of the two laſt Equations give a Canon to find out the numbers required.

QUEST. 19.

There are two numbers, the greater of which hath ſuch proportion to the leſſer, as 3 to 2, (or as r to s;) and the ſumm of the ſaid numbers hath ſuch proportion to the Product of their multiplication, as 1 to 6, (or as b to c.) What are the numbers?

1. For the greater number ſought put . . .	a	a
2. Then (according to *Queſt.* 2. in *Sect.* 4. *Chap.* 10.) the ſumm of the two numbers will be	$\frac{5a}{3}$	$a + \frac{sa}{r}$
3. And (by *Queſt.* 4. in *Sect.* 4. *Chap.* 10.) the Product of their multiplication is . .	$\frac{2aa}{3}$	$\frac{saa}{r}$
4. Again, by the help of the latter proportion given in the Queſtion, and of the ſumm found in the ſecond ſtep, ſearch out the Product of the multiplication of the two numbers ſought; *viz.* ſay by the Rule of Three, If $1 \,.\, 6 :: \frac{5a}{3} \,.\, 10a$, Or, if $b \,.\, c :: a + \frac{sa}{r} \,.\, \frac{cra + csa}{br}$; whence the Product is found	$10a$	$\frac{cra + csa}{br}$

5. But the Products found out in the two laſt ſteps muſt be equal to one another; hence this Equation, *viz.*

$$\frac{2aa}{3} = 10a,$$

$$\text{Or, } \ldots \frac{saa}{r} = \frac{cra + csa}{br}.$$

6. Which Equation, after due Reduction, diſcovers the greater of the two numbers ſought, *viz.*

$$a = 15 = \frac{cr + cs}{bs}.$$

7. Whence, by the help of the firſt proportion given in the Queſtion, the leſſer number ſought will alſo be made known, *viz.*

$$10 = \frac{cr + cs}{br}.$$

So the numbers ſought are found 15 and 10; but that they will ſolve the Queſtion the Proof will make manifeſt: For the greater is to the leſſer as 3 to 2; and their ſumm 25, is to 150 the Product of their Multiplication, as 1 to 6; as was preſcribed.

Moreover, the two laſt Equations give a Canon to find out the number ſought.

QUEST. 20.

There are two numbers, the greater of which hath ſuch proportion to the leſſer as 2 to 1 (or as r to s,) and the ſumm of the Squares of the ſaid numbers is 125 (or b;) What are the numbers?

1. For the greater number ſought put	a	a
2. Then (according to *Queſt.* 1. in *Sect.* 4. *Chap.* 10.) the leſſer number will be found	$\frac{a}{2}$	$\frac{sa}{r}$
3. Therefore the ſumm of their Squares ſhall be	$\frac{5aa}{4}$	$aa + \frac{ssaa}{rr}$

4. Which

4. Which summ must be equal to 125 (or b) the given summ of the Squares; hence this Equation, ...	$\frac{5aa}{4} = 125$	$aa + \frac{ssaa}{rr} = b$
5. Which Equation, after due Reduction (according to *Sect.* 2, 5, and 7. of *Chap.* 12.) will discover the greater number sought, *viz.*	$a = 10$	$a = \sqrt{\frac{rrb}{rr + ss}}$
6. But if a had been put for the lesser number, it would by the like process have been found	$= 5$	$= \sqrt{\frac{ssb}{rr + ss}}$

From the two last steps the numbers sought are found 10 and 5, which will solve the Question: For the greater is to the lesser as 2 to 1, and the summ of their Squares is 125; as was prescribed.

Moreover, to find out the numbers sought, the two last steps of the literal Resolution give this

CANON.

Multiply severally the Squares of the Terms of the given Reason, by the given summ of the Squares of the number sought; then divide the Products severally by the summ of the Squares of the said Terms; lastly, extract the square Root out of each Quotient, so shall these square Roots be the numbers sought.

QUEST. 21.

There are two numbers, the greater of which hath such proportion to the lesser as 2 to 1, (or as r to s;) and the difference of their Squares is 75, (or d:) What are the numbers?

1. For the greater number sought put . . .	a	a
2. Then (according to *Quest.* 1. in *Sect.* 4. *Chap.* 10.) the lesser number will be . . .	$\frac{a}{2}$	$\frac{sa}{r}$
3. Therefore the difference of their Squares is	$\frac{3aa}{4}$	$aa - \frac{ssaa}{rr}$
4. Which Difference must be equal to the given Difference 75 (or d;) hence this Equation, *viz.*	$\frac{3aa}{4} = 75$	$aa - \frac{ssaa}{rr} = d$
5. Which Equation, after due Reduction, discovers the greater number, *viz.* . . .	$a = 10$	$a = \sqrt{\frac{rrd}{rr - ss}}$
6. But if a had been put for the lesser number it would have been found by the like process	$= 5$	$= \sqrt{\frac{ssd}{rr - ss}}$

So the numbers sought are 10 and 5, which will solve the Question: For the greater is to the lesser as 2 to 1, and the Difference of their Squares is 75; as was prescribed.

Moreover, to find out the numbers sought, the two last steps of the literal Resolution give this

CANON.

Multiply severally the Squares the Terms of the given Reason by the given Difference of the Squares, then divide the Products severally by the Difference of the Squares of the said Terms; lastly, extract the square Root of each Quotient, so shall these square Roots be the numbers sought.

QUEST. 22.

There are two numbers, the summ of whose Squares is 125 (or b,) and the difference of their Squares is 75 (or d;) what are the numbers?

1. For the greater number put	a	a
2. Then its Square will be	aa	aa
3. Which subtracted from 125 (or b) the given summ, leaves the Square of the lesser number, to wit, ...	$125 - aa$	$b - aa$

4. And

4. And from the ſecond and third ſteps by ſubtracting the leſſer Square from the greater, their difference is	$2aa - 125$	$2aa - b$
5. Which Difference muſt be equal to the given Difference 75 (or d,) whence this Equation ariſeth, ...	$2aa - 125 = 75$	$2aa - b = d$
6. From which Equation after due Reduction, according to *Sect.* 3, 5, and 7. of *Chap.* 12. the greater number ſought will be made known, *viz.*	$a = 10$	$a = \sqrt{\frac{b+d}{2}}$
7. But if a had been put for the leſſer number ſought, it would by the like proceſs have been found	$= 5$	$= \sqrt{\frac{b-d}{2}}$

So the numbers ſought are found 10 and 5, which will ſolve the Queſtion; for the ſumm of their Squares is 125, and the difference of their Squares is 75, as was preſcribed.

Moreover, to find out the numbers ſought, the two laſt ſteps of the literal Reſolution give this

CANON.

The ſquare Root of half the ſumm of the given ſumm and difference of the Squares of the two numbers ſought, is equal to the greater number; and the ſquare Root of half the difference of the ſaid given ſumm and difference gives the leſſer number.

QUEST. 23.

There are two numbers, the ſumm of whoſe Squares is 340 (or b;) and the Product made by the multiplication of the two numbers is equal to $\frac{6}{7}$ (or c) parts of the Square of the greater number; what are the numbers?

1. For the greater number put	a	a
2. Then its Square is	aa	aa
3. And $\frac{6}{7}$ (or c) parts of that Square is . .	$\frac{6aa}{7}$	caa
4. Therefore alſo (according to the condition in the Queſtion) the Product of the mutiplication of the two numbers ſought, ſhall be	$\frac{6aa}{7}$	caa
5. Which Product divided by the greater number a will give the leſſer number, to wit,	$\frac{6a}{7}$	ca
6. Therefore from the laſt ſtep the Square of the leſſer number is	$\frac{36aa}{49}$	$ccaa$
7. And by adding together the Squares in the ſecond and ſixth ſteps, their ſumm will be . .	$\frac{85aa}{49}$	$ccaa + aa$
8. Which ſumm muſt be equal to the given ſumm 340 (or b,) whence this Equation ariſeth,	$\frac{85aa}{49} = 340$	$ccaa + aa = b$
9. From which Equation, after it is duly reduced according to *Sect.* 2, 5, and 7. of *Chap.* 12. the greater number ſought will be made known, *viz.*	$a = 14$	$a = \sqrt{\frac{b}{cc+1}}$
10. And from the ninth and fifth ſteps the leſſer number will alſo be diſcovered, . . .	$= 12$	$= \sqrt{\frac{bcc}{cc+1}}$

So the two numbers ſought are found 14 and 12, which will ſolve the Queſtion; for the ſumm of their Squares 196 and 144 is 340; alſo, 14 multiplyed by 12 makes 168, which is equal to $\frac{6}{7}$ of the greater Square 196.

QUEST. 24.

A Merchant bought a certain number of Yards of linnen Cloth at 12 pence (or b) *per* Yard; and if the number of pence paid for all the Cloth be multiplyed by the number of

of Yards bought, the Product will be 30000, (or c.) The Question is, to find the number of Yards bought.

1. For the number of yards bought put . . .	a	a
2. Then the number of pence paid for the whole Cloth will be	$12a$	ba
3. Which number multiplyed by a (the number of yards bought,) produceth	$12aa$	baa
4. Which Product must, according to the Question, be equal to 30000 (or c;) therefore	$12aa = 30000$	$baa = c$
5. From which Equation, after due Reduction, the number of yards sought will be discovered, *viz.*	$a = 50$	$a = \sqrt{\frac{c}{b}}$

So it is found that the Merchant bought 50 yards of Cloth, which at 12. *d. per* yard makes 600. *d.* this 600 multiplyed by 50 (the number of yards bought,) produceth 30000; as was prescribed in the Question.

QUEST. 25.

Two Merchants, *A* and *B*, were Copartners in traffick; *A* brought in a certain number of pounds, which continued in Company 4 (or c) months, *B* brought in 100 (or b) pounds, which continued in Company such a time, that if it be multiplyed by the Stock of *A* it makes 50 (or d.) At the end of their Partnership they had gained 60 pounds; whereof *A* had 40 (or r) pounds for his share, and *B* the rest, to wit, 20 (or s) pounds. What was the Stock which *A* put in at first, and how many months did the Stock of *B* continue in Company?

1. For the Stock of *A* put	a	a
2. Then multiplying that stock by the time it continued in company, to wit, by 4 (or c,) it makes	$4a$	ca
3. Then divide 50 (or d) the Product given in the Question, by a the (Stock of *A*,) and the Quotient will give the time that the Stock of *B* continued in Company, to wit, . .	$\frac{50}{a}$	$\frac{d}{a}$
4. The Stock of *B*, to wit, 100 *l.* (or b) multiplyed by its time $\frac{50}{a}$ (or $\frac{d}{a}$) produceth	$\frac{5000}{a}$	$\frac{bd}{a}$

5. Then according to the nature of the Rule of Fellowship with Time, this Analogy will arise, *viz.* As the Product made by the mutual multiplication of the Stock and Time of *A*, is to the Product of the Stock and Time of *B*; so is the gain of *A* to the gain of *B*: *viz.*

$$\text{As, } 4a : \frac{5000}{a} :: 40 : 20;$$

$$\text{Or, } ca : \frac{bd}{a} :: r : s.$$

6. Which Analogy (according to *Sect.* 1. *Chap.* 13.) may be converted into this Equation, *viz.*

$$80a = \frac{200000}{a},$$

$$\text{Or, } sca = \frac{rbd}{a}.$$

7. From which Equation, (after due Reduction according to *Sect.* 2, 5, and 7. of *Chap.* 12.) the Stock of *A* will be discovered, *viz.*

$$a = 50 = \sqrt{\frac{rbd}{sc}}.$$

8. And

8. And from the seventh and third steps, the Time that the Stock of *B* continued in Company will also be made known, *viz.*

$$\frac{50}{50} = 1 = \sqrt{\frac{scd}{rb}}.$$

9. So it is found that the Stock which *A* put in at first was 50 *l.* and the Time during which the Stock of *B* continued in Company was one month; as will appear by

The Proof.

$$50 \times 4 = 200$$
$$100 \times 1 = 100$$

Then if . . . . . . 300 . 60 :: $\left\{ \begin{array}{l} 200 \;.\; 40 \\ 100 \;.\; 20 \end{array} \right.$

QUEST. 26.

Certain Noble-men made a Progress for their pleasure; every Noble-man carried along with him the same summ of pounds; the number of the Noble-men was equal to the number of Servants which attended upon each Noble-man; the number of pounds that each Noble-man had was the double of the number of all their Servants; and the summ of all their money was 3456 pounds: the Question is, to find out the number of Noble-men; also, how many pounds and Servants each Noble-man had?

1. For the number of Noble-men put . . . . a
2. Then (according to the Question) the number of Servants that attended upon each Noble man was also . . . . a
3. Therefore the number of all the Servants was . . . . aa
4. Which last number doubled gives the number of pounds that each Nobleman had, to wit, ... $2aa$
5. And if the said number of pounds be multiplyed by the number of Noble-men, it produceth the summ of all their money, to wit, . $2aaa$
6. Which summ must be equal to the given summ 3456, therefore . . $2aaa = 3456$
7. Therefore by taking the half of that Equation, there ariseth . . . $aaa = 1728$
8. Lastly, by extracting the Cubick root of each part of the last Equation, the number of Noble-men is discovered, to wit, ... $a = 12$

So it is found that there were 12 Noble men; also, every one of them had 12 Servants and 288 pounds, as will appear by

The Proof.

$$12 \times 12 = 144$$
$$144 \times 2 = 288$$
$$288 \times 12 = 3456.$$

QUEST. 27.

A Merchant bought as many pounds of Pepper for one Crown as was half the number of Crowns he laid out, then in selling the Pepper he received for every 25 ℔ of Pepper as many Crowns as he paid for all the Pepper; and, in conclusion he had 20 Crowns. The question is, to find how many Crowns he laid out.

1. For the number of Crowns which the Merchant laid out, let there be put . . . . a
2. Then the number of pounds of Pepper which he bought for one Crown was . . . . $\frac{a}{2}$
3. Whence the whole quantity of Pepper bought will be found $\frac{aa}{2}$, for, If $1 \;.\; \frac{a}{2} :: a \;.\; \left(\frac{aa}{2}\right.$ . . . . . $\frac{aa}{2}$

4. Then

4. Then find how many Crowns the Merchant received for the total quantity of Pepper sold; saying by the Rule of Three, If $25 . a :: \frac{aa}{2} . \left(\frac{aaa}{50}\right.$; whence the number of Crowns for which all the Pepper was sold is found . . . . $\frac{aaa}{50}$

5. Which number of Crowns found out in the last step, must be equal to 20 the number of Crowns given in the Question; hence this Equation, ... $\frac{aaa}{50} = 20$

6. From which Equation, after it is reduced according to *Sect.* 2, and 7. of *Chap.* 12. there will come forth the first cost of the Pepper, to wit, ... $a = 10$

So the number of Crowns which the Merchant laid out was 10, as will appear by the Proof; for first, the half of 10, to wit, 5, will be the number of pounds of Pepper which he bought for 1 Crown; then say,

If 1 . 5 :: 10 . 50 || pounds of Pepper bought,
If 25 . 10 :: 50 . 20 || Crowns received for Pepper sold.

QUEST. 28.

There are two numbers, the greater of which hath such proportion to the lesser as 3 to 2, (or as r to s;) and the summ of the Cubes of the two numbers is 4375, (or b;) what are the numbers?

1. For the greater number put	a	a
2. Then (according to *Quest.* 1. in *Sect.* 4. of *Chap.* 10.) the lesser number will be found	$\frac{2a}{3}$	$\frac{sa}{r}$
3. Therefore from the first step, the Cube of the greater number is	aaa	aaa
4. And from the second step the Cube of the lesser number is	$\frac{8aaa}{27}$	$\frac{sssaaa}{rrr}$
5. Therefore from the third and fourth steps, the summ of the Cubes of both numbers is	$\frac{35aaa}{27}$	$\frac{sssaaa}{rrr} + aaa$

6. Which summ must be equal to the given summ 4375, (or b;) whence this Equation ariseth, *viz.*

$$\frac{35aaa}{27} = 4375. \quad \text{Or,} \quad \frac{sssaaa}{rrr} + aaa = b.$$

7. From which Equation, after due Reduction, (according to *Sect.* 2, 5, and 7. of *Chap.* 12.) the greater number sought will be made known, *viz.*

$$a = 15 = \sqrt{(3)}\,\frac{rrrb}{sss + rrr}.$$

8. And from the seventh and second steps, the lesser number will also be discovered, to wit,

$$10 = \sqrt{(3)}\,\frac{sssb}{sss + rrr}.$$

So the numbers sought are found 15 and 10, which will solve the Question; for they are in the given Reason of 3 to 2; and the summ of the Cubes of the said 15 and 10, to wit, of 3375 and 1000 makes 4375; as was prescribed.

Moreover, to find the numbers sought, the latter parts of the Equations in the seventh and eighth steps give this

CANON.

Multiply severally the Cubes of the Terms of the given Reason (or Proportion) by the given summ of the Cubes of the numbers sought; divide the Products severally by the summ of the Cubes of the said Terms; lastly, extract the Cubick Root of each of the Quotients, so these Roots shall be the numbers sought.

CHAP. XV.

Concerning the Resolution of such adfected or compounded Equations wherein there are two different Powers of the quantity sought, and those Powers such, that the higher of them is a Square whose Side or Square Root is the lower Power.

I. THe Equations treated of in this Chapter fall under three heads or forms here-under specified, which I shall first explain, and then shew how they may be Arithmetically resolved.

Equations of the first form.

$aa + 6a = 55.$	$aa + ca = b.$
$aaaa + 8aa = 48.$	$aaaa + daa = f.$
$aaaaaa + 4aaa = 837.$	$aaaaaa + gaaa = h.$

Equations of the second form.

$aa - 10a = 24.$	$aa - ba = k.$
$aaaa - 6aa = 27.$	$aaaa - paa = d.$
$aaaaaa - 2aaa = 48.$	$aaaaaa - maaa = g.$

Equations of the third form.

$10a - aa = 24.$	$ca - aa = n.$
$5aa - aaaa = 4.$	$raa - aaaa = s.$
$9aaa - aaaaaa = 8.$	$daaa - aaaaaa = t,$

II. Every Equation which falleth under any of the said three forms, consists of three distinct Terms or Members, whereof two are unknown and the third is known; of the two unknown terms, one is a Square, (by which in this place I mean a square number) which is called the highest term in the Equation; and the other unknown term is the Product made by the multiplication of the square Root of the said square number by some known number, which Product is called the middle term; and the third or lowest term is a number purely known: So in this Equation $aa + 6a = 55$, the highest term is aa, which may represent an unknown square number whose Root is a; the middle term is $6a$, which is the Product of the multiplication of the said unknown Root a by the known number 6; and the lowest term (or known part of the said Equation) is the number 55, which for distinction sake is usually called the Absolute number given.

The like may be observed in this Equation $aa + ca = b$, where we may suppose b and c to represent two known numbers, and a some number unknown; then the highest term is the Square aa; the middle term is ca, to wit, the Product made by the multiplication of a the Root of the said Square aa by the known number c; and the lowest term of the said Equation is the known absolute number b.

Again, in this Equation $5aa - aaaa = 4$, the highest term is the square number $aaaa$; the middle term is $5aa$, to wit, the Product made by the multiplication of aa the square Root of the said square number $aaaa$ into the known number 5; and the lowest term is the absolute number 4.

III. In every Equation which falls under any of the three before-mentioned forms, there are two different Powers or Degrees of the number sought, and those such, that the Index or Exponent of the higher Power is the double of the Index of the lower: As in this Equation $aa + 6a = 55$, the Index or number of dimensions in aa is 2, which is the double of 1 the Index of a (in the middle term $6a$:) so also in this Equation $5aa - aaaa = 4$, the Index of the highest term $aaaa$ is 4, which is the double of 2 the Index of aa in the middle term. Likewise in this Equation $aaaaaa + 4aaa = 837$, the Index of the highest term $aaaaaa$ is 6, which is the double of 3 the Index of aaa in the

the middle term. But in this Equation $aaa + 6a = 39$ the Index of the highest term aaa is not the double of the Index of a in the middle term, (for the Index of the former is 3, and of the latter 1;) and therefore the Equation last proposed cannot be ranked under any of the three Forms aforesaid, and consequently it is not resolvable by the following Rules of this Chapter, but belongs to the 10, and 11. Chapters of my second Book.

IV. Known numbers which are drawn into, or multiplyed by some Degree or Power of the number sought are by *Vieta* and others called Coefficients, *viz.* fellow-factors, or copartners in multiplication with unknown Powers: So in this Equation $aa + 6a = 55$ the number 6 is called the Coefficient, to wit, the fellow-multiplyer with the unknown number a to make the Product $6a$. Likewise in this Equation $aa + ca = b$, we may suppose the letters b and c to represent known numbers, and the letter a some unknown number whose Coefficient is c.

But sometimes the Coefficient will happen to be exprest by many letters, as in this Equation $aa + \frac{sca}{r}$ (or $\frac{sc}{r}a$) $= \frac{15sscc}{4rr}$, where a only is supposed to be unknown, and the known number $\frac{sc}{r}$ is the Coefficient, which signifies but one number, to wit, the Quotient that ariseth, when the Product of the number s multiplyed by the number c is divided by the number r, *viz.* if $s = 2$; $c = 4$; and $r = 1$, then $\frac{sc}{r}$ or 8 is the Coefficient, and consequently $\frac{sc}{r}a$ is the same with $8a$.

Likewise in this Equation $\frac{2r + s}{s}a$ (or $\frac{2ra + sa}{s}$) $- aa = \frac{2r}{s}$, the Coefficient is $\frac{2r + s}{s}$, which is to be esteemed but as one number, to wit, the Quotient that ariseth by dividing the summ of $2r$ and s by s; so that if we suppose $r = 3$ and $s = 2$, then the Equation last proposed may be exprest thus, $4a - aa = 3$.

Note. When no known number appears to be drawn into the middle term of the Equation, then 1 (or Unity) must in that case be alwayes taken for the Coefficient; so in this Equation $aa + a = 30$, the middle term a implies $1a$, to wit, the Product of a multiplyed by 1, and therefore 1 is the Coefficient.

Note also. When the highest unknown Power or Degree is multiplyed by any number greater than 1, then every term or member of the Equation must be divided by that number, to the end the said highest unknown Power may be clear'd from any Coefficient unless it be 1; as before hath been shewn in *Sect.* 5. *Chap.* 12.

These things being premised by way of Explication, I proceed to the Resolution of Equations which fall under any of the three forms before specified.

V. *The Arithmetical Resolution of Equations which fall under the first of the three Forms before specified in* Sect. I. *of this* Chapter.

QUEST. 1.

1. What is the number represented by a in this Equation? . . . $aa + 6a = 55$
2. Which Equation, if c be assumed to signifie 6, and b 55, may be exprest thus, ... $aa + ca = b$

RESOLUTION.

3. To resolve the said Equation imports the same thing as to solve this Question, *viz.* There is an unknown number (represented by a) which is such, that if to its Square you add the Product made by the multiplication of that unknown number by 6, (or c,) the summ will be 55, (or b;) what is that unknown number a? *Answ.* 5; found out thus,
4. Let the Square of half the Coefficient 6 (or c) be added to each part of the Equation proposed, to the end its first part may be made a compleat Square, (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariseth,

$$aa + 6a + 9 = 64; \quad \text{or,} \quad aa + ca + \tfrac{1}{4}cc = b + \tfrac{1}{4}cc.$$

5. Then by extracting the square Root of each part of the last Equation (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariseth ;

$$a + 3 = 8,$$

Or, $$a + \tfrac{1}{2}c = \sqrt{:b + \tfrac{1}{4}cc:}$$

6. Wherefore by transposition (or equal subtraction) of 3 , or $\frac{1}{2}c$, the number *a* sought will be made known , *viz.*

$$a = 5 = \sqrt{:b + \tfrac{1}{4}cc:} - \tfrac{1}{2}c.$$

I say the number *a* sought is 5 , which will solve the Question proposed, as will appear by

The Proof.

If	$a = 5,$
Then consequently . . .	$aa = 25,$
And	$6a = 30;$
Therefore	$aa + 6a = 55.$

Which was the Equation proposed.

Note. Every Equation which falls under this first Form may be expounded by either of two Roots, whereof one is Affirmative or greater than nothing , and the other Negative or less than nothing. As in the Equation proposed, to wit, $aa + 6a = 55$; forasmuch as according to the Rules of Algebraical Multiplication , — multiplyed by — produceth +, and so in this sense the square Root of 64 may be — 8 as well as + 8 ; therefore the square Root of the Equation $aa + 6a + 9 = 64$ in the fourth step may be this , to wit , ... $a + 3 = -8.$

Whence , by transposition of + 3, a Negative Root or value of *a* is discovered , to wit , ... } . . $a = -11.$

I say the Root *a* in the Equation $aa + 6a = 55$ may be expounded by -11, (besides + 5,) as will be manifest by

The Proof.

If	$a = -11,$	Here the Rules of + and — in Algebraical Multiplication and Addition are to be respected.
Then	$aa = +121,$	
And	$6a = -66.$	
Therefore, as before,	$aa + 6a = +55.$	

Negative Roots are oftentimes of good use to find out Affirmative Roots , as hereafter will appear in *Chap.* 11. of the second *Book.*

QUEST. 2.

1. What is the number represented by *a* in this Equation ? . . } $aaaa + 8aa = 48,$
2. Which Equation , if *d* be put for 8 , and *f* for 48, may be exprest thus , ... } $aaaa + daa = f.$

RESOLUTION.

3. To resolve the said Equation imports the same thing as to solve this Question, *viz.* There is an unknown number represented by *a*, which is such, that if to its Biquadrate or squared Square you add the Product made by the multiplication of the Square of that unknown number *a* by 8, (or *d*,) the summ will be 48, (or *f*;) what is the unknown number *a* ? *Answ.* 2. found out in the same manner as before in *Quest.* 1. viz.
4. Let the Square of half the Coefficient 8 (or *d*) be added to each part of the Equation proposed, to the end its former part may be made a compleat Square, according to *Sect.* 4. *Chap.* 9.) whence this Equation ariseth ;

$$aaaa + 8aa + 16 = 64,$$

Or, $$aaaa + daa + \tfrac{1}{4}dd = f + \tfrac{1}{4}dd.$$

5. Then by extracting the square Root of each part of the last Equation (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariseth ,

$$aa + 4 = 8,$$

Or, $$aa + \tfrac{1}{2}d = \sqrt{:f + \tfrac{1}{4}dd:}$$

6. Whence by equal subtraction or transposition of 4 (or $\frac{1}{2}d$) there will arise

$$aa = 4$$

Or, $$aa = \sqrt{:f + \tfrac{1}{4}dd:} - \tfrac{1}{2}d.$$

7. There-

7. Therefore by extracting the Square Root of each part of the last Equation, the number a sought, will be made known, *viz.*

$$a = 2 = \sqrt{(2)} : \sqrt{f + \tfrac{1}{4}dd} - \tfrac{1}{2}d :$$

I say the number a sought is 2, which will solve the Question proposed, as will appear by

The Proof.

If	$a = 2$,
Then consequently	$aa = 4$,
And	$aaaa = 16$,
Also	$8aa = 32$,
Therefore	$aaaa + 8aa = 48$.

Which was the Equation propos'd to be resolved.

QUEST. 3.

1. What is the number represented by a in this Equation? . . . . $aaaaaa + 4aaa = 837$.
2. Which Equation, if g be put for 4, and h for 837, may be exprest thus . . . . $aaaaaa + gaaa = h$.

RESOLUTION.

3. To resolve the said Equation imports the same thing as to solve this Question, *viz.* There is an unknown number represented by a, which is such, that if to its cubed Cube or sixth Power, you add the Product made by the multiplication of the Cube of that unknown number by 4 (or g) the summ will be 837, what is that unknown number a? *Answ.* 3. found out in the same manner as before, *viz.*
4. By adding the Square of half the Coefficient 4 (or g) to each part of the Equation proposed, this Equation ariseth;

$$aaaaaa + 4aaa + 4 = 841.$$
Or, $$aaaaaa + gaaa + \tfrac{1}{4}gg = h + \tfrac{1}{4}gg.$$

5. And by extracting the Square Root of each part of the last Equation this ariseth;

$$aaa + 2 = 29.$$
Or, $$aaa + \tfrac{1}{2}g = \sqrt{:h + \tfrac{1}{4}gg:}$$

6. Whence by transposition of 2 (or $\tfrac{1}{2}g$) this Equation ariseth;

$$aaa = 27.$$
Or, $$aaa = \sqrt{:h + \tfrac{1}{4}gg:} - \tfrac{1}{2}g.$$

7. Therefore by extracting the Cubick Root of each part of the last Equation the number a sought will be made known, *viz.*

$$a = 3 = \sqrt{(3)} : \sqrt{h + \tfrac{1}{4}gg} - \tfrac{1}{2}g :$$

I say the number a sought is 3, which will solve the Question proposed, as will appear by

The Proof.

If	$a = 3$,
Then consequently	$aaa = 27$,
And	$aaaaaa = 729$,
Also	$4aaa = 108$,
Therefore	$aaaaaa + 4aaa = 837$.

Which was the Equation propos'd to be resolved.

VI. From the Resolution of the three last Questions the following Canon is deduced for the resolving of all Equations which fall under the first of the three forms before specified in *Sect.* 1. of this *Chapter.*

CANON.

Add the Square of half the Coefficient, or (which is the same thing) a quarter of the Square of the whole Coefficient, to the given Absolute number.

Extract the Square Root of that summ.

From

From the ſaid Square Root ſubtract half the Coefficient, and reſerve the Remainder.

Laſtly, when the unknown number which is multiplyed by the Coefficient in the middle term of the Equation is expreſt by a ſingle letter only, as a, then the Remainder before reſerved is the number ſought; but if the ſaid unknown number in the middle term be a Square, as aa, then the Square Root of the Remainder reſerved is the number ſought; if a Cube, as aaa, then the Cubick Root of the ſaid Remainder ſhall be the number ſought; if any higher Power, then the Root for the kind muſt be extracted out of the ſaid Remainder, which Root ſhall be the number ſought.

An Example of the Canon.

1. Let the preceding *Queſt.* 1. be here repeated, *viz.* What is the number repreſented by a in this Equation?	$aa + 6a = 55$
2. Or, what is the value of a in this Equation, . . .	$aa + ca = b$

RESOLVTION.

3. To the given abſolute number	55	$b.$
4. Add the Square of half the Coefficient 6, to wit, the Square of 3, which is . . .	9	$\frac{1}{4}cc.$
5. The ſumm is	64	$b + \frac{1}{4}cc.$
6. The Square Root of that ſumm is	8	$\sqrt{:b + \frac{1}{4}cc:}$
7. From that Square Root ſubtract half the Coefficient 6, to wit, ...	3	$\frac{1}{2}c.$
8. The Remainder is the number a ſought, to wit,	5	$\sqrt{:b + \frac{1}{4}cc:} - \frac{1}{2}c.$

Whence it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 1.

A ſecond Example of the Canon.

1. Let the preceding *Queſt.* 2. be here repeated, *viz.* What is the number repreſented by a in this Equation?	$aaaa + 8aa = 48$
2. Or what is the value of a in this Equation, . . .	$aaaa + daa = f$

RESOLVTION.

3. To the given abſolute number	48	$f.$
4. Add the Square of half the Coefficient 8, to wit, the Square of 4, which is . . .	16	$\frac{1}{4}dd$
5. The ſumm is	64	$f + \frac{1}{4}dd.$
6. The ſquare root of that ſumm is	8	$\sqrt{:f + \frac{1}{4}dd:}$
7. From which ſquare root ſubtract half the Coefficient 8, to wit, ...	4	$\frac{1}{2}d.$
8. The Remainder is the value of aa, to wit, .	4	$\sqrt{f + \frac{1}{4}dd} - \frac{1}{2}d$
9. Laſtly, the ſquare root of the ſaid Remainder gives the number a, ...	2	$\sqrt{(2)\,:\sqrt{f + \frac{1}{4}dd} - \frac{1}{2}d:}$

Whence it is evident that the Anſwer is the ſame as was before found to *Queſt.* 2.

A third Example of the Canon.

1. Let the preceding *Queſt.* 3. be here repeated, *viz.* What is the number repreſented by a in this Equation?	$aaaaaa + 4aaa = 837.$
2. Or what is the value of a in this Equation, . . .	$aaaaaa + gaaa = h.$

RESOLVTION.

3. To the abſolute number	837	$h.$
4. Add the Square of half the Coefficient 4, to wit,	4	$\frac{1}{4}gg.$
5. The ſumm is	841	$h + \frac{1}{4}gg.$
6. The ſquare root whereof is	29	$\sqrt{:h + \frac{1}{4}gg:}$
7. From that ſquare root ſubtract half the Coefficient 4, to wit, ...	2	$\frac{1}{2}g.$

8. The

8. The Remainder is the value of aaa, to wit, . 27 | $\sqrt{b+\frac{1}{4}gg}-\frac{1}{2}g$

9. Therefore the Cubick Root of that Remainder shall be the number a sought, . . . 3 | $\sqrt{(3)}:\sqrt{b+\frac{1}{4}gg}-\frac{1}{2}g:$

Whereby it is manifest that the Answer is the same as was before found to *Quest.* 3.

Example 4.

If . . . . $aa+a=b$ (or 35,) what is $a=$?

Answ. . . . . . $a=\sqrt{:b+\frac{1}{4}:}-\frac{1}{2}=5\frac{4371}{10000}$, &c.

For the Coefficient drawn into the middle term a being 1, its half is $\frac{1}{2}$, the Square whereof is $\frac{1}{4}$, which added to the absolute number 35 makes $35\frac{1}{4}$, whose Square Root is $5\frac{9371}{10000}$, &c. from which subtracting $\frac{1}{2}$, (or $\frac{5}{10}$) to wit, half the Coefficient 1, the Remainder $5\frac{4371}{10000}$, &c. is the number a sought, which here happens to be irrational, that is, inexpressible by any true number, but by continuing the extraction of the said Square Root of the said $35\frac{1}{4}$, you may approach infinitely near the exact number a.

Example 5.

If . . . . $aa+\frac{15}{2}a=\frac{143}{2}$, what is $a=$?

Answ. . . . . . . $a=\sqrt{:\frac{143}{2}+\frac{225}{16}:}-\frac{15}{4}=\frac{11}{2}$.

The Learner must remember to reduce a Fraction to its least Terms, before he goes about to extract any Root out of it.

Example 6.

If . . . . $\begin{cases} r=1, \\ s=2, \\ c=4, \end{cases}$

And if . . . . $aa+\dfrac{sc}{r}a=\dfrac{15sscc}{4rr}$;

What is . . . . $a=$?

Answ. . . . . $a=\dfrac{3sc}{2r}=12$.

Example 7.

If . . . $aaaa+\frac{13}{5}aa=\frac{87362}{405}$, what is $a=$?

Answ. . . . . . . $a=\frac{11}{3}$.

VII. *The Arithmetical Resolution of Equations which fall under the second of the three Forms before expressed in* Sect. 1. *of this* Chapter.

QUEST. 1.

1. What is the number represented by a in this Equation? . . . . $aa-10a=24$.
2. Which Equation, by assuming b to represent 10, and k to signifie 24, may be exprest thus, ... $aa-ba=k$.

RESOLUTION.

3. Let the Square of half the Coefficient 10 (or b) be added to each part of the Equation proposed, to the end its first part may be made a compleat Square, (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariseth;

$$aa-10a+25=49,$$

Or, $$aa-ba+\tfrac{1}{4}bb=k+\tfrac{1}{4}bb.$$

4. Then by extracting the Square Root of each part of the last Equation (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariseth;

$$a-5=7,$$

Or, $$a-\tfrac{1}{2}b=\sqrt{:k+\tfrac{1}{4}bb:}$$

5. Wherefore by equal addition of 5, or $\frac{1}{2}b$, the number a sought will be made known, *viz.* $a=12=\frac{1}{2}b+\sqrt{:k+\frac{1}{4}bb:}$
6. But forasmuch as the square Root of $aa-10a+25$ in the third step may be $5-a$ as well as $a-5$, (for either of those Roots being multiplyed by it self will produce the

the same Square $aa - 10a + 25$,) therefore let $5 - a$ be set instead of $a - 5$ in the fourth step; whence this Equation ariseth, *viz.*

$$5 - a = 7,$$

$$\text{Or,}\quad \tfrac{1}{2}b - a = \sqrt{:k + \tfrac{1}{4}bb:}$$

7. Therefore by transposition, another value of a ariseth, to wit,

$$a = -2 = \tfrac{1}{2}b - \sqrt{:k + \tfrac{1}{4}bb:}$$

Which latter value of a is less than nothing, and such it will alwayes be, as may easily be proved from the last Equation. For $k + \frac{1}{4}bb$ is manifestly greater than $\frac{1}{4}bb$, and consequently the square Root of the former will be greater than the square Root of the latter, *viz.* $\sqrt{:k + \frac{1}{4}bb:}$ is greater than $\frac{1}{2}b$, therefore $\frac{1}{2}b - \sqrt{:k + \frac{1}{4}bb:}$ (that is a) will be less than nothing, for if a greater quantity be subtracted from a less, the Remainder will be a negative quantity, that is less than nothing, as before hath been shewn in *Algebraical* Subtraction. From the premises it is evident that the Equation propounded, to wit, $aa - 10a = 24$ (and likewise every Equation which falleth under the second form of Equations before-mentioned) is explicable by two Roots, whereof one is real or affirmative, whose value is before exprest in the fifth step; and the other negative or less than nothing, the value whereof is exprest in the seventh step.

I say the real or true number a sought in the Question proposed is 12, as will appear by

The Proof.

If . . . . $a = 12$,
Then consequently . . . . $aa = 144$,
And . . . . $10a = 120$,
Therefore . . . . $aa - 10a = 24$.
Which was the Equation proposed.

Moreover, according to the Rules of *Algebraical* Multiplication and Subtraction, the negative value of a, to wit -2 before found, will constitute the Equation first proposed:

For if . . . . $a = -2$,
Then consequently . . . . $aa = +4$,
And . . . . $10a = -20$,
Therefore . . . . $aa - 10a = +24$; as before.

QUEST. 2.

1. What is the number represented by a in this Equation? . . . . } . . . $aaaa - 6aa = 27$
2. Which Equation, if p be put for 6, and d for 27, may be exprest thus, ... } . . . $aaaa - paa = d$

RESOLUTION.

3. Let the Square of half the Coefficient 6 (or p) be added to each part of the Equation proposed, to the end its first part may be made a compleat Square (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariseth;

$$aaaa - 6aa + 9 = 36,$$

$$\text{Or,}\quad aaaa - paa + \tfrac{1}{4}pp = d + \tfrac{1}{4}pp.$$

4. Then by extracting the square Root of each part of the last Equation (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariseth, *viz.*

$$aa - 3 = 6,$$

$$aa - \tfrac{1}{2}p = \sqrt{:d + \tfrac{1}{4}pp:}$$

5. Whence, by equal addition of 3 (or $\frac{1}{2}p$,) there will arise

$$aa = 9,$$

$$\text{Or,}\quad aa = \sqrt{:d + \tfrac{1}{4}pp:} + \tfrac{1}{2}p.$$

6. Wherefore by extracting the square Root of each part of the last Equation, the number a sought will be made known, *viz.*

$$a = 3 = \sqrt{(2)}:\sqrt{d + \tfrac{1}{4}pp} + \tfrac{1}{2}p:$$

I say the number a sought is 3, which will solve the Question proposed; as will appear by

The

The Proof.

If . . . . $a = 3$,
Then consequently . . . . $aa = 9$,
And . . . . $aaaa = 81$,
Also . . . . $6aa = 54$,
Therefore . . . . $aaaa - 6aa = 27$.
Which was the Equation proposed to be resolved.

QUEST. 3.

1. What is the number represented by a in this Equation? . . . . $aaaaaa - 2aaa = 48$
2. Which Equation, if m be put for 2, and g for 48, may be exprest thus, . . . $aaaaaa - maaa = g$

RESOLUTION.

3. Let the Square of half the Coefficient 2 (or m) be added to each part of the Equation proposed, to the end its former part may be made a compleat Square (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariseth;

$$aaaaaa - 2aaa + 1 = 49,$$
Or, $$aaaaaa - maaa + \tfrac{1}{4}mm = g + \tfrac{1}{4}mm.$$

4. Then by extracting the square Root of each part of the last Equation (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariseth;

$$aaa - 1 = 7,$$
Or, $$aaa - \tfrac{1}{2}m = \sqrt{:g + \tfrac{1}{4}mm:}$$

5. Whence by equal addition of 1 (or $\frac{1}{2}m$) there ariseth

$$aaa = 8,$$
Or, $$aaa = \sqrt{:g + \tfrac{1}{4}mm:} + \tfrac{1}{2}m.$$

6. Wherefore by extracting the Cubick Root of each part of the last Equation, the number a sought will be made known, *viz.*

$$a = 2 = \sqrt{(3)}:\overline{\sqrt{g + \tfrac{1}{4}mm.} + \tfrac{1}{2}m}:$$

I say the number a sought is 2, which will solve the Question proposed; as will appear by

The Proof.

If . . . . $a = 2$,
Then consequently . . . . $aaa = 8$,
And . . . . $aaaaaa = 64$,
Also . . . . $2aaa = 16$,
Therefore . . . . $aaaaaa - 2aaa = 48$.
Which was the Equation proposed to be resolved.

VIII. From the Resolution of the three last Questions the following Canon is deduced, for the resolving of all Equations which fall under the second of the three Forms before specified in *Sect.* 1. of this *Chapt.*

CANON.

Add the Square of half the Coefficient, or, (which is the same thing) a quarter of the Square of the whole Coefficient, to the given Absolute number.

Extract the Square Root of that summ.

To the said Square Root add half the Coefficient, and reserve this summ.

Lastly, when the unknown number which is drawn into the Coefficient in the middle term of the Equation is exprest by a single letter only, as a, then the Summ before reserved is the number sought; but if the said unknown number in the middle term be a Square, as aa, then the Square Root of the Summ reserved is the number sought; if a Cube, as aaa, then the Cubick Root of the said Summ shall be the number sought; if any higher Power, then the Root for the kind must be extracted out of the said Summ, which Root shall be the number sought.

An Example of the ſaid Canon.

1. Let the preceding *Queſt.* 1. in *Sect.* 7. of this *Chapt.* be here repeated, *viz.* What is the number repreſented by a in this Equation?	. .	$aa - 10a = 24$
2. Or, what is the value of a in this Equation?	. .	$aa - ba = k$

RESOLVTION.

3. To the given abſolute number . . .	24	$k.$
4. Add the Square of half the Coefficient 10, to wit, the Square of 5, which is . . .	25	$\frac{1}{4}bb.$
5. The ſumm is	49	$k + \frac{1}{4}bb.$
6. The Square Root of that ſumm is . . .	7	$\sqrt{: k + \frac{1}{4}bb:}$
7. To which Square Root add half the Coefficient 10, to wit, ...	5	$\frac{1}{2}b.$
8. The Summ is the number a ſought, to wit,	12	$\sqrt{: k + \frac{1}{4}bb:} + \frac{1}{2}b.$

Whence it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 1. in *Sect.* 7.

A ſecond Example of the Canon in Sect. 8.

1. Let the preceding *Queſt.* 2. in *Sect.* 7. of this *Chapt.* be here repeated, *viz.* What is the number repreſented by a in this Equation?	. .	$aaaa - 6aa = 27$
2. Or, What is the value of a in this Equation?	. .	$aaaa - paa = d.$

RESOLVTION.

3. To the given abſolute number	27	$d.$
4. Add the Square of half the Coefficient 6, to wit, the Square of 3, which is . . .	9	$\frac{1}{4}pp.$
5. The ſumm is	36	$d + \frac{1}{4}pp.$
6. The ſquare Root of that ſumm is . . .	6	$\sqrt{: d + \frac{1}{4}pp:}$
7. To which ſquare Root add half the Coefficient 6, to wit, ...	3	$\frac{1}{2}p$
8. The Summ is the value of aa, to wit, .	9	$\sqrt{: d + \frac{1}{4}pp:} + \frac{1}{2}p.$
9. Therefore the ſquare Root of the ſaid Summ ſhall be the number ſought, to wit, . .	3	$\sqrt{(2)}: \sqrt{d + \frac{1}{4}pp} + \frac{1}{2}p:$

Whence it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 2. in *Sect.* 7.

A third Example of the Canon in Sect. 8.

1. Let the preceding *Queſt.* 3. in *Sect.* 7. of this *Chapt.* be here repeated, *viz.* What is the number repreſented by a in this Equation?	. .	$aaaaaa - 2aaa = 48,$
2. Or, What is the value of a in this Equation?	. .	$aaaaaa - maaa = g.$

RESOLVTION.

3. To the given abſolute number . . .	48	$g.$
4. Add the Square of half the Coefficient 2, to wit, the Square of 1, which is	1	$\frac{1}{4}mm.$
5. The ſumm is	49	$g + \frac{1}{4}mm.$
6. The ſquare Root of that Summ is . .	7	$\sqrt{: g + \frac{1}{4}mm:}$
7. To which ſquare Root add half the Coefficient 2, to wit, ...	1	$\frac{1}{2}m.$
8. The ſumm is the value of aaa, to wit, . .	8	$\sqrt{. g + \frac{1}{4}mm.} + \frac{1}{2}m.$
9. Therefore the cubick Root of the ſaid ſumm ſhall be the number a ſought, to wit, . .	2	$\sqrt{(3)}: \sqrt{g + \frac{1}{4}mm} + \frac{1}{2}m:$

Whereby it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 3. in *Sect.* 7.

Example

Example 4.

If . . . $aa - a = g$ (or 1122,) what is $a = ?$
Answ. . . . $a = \sqrt{:g + \frac{1}{4}:} + \frac{1}{2} = 34.$

Example 5.

If . . . . $aa - \frac{13}{3}a = 373\frac{12}{45}$, what is $a = ?$
Answ . . . . $a = 20\frac{2}{3}$.

Example 6.

If . . . . $\left\{\begin{array}{l} r = 1, \\ s = 2, \\ c = 4. \end{array}\right.$

And if . . . . . . $aa - \frac{sc}{r}a = \frac{15ssc c}{4rr}$,

What is . . . . $a = ?$

Answ. . . . . $a = \frac{5sc}{2r} = 20.$

IX. *The Arithmetical Resolution of Equations which fall under the last of the three Forms before exprest in* Sect. I. *of this* Chapter.

QUEST. 1.

1. What is the number represented by a in this Equation? . . $10a - aa = 24$;
2. Which Equation, if c be assumed to signifie 10, and n put for 24, may be exprest thus, ... $ca - aa = n$.

RESOLUTION.

3. Let the Equation proposed, by transposition of its Terms, be reduced to an Equation of the second of the three Forms before exprest in *Sect.* 1. *viz.* First, by transposition of $-aa$, this Equation ariseth;

$$10a = 24 + aa,$$
Or, $$ca = n + aa.$$

4. Likewise by transposition of 24 (or n) this Equation ariseth;

$$10a - 24 = aa,$$
Or, $$ca - n = aa.$$

5. And from the last Equation by transposition of $10a$ (or ca) there will arise

$$-24 = aa - 10a,$$
Or, $$-n = aa - ca.$$

6. Which last Equation, by transposing each part of it to the contrary coast, may be exprest thus;

$$aa - 10a = -24,$$
Or, $$aa - ca = -n.$$

7. Now let the following process be made as before in the Resolution of Equations of the second Form (in *Sect.* 7.) *viz.* Let the Square of half the Coefficient 10 (or c) be added to each part of the last Equation, to the end its former part may be made a compleat Square (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariseth;

$$aa - 10a + 25 = 25 - 24 = 1,$$
Or, $$aa - ca + \tfrac{1}{4}cc = \tfrac{1}{4}cc - n.$$

8. Then by extracting the Square root of each part of the last Equation, (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariseth, *viz.*

$$a - 5 = 1,$$
Or, ... $$a - \tfrac{1}{2}c = \sqrt{:\tfrac{1}{4}cc - n:}$$

9. Whence by equal addition of 5 (or $\frac{1}{2}c$) one value of a will be made known, *viz.*

$$a = 6 = \tfrac{1}{2}c + \sqrt{:\tfrac{1}{4}cc - n:}$$

10. But forasmuch as the Square root of $aa - 10a + 25$ in the seventh step may be $5 - a$ as well as $a - 5$, (for either of those Roots being multiplyed into it self, will

 produce

produce $aa - 10a + 25$,) therefore let $5 - a$ be fet inftead of $a - 5$ in the eighth ftep, whence this Equation will arife, *viz.*

$$5 - a = 1,$$

Or, $$\tfrac{1}{2}c - a = \sqrt{:\tfrac{1}{4}cc - n:}$$

11. Whence by due tranfpofition another value of a is difcovered, to wit,

$$a = 4 = \tfrac{1}{2}c - \sqrt{:\tfrac{1}{4}cc - n:}$$

12. I fay the number a fought may be either 6 or 4, for either of thefe numbers will conftitute the Equation propofed, as will appear by.

The Proof.

If . . . . $a = 6$,
Then confequently . . . . . . $aa = 36$,
And . . . . $10a = 60$,
Therefore . . . . $10a - aa = 24$.

Which was the Equation propos'd to be refolved.

Again,

If . . . . $a = 4$,
Then confequently . . . . . . $aa = 16$,
And . . . . $10a = 40$,
Therefore . . . . $10a - aa = 24$; as before.

13. But to the end that both the values of a before expreft in the ninth and eleventh Equations may be real or Affirmative numbers, (that is, each greater than nothing) the given numbers in the Equation propofed, and likewife in every Equation of the third Form aforefaid muft be fubject to this following

DETERMINATION.

The Abfolute number given muft not exceed the Square of half the Coefficient.

The reafon of this Determination is evident by the faid ninth and eleventh Equations; for the latter part of each of them fhews, that the given Abfolute number is to be fubtracted from the Square of half the Coefficient, and therefore it ought to be lefs, or equal to the faid Square: Therefore when in any Equation of the third form, the given Abfolute number exceeds the Square of half the Coefficient that Equation is impoffible, and likewife the Queftion that produced it.

It is alfo evident by the faid ninth and eleventh Equations, That when it happens that $n = \tfrac{1}{4}cc$, then $\tfrac{1}{4}cc - n = 0$, and confequently each value of a is equal to $\tfrac{1}{2}c$; *viz.* When the Abfolute number happens to be equal to the Square of half the Coefficient, then the two values of a will be equal to one another, each value in that cafe being equal to half the Coefficient: But when it happens that the Abfolute number is lefs than the Square of half the Coefficient, then thofe two Roots or values of a will be unequal. But here is to be noted, that although in this latter cafe the Equation be alwayes explicable by either of thofe two unequal Roots or numbers, yet the Queftion that produced the Equation will fometimes be anfwered only by one of thofe Roots or numbers, (as hereafter will appear in *Queft.* 10. *Chap.* 16. and by the latter way of refolving the 16. *Queft.* of the fame *Chapt.*)

QUEST. 2.

1. What is the number reprefented by a in this Equation? . . . . $5aa - aaaa = 4$.
2. Which Equation, if r be put for 5, and s for 4, may be expreft thus . . . . . $raa - aaaa = s$.

RESOLUTION.

3. Let the Equation propos'd, by Tranfpofition of its Terms (after the fame manner as in the third, fourth, fifth, and fixth fteps of the preceding *Queft.* 1. *Sect.* 9.) be reduced to an Equation of the fecond of the three Forms before expreft in *Sect.* 1. fo this Equation will arife, *viz.*

$$aaaa - 5aa = -4,$$

Or, $$aaaa - raa = -s.$$

4. Then

4. Then by adding (as in the former Examples) the Square of half the Coefficient 5 (or r) to each part of the last Equation, there ariseth

$$aaaa - 5aa + \tfrac{25}{4} = \tfrac{25}{4} - 4 = \tfrac{9}{4},$$

Or, $$aaaa - raa + \tfrac{1}{4}rr = \tfrac{1}{4}rr - s.$$

5. And by extracting the Square Root of each part of the last Equation this ariseth;

$$aa - \tfrac{5}{2} = \tfrac{3}{2},$$

Or, $$aa - \tfrac{1}{2}r = \sqrt{:\tfrac{1}{4}rr - s:}$$

6. Whence by equal addition of $\frac{5}{2}$ (or $\frac{1}{2}r$,) this Equation ariseth, *viz.*

$$aa = \tfrac{8}{2} \text{ or } 4,$$

Or, $$aa = \tfrac{1}{2}r + \sqrt{:\tfrac{1}{4}rr - s:}$$

7. Therefore by extracting the Square Root of each part of the last Equation, one value of a will be made known, *viz.*

$$a = 2 = \sqrt{(2):\tfrac{1}{2}r + \sqrt{\tfrac{1}{4}rr - s}:}$$

8. But forasmuch as the square Root of $aaaa - 5aa + \frac{25}{4}$ in the fourth step may be $\frac{5}{2} - aa$, as well as $aa - \frac{5}{2}$, (for either of those Roots being multiplyed by it self will produce $aaaa - 5aa + \frac{25}{4}$;) therefore let $\frac{5}{2} - aa$ be set instead of $aa - \frac{5}{2}$ in the fifth step, whence this Equation will arise;

$$\tfrac{5}{2} - aa = \tfrac{3}{2},$$

Or, $$\tfrac{1}{2}r - aa = \sqrt{:\tfrac{1}{4}rr - s:}$$

9. Whence by due transposition this Equation ariseth;

$$aa = \tfrac{2}{2} \text{ or } 1,$$

Or, $$aa = \tfrac{1}{2}r - \sqrt{:\tfrac{1}{4}rr - s:}$$

10. Wherefore by extracting the square Root of each part of the last Equation, another value of a is discovered, to wit,

$$a = 1 = \sqrt{(2):\tfrac{1}{2}r - \sqrt{\tfrac{1}{4}rr - s}:}$$

I say the number a sought may be either 2 or 1, for either of these numbers will constitute the Equation proposed, as will appear by

The Proof.

	If	$a =$	2,
	Then consequently	$aa =$	4,
	And	$aaaa =$	16,
	Also	$5aa =$	20,
	Therefore	$5aa - aaaa =$	4.
Which was the Equation propos'd to be resolved.			
Again,	If	$a =$	1,
	Then	$aa =$	1,
	And	$aaaa =$	1,
	Also	$5aa =$	5,
	Therefore, ...	$5aa - aaaa =$	4; as before.

QUEST. 3.

1. What is the number represented by a in this Equation? } $9aaa - aaaaaa = 8.$
2. Which Equation, if d be put for 9, and t for 8, may be exprest thus, ... } $daaa - aaaada = t.$

RESOLUTION.

3. Let the Equation propos'd, by transposition of its Terms (after the same manner as in the third, fourth, fifth, and sixth steps of the preceding *Quest.* 1. *Sect.* 9.) be reduced to an Equation of the second of the three forms before exprest in *Sect.* 1. so this Equation will arise, *viz.*

$$aaaaaa - 9aaa = -8,$$

Or, $$aaaaaa - daaa = -t.$$

4. Then by adding the Square of half the Coefficient 9 (or d) to each part of the last Equation, there ariseth

$$aaaaaa - 9aaa + \tfrac{81}{4} = \tfrac{81}{4} - 8 = \tfrac{49}{4},$$

Or, $$aaaaaa - daaa + \tfrac{1}{4}dd = \tfrac{1}{4}dd - t.$$

5. And

5. And by extracting the Square Root of each part of the last Equation this ariseth,

$$aaa - \tfrac{9}{2} = \tfrac{7}{2},$$

Or, $$aaa - \tfrac{1}{2}d = \sqrt{:\tfrac{1}{4}dd - t}:$$

6. Whence by equal addition of $\frac{9}{2}$ (or $\frac{1}{2}d$) this Equation ariseth;

$$aaa = \tfrac{16}{2} \text{ or } 8,$$

Or, $$aaa = \tfrac{1}{2}d + \sqrt{:\tfrac{1}{4}dd - t}:$$

7. Therefore by extracting the Cubick Root of each part of the Equation, one value of a will be made known, *viz.*

$$a = 2 = \sqrt{(3)}:\overline{\tfrac{1}{2}d + \sqrt{:\tfrac{1}{4}dd - t}}:$$

8. But forasmuch as the Square Root of $aaaaaa - 9aaa + \frac{81}{4}$ in the fourth step may be $\frac{9}{2} - aaa$ as well as $aaa - \frac{9}{2}$, (for either of these Roots being multiplyed by it self, will produce the same Square $aaaaaa - 9aaa + \frac{81}{4}$,) therefore let $\frac{9}{2} - aaa$ be set instead of $aaa - \frac{9}{2}$ in the fifth step, whence this Equation will be made, *viz.*

$$\tfrac{9}{2} - aaa = \tfrac{7}{2},$$

Or, $$\tfrac{1}{2}d - aaa = \sqrt{:\tfrac{1}{4}dd - t}:$$

9. Whence by due transposition this Equation ariseth, *viz.*

$$aaa = \tfrac{2}{2} = 1,$$

Or, $$aaa = \tfrac{1}{2}d - \sqrt{:\tfrac{1}{4}dd - t}:$$

10. Wherefore by extracting the Cubick Root of each part of the last Equation, another value of a is made known, *viz.*

$$a = 1 = \sqrt{(3)}:\overline{\tfrac{1}{2}d - \sqrt{\tfrac{1}{4}dd - t}}:$$

I say the number a sought is either 2 or 1, for either of these numbers will constitute the Equation proposed; as will appear by

The Proof.

If	$a =$	2,
Then consequently	$aaa =$	8,
And	$aaaaaa =$	64,
Also	$9aaa =$	72,
Therefore	$9aaa - aaaaaa =$	8.

Which was the Equation proposed to be resolved.

Again,

If	$a =$	1,
Then consequently	$aaa =$	1,
And	$aaaaaa =$	1,
Also	$9aaa =$	9,
Therefore	$9aaa - aaaaaa =$	8; as before.

X. From the Resolution of the three last Questions the following Canon is deduced for the resolving of all Equations which fall under the last of the three Forms before specified in *Sect.* 1. of this *Chapt.*

CANON.

From the Square of half the Coefficient, or (which is the same thing) from a quarter of the Square of the whole Coefficient, subtract the Absolute number given.

Extract the Square Root of that Remainder.

Add the said Square Root to half the Coefficient, and also subtract it from half the Coefficient, reserving the Summ and Remainder.

Lastly, when the unknown number which is multiplyed by the Coefficient in the middle term of the Equation is exprest by a single letter only, as a, then the Summ and Remainder before reserved are the two numbers sought, each of which will constitute the Equation proposed; but if the said unknown number in the middle term be a Square, as aa, then the Square Root severally extracted out of the Summ and Remainder reserved shall be the two numbers sought; if a Cube, as aaa, then the Cubick Root severally extracted out of the said Summ and Remainder shall be the two numbers sought; if any higher Power, then the Root for the kind must be extracted severally out of the said Summ and Remainder, which Roots shall be the two numbers sought.

An

An Example of the said Canon.

1. Let the preceding *Quest.* 1. in *Sect.* 9. of this *Chapt.* be here repeated, *viz.* What is the number represented by a in this Equation? $\}$ $10a - aa = 24$
2. Or, What is the value of a in this Equation? . . . . . . $ca - aa = n$

RESOLUTION.

3. From the square of half the Coefficient 10, to wit, the square of 5, which is . . .	25	$\frac{1}{4}cc$.
4. Subtract the given absolute number . . .	24	n.
5. The remainder is	1	$\frac{1}{4}cc - n$.
6. The square root of that remainder is . .	1	$\sqrt{:\frac{1}{4}cc - n:}$
7. To which square root add half the Coefficient 10, to wit, ...	5	$\frac{1}{2}c$.
8. The summ is the greater value of a sought, to wit, ...	6	$\frac{1}{2}c + \sqrt{:\frac{1}{4}cc - n:}$
9. But subtracting the said square root from half the Coefficient, the remainder is the lesser value of a, to wit, ...	4	$\frac{1}{2}c - \sqrt{:\frac{1}{4}cc - n:}$

Either of which numbers 6 and 4 found out in the two last steps will constitute the Equation proposed, as before hath been proved in the *Answer* to *Quest.* 1. in *Sect.* 9. of this *Chapt.*

A second Example of the Canon in Sect. 10.

1. Let the preceding *Quest.* 2. in *Sect.* 9. of this *Chapt.* be here repeated, *viz.* What is the number represented by a in this Equation? . . . . $5aa - aaaa = 4$
2. Or, What is the value of a in this Equation? . . . $raa - aaaa = s$.

RESOLUTION.

3. From the square of half the Coefficient 5, to wit, the square of $\frac{5}{2}$, which is . . .	$\frac{25}{4}$	$\frac{1}{4}rr$.
4. Subtract the given absolute number . .	4	s.
5. The remainder is	$\frac{9}{4}$	$\frac{1}{4}rr - s$.
6. The square root of that remainder is . .	$\frac{3}{2}$	$\sqrt{:\frac{1}{4}rr - s:}$
7. To which square root add half the Coefficient 5, to wit, ...	$\frac{5}{2}$	$\frac{1}{2}r$.
8. The summ is the greater value of aa, to wit,	4	$\frac{1}{2}r + \sqrt{:\frac{1}{4}rr - s:}$
9. But subtracting the said square root from half the Coefficient, the remainder is the lesser value of aa, to wit, ...	1	$\frac{1}{2}r - \sqrt{:\frac{1}{4}rr - s:}$
10. Therefore the square root of the summ in the 8th step is the greater value of a, to wit,	2	$\sqrt{(2):\frac{1}{2}r + \sqrt{\frac{1}{4}rr - s}:}$
11. And the square root of the remainder in the ninth step is the lesser value of a, to wit,	1	$\sqrt{(2):\frac{1}{2}r - \sqrt{\frac{1}{4}rr - s}:}$

Either of which numbers 2 and 1 found out in the two last steps will constitute the Equation proposed, as before hath been proved in the *Answer* to *Quest.* 2. in *Sect.* 9. of this *Chap.*

A third Example of the Canon in Sect. 10.

1. Let the preceding *Quest.* 3. in *Sect.* 9. of this *Chapt.* be here repeated, *viz.* What is the number represented by a in this Equation? . . . . $9aaa - aaaaaa = 8$
2. Or, What is the value of a in this Equation? . . . . $daaa - aaaaaa = t$

RESOLUTION.

3. From the square of half the Coefficient 9, to wit, the square of $\frac{9}{2}$, which is	$\frac{81}{4}$	$\frac{1}{4}dd$.
4. Subtract the given absolute number . .	8	t.

5. The

5. The remainder is	$\frac{49}{4}$	$\frac{1}{4}dd - t.$
6. The square root of that remainder is . .	$\frac{7}{2}$	$\sqrt{:\frac{1}{4}dd - t:}$
7. To which square root add half the Coefficient 9, to wit, ...	$\frac{9}{2}$	$\frac{1}{2}d.$
8. The summ is the greater value of *aaa*, to wit,	8	$\frac{1}{2}d + \sqrt{:\frac{1}{4}dd - t:}$
9. But subtracting the said square root from half the Coefficient, the remainder is the lesser value of *aaa*, to wit, ...	1	$\frac{1}{2}d - \sqrt{:\frac{1}{4}dd - t:}$
10. Therefore the Cubick root of the summ in the eighth step is the greater value of *a*, to wit,	2	$\sqrt{(3)\,\frac{1}{2}d + \sqrt{\frac{1}{4}dd - t}:}$
11. And the Cubick root of the remainder in the ninth step is the lesser value of *a*, to wit,	1	$\sqrt{(3):\frac{1}{2}d - \sqrt{\frac{1}{4}dd - t}:}$

Either of which numbers 2 and 1 found out in the two last steps will constitute the Equation proposed, as before hath been proved in the *Answer* to *Quest.* 3. in *Sect.* 9. of this *Chapt.*

Example 4.

1. If b, d, f, g represent such known numbers that bf is greater than dg; and,
2. If $\frac{bg + 2bf + df}{bg + dg + bf + df}a - aa = \frac{bf - dg}{bg + dg + bf + df}$;

What is a equal to?

Answ. a is equal to 1, and also to $\frac{bf - dg}{bg + dg + bf + df}$.

Which values of a are also found out by the Canon in the tenth *Section* of this *Chapt.* but I shall leave the Operation as an exercise for the industrious Learner, and in the next place shew the use of the Rules before delivered in this fifteenth *Chapt.* in the Resolution of various Arithmetical Questions.

CHAP. XVI.

Various Arithmetical Questions, producing Equations that fall under some of the three Forms in Sect. 1. *of the foregoing* Chap. 15. *and are resolvable by their respective Canons in* Sect. 6, 8, *and* 10. *of the same* Chapt.

QUEST. 1.

There are two numbers whose difference is 16 (or c,) and the Product of their multiplication is 36 (or b;) what are the numbers?

RESOLUTION.	*Numeral.*	*Literal.*
1. For the lesser of the two numbers sought put	a	a
2. Then by adding to the said lesser number the given difference 16 (or c,) the greater number sought will be	$a + 16$	$a + c$
3. Therefore from the two last steps the Product made by the mutual multiplication of the two numbers sought will be	$aa + 16a$	$aa + ca$

4. Which Product must be equal to the given Product 36 (or b) whence this Equation ariseth, *viz.* $aa + 16a = 36$,

Or, $aa + ca = b$.

5. Which Equation being resolved by the Canon in *Sect.* 6. of *Chap.* 15. the value of a, or the lesser number sought by this Question will be discovered, *viz.*

$$a = 2 = \sqrt{:b + \tfrac{1}{4}cc:} - \tfrac{1}{2}c.$$

6. To

6. To which lesser number adding the given difference 16 (or c) the greater number sought will also be made known, *viz.*

$$2 + 16 = 18 = \sqrt{:b + \tfrac{1}{4}cc:} + \tfrac{1}{2}c.$$

Otherwise thus,

1. For the greater of the two numbers sought put	a	a
2. Then by subtracting from the said greater number the given difference 16, (or c) the lesser number sought will be	$a - 16$	$a - c$
3. Therefore from the two last steps, the Product made by the mutual multiplication of the two numbers sought will be	$aa - 16a$	$aa - ca$

4. Which Product must be equal to the given Product 36, (or b;) whence this Equation ariseth, *viz.*

$$aa - 16a = 36, \quad \text{or,} \quad aa - ca = b.$$

5. Which Equation being resolved by the Canon in *Sect.* 8. of *Chap.* 15. the value of a, to wit, the greater number sought will be discovered, *viz.*

$$a = 18 = \sqrt{:b + \tfrac{1}{4}cc:} + \tfrac{1}{2}c.$$

6. And by subtracting from the said greater number the given difference 16 (or c;) the lesser number sought will also be discovered, *viz.*

$$18 - 16 = 2 = \sqrt{:b + \tfrac{1}{4}cc:} - \tfrac{1}{2}c.$$

From either of those wayes of Resolution, the numbers sought are found 18 and 2, which will solve the Question proposed; for their difference is 16, and the Product of their multiplication is 36, as was prescribed.

Moreover, the two last steps of each Resolution by Literal Algebra give one and the same Canon to solve the Question proposed.

CANON.

To the given Product add the Square of half the given difference; and extract the square Root of that summ; then to the said square Root adding half the given difference, and from the said square Root subtracting the said half difference, the Summ and Remainder shall be the two numbers sought.

Therefore the difference and the Rectangle (or Product of the multiplication) of any two numbers being severally given, the numbers themselves shall also be given by the said Canon.

QUEST. 2.

There are three numbers in Geometrical proportion continued; the difference of the extremes, that is, of the first and third is 16 (or c,) and the mean is 6 (or m;) what are the extreme Proportionals?

RESOLUTION.

1. For the lesser of the two extreme Proportionals sought put	a	a
2. Then by adding to the said lesser extreme the given difference of the extremes, to wit, 16 (or c,) the greater extreme will be . . .	$a + 16$	$a + c$
3. Therefore the Rectangle contained under the extreme Proportionals,) to wit, the Product made by their mutual multiplication) shall be	$aa + 16a$	$aa + ca$

4. Which Rectangle (or Product) must (by *Sect.* 1. *Chap.* 13.) be equal to the Square of the given mean Proportional 6 (or m,) hence this Equation;

$$aa + 16a = 36, \quad \text{or,} \quad aa + ca = mm.$$

5. Which Equation being resolved by the Canon in *Sect.* 6. *Chap.* 15. the value of a, or the lesser of the two extreme Proportionals sought will be made known, *viz.*

$$a = 2 = \sqrt{:mm + \tfrac{1}{4}cc:} - \tfrac{1}{2}c$$

6. To which lesser extreme Proportional adding 16 (or c) the given difference of the extremes, the greater of the two extreme Proportionals will also be discovered, *viz.*

$$2 + 16 = 18 = \sqrt{:mm + \tfrac{1}{4}cc:} + \tfrac{1}{2}c.$$

I say the two extreme Proportionals sought are 2 and 18, between which the given number 6 is a mean Proportional; for, as 2 is to 6, so is 6 to 18.

Moreover, the two last steps of the Resolution give the following Canon to find out the extreme Proportionals sought.

CANON.

To the Square of the given mean Proportional add the Square of half the given difference of the extremes, and extract the square Root of that summ; then to the said square Root adding half the said difference, and from the said square Root subtracting the same half difference, the Summ and Remainder shall be the extreme Proportionals sought.

Therefore if of three numbers in continual proportion the mean be given, as also the difference of the extremes, the extremes shall be given severally by the said Canon.

QUEST. 3.

There are two numbers whose summ is 20 (or c,) and the Product of their multiplication is 36 (or n;) what are the numbers?

RESOLUTION.

1. For one of the numbers sought put . .	a	a
2. Then by subtracting that number from the given summ 20 (or c,) the Remainder will be the other number sought, to wit, . . .	$20 - a$	$c - a$
3. Therefore the Product of the multiplication of those two numbers will be	$20a - aa$	$ca - aa$

4. Which Product must be equal to the given Product 36 (or n,) whence this Equation ariseth, *viz.*

$$20a - aa = 36,$$

Or,

$$ca - aa = n.$$

5. Which Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers sought by this Question will be discovered, *viz.*

$$a = \begin{cases} 18 = \tfrac{1}{2}c + \sqrt{:\tfrac{1}{4}cc - n:} \\ 2 = \tfrac{1}{2}c - \sqrt{:\tfrac{1}{4}cc - n:} \end{cases}$$

I say the numbers sought are 18 and 2, for their summ is 20, and the Product of their multiplication is 36, as was prescribed.

Moreover, if the two values of a, which are exprest by letters in the last step of the Resolution, be exprest by words, they will give the following Canon to solve the Question proposed.

CANON.

From the Square of half the given Summ subtract the given Product, and extract the square Root of the Remainder; then to the said half Summ adding the said square Root, and from the said half Summ subtracting the same square Root, the Summ and Remainder shall be the two numbers sought.

Therefore the Summ and Rectangle (or Product of the multiplication) of any two numbers being severally given, the numbers themselves shall also be given severally by the said Canon.

QUEST. 4.

There are three numbers in continual proportion; the summ of the extremes is 20, (or c,) and the mean proportional is 6, (or m;) what are the extremes?

RESOLUTION.

1. For one of the two extreme proportionals sought put	a	a

2. Then

2. Then by subtracting that extreme from 20 (or c) the given summ, the Remainder will be the other extreme, to wit, ...	$20 - a$	$c - a$
3. Therefore the Rectangle contained under the extreme proportionals, (to wit, the Product of their multiplication) shall be	$20a - aa$	$ca - aa$

4. Which Rectangle (or Product) must (according to *Sect.* 1. *Chap.* 13.) be equal to the Square of the given mean Proportional 6 (or m,) whence this Equation ariseth, *viz.*

$$20a - aa = 36,$$

Or, $$ca - aa = mm.$$

5. Which Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers sought by this Question will be discovered, *viz.*

$$a = \begin{cases} 18 = \frac{1}{2}c + \sqrt{:\frac{1}{4}cc - mm:} \\ 2 = \frac{1}{2}c - \sqrt{:\frac{1}{4}cc - mm:} \end{cases}$$

I say the two extreme Proportionals sought are 18 and 2, between which the given number 6 is a mean Proportional; for, as 18 is to 6, so is 6 to 2.

Moreover, if the two values of a which are exprest by letters in the last step of the Resolution be exprest by words, they will give the following Canon to find out the extreme Proportionals sought:

CANON.

From the Square of half the given summ of the extreme Proportionals subtract the Square of the given mean, and extract the square Root of the Remainder; then to the said half summ adding the said square Root, and from the said half summ subtracting the same square Root, the Summ and Remainder shall be the two extreme Proportionals sought.

Therefore if of three numbers in continual proportion the mean be given, as also the summ of the extremes, the extremes themselves shall be given severally by the said Canon.

QUEST. 5.

There are two numbers whose difference is 15, (or d,) and if the Product of the multiplication of the said two numbers be divided by 2, (or c,) the Quotient will give the Cube of the lesser number; what are the numbers?

RESOLUTION.

1. For the lesser number sought put . . .	a	a
2. To which adding the given difference 15 (or d,) the summ shall be the greater number, to wit, ...	$a + 15$	$a + d$
3. Therefore the Product of the multiplication of the two numbers is	$aa + 15a$	$aa + da$
4. Which Product being divided by 2 (or c) the Quotient will be	$\frac{aa + 15a}{2}$	$\frac{aa + da}{c}$
5. From the first step the Cube of the lesser number is	aaa	aaa

6. Which Cube must (as the Question requires) be equal to the Quotient in the fourth step, whence this Equation;

$$aaa = \frac{aa + 15a}{2},$$

Or, $$aaa = \frac{aa + da}{c}.$$

7. Which Equation being duly reduced (according to *Sect.* 2, 4, 3, 5 of *Chap.* 12.) there will arise

$$aa - \tfrac{1}{2}a = \tfrac{15}{2},$$

Or, $$aa - \frac{1}{c}a = \frac{d}{c}.$$

8. Therefore the last Equation being resolved by the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the lesser number sought will be discovered, *viz.*

$$a = 3 = \sqrt{:\frac{d}{c} + \frac{1}{4cc}:} + \frac{1}{2c}.$$

9. To

9. To which leſſer number adding the given difference 15 (or d) the ſumm ſhall be the greater number ſought, to wit,

$$3 + 15 = 18 = \sqrt{:\frac{d}{c} + \frac{1}{4cc}:} + \frac{1}{2c} + d.$$

10. I ſay the two numbers ſought are 3 and 18, which will ſatisfie the conditions in the Queſtion, for their difference is 15, and if the Product of their multiplication 54 be divided by 2, the Quotient is 27, which is the Cube of the leſſer number 3, as was required.

11. But if the Equation in the eighth ſtep be expreſt by words, it will give the following Canon to find out the leſſer number ſought, to which adding the given difference, the greater number is alſo given.

CANON.

Divide the given difference by the given Diviſor, alſo divide 1 (or Unity) by the quadruple of the Square of the given Diviſor, add thoſe two Quotients together, and extract the ſquare Root of the ſumm; then to this ſquare Root add the Quotient that ariſeth by dividing 1 by the double of the given Diviſor; ſo ſhall the ſumm be the leſſer of the two numbers ſought, which increaſed with their given difference will give the greater number.

QUEST. 6.

There are two numbers whoſe difference is 2 (or d,) and the ſumm of their Squares is 130 (or c;) what are the numbers?

RESOLUTION.

1. For the leſſer number ſought put . . .	a	a
2. Then to that leſſer number adding the given difference 2 (or d) the ſumm ſhall be the greater number, to wit, ...	$a + 2$	$a + d$
3. Therefore from the firſt ſtep the Square of the leſſer number is	aa	aa
4. And from the ſecond ſtep the Square of the greater number is	$aa + 4a + 4$	$aa + 2da + dd$
5. Therefore from the two laſt ſteps the ſumm of the Squares of the two numbers ſought is	$2aa + 4a + 4$	$2aa + 2da + dd$

6. Which ſumm muſt be equal to the given ſumm of the Squares 130 (or c,) whence this Equation ariſeth, *viz.*

$$2aa + 4a + 4 = 130,$$

Or, $$2aa + 2da + dd = c.$$

7. Which Equation, after due Reduction according to the Rules of the twelfth *Chapt.* will give this Equation, *viz.* $aa + 2a = 63,$

Or, $$aa + da = \tfrac{1}{2}c - \tfrac{1}{2}dd.$$

8. Therefore the Equation in the laſt ſtep being reſolved according to the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit, the leſſer number ſought by the Queſtion will be made known, *viz.*

$$a = 7 = \sqrt{:\tfrac{1}{2}c - \tfrac{1}{4}dd:} - \tfrac{1}{2}d.$$

9. To which leſſer number adding the given difference 2 (or d) the ſumm ſhall be the greater number ſought, to wit,

$$7 + 2 = 9 = \sqrt{:\tfrac{1}{2}c - \tfrac{1}{4}dd:} + \tfrac{1}{2}d.$$

10. I ſay the two numbers ſought are 9 and 7; for their difference is 2, and the ſumm of their Squares is 130, as was preſcribed by the Queſtion.

11. Moreover, from the eighth and ninth ſtep ariſeth this

CANON.

From half the given ſumm ſubtract the Square of half the given difference, and extract the ſquare Root of the Remainder; then from this ſquare Root ſubtract half the given difference, the Remainder ſhall be the leſſer number ſought, to which adding the given difference the ſumm ſhall be the greater number.

QUEST.

QUEST. 7.

There are two numbers whoſe ſumm is 14 (or b,) and the ſumm of their Squares is 100 (or c,) what are the numbers ?

RESOLUTION.

1. For one of the numbers ſought put . . .	a	a
2. Which ſubtracted from the given ſumm 14 (or b) leaves the other number	$14 - a$	$b - a$
3. The Square of the firſt number is . . .	aa	aa
4. The Square of the other number is . . .	$aa - 28a + 196$	$aa - 2ba + bb$
5. The ſumm of the ſaid Squares is	$2aa - 28a + 196$	$2aa - 2ba + bb$

6. Which ſumm muſt be equal to 100 (or c) the given ſumm of the Squares, whence this Equation ariſeth, *viz.* $2aa - 28a + 196 = 100$,
Or, $2aa - 2ba + bb = c$.

7. Which Equation, after due Reduction, according to the Rules of the twelfth *Chapt.* will give this following Equation;

$$14a - aa = 48,$$
Or, $$ba - aa = \tfrac{1}{2}bb - \tfrac{1}{2}c.$$

8. Which Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15: the two values of a, which are the numbers ſought by this Queſtion, will be diſcovered, *viz.*

$$a = \begin{cases} 8 = \tfrac{1}{2}b + \sqrt{:\tfrac{1}{2}c - \tfrac{1}{4}bb}. \\ 6 = \tfrac{1}{2}b - \sqrt{:\tfrac{1}{2}c - \tfrac{1}{4}bb}. \end{cases}$$

9. I ſay the numbers ſought are 8 and 6; for their ſumm is 14, and the ſumm of their Squares is 100, as was preſcribed.

10. Moreover, if the two values of a which are expreſt by letters in the eighth ſtep be expreſt by words there will ariſe this

CANON.

From half the given ſumm of the Squares ſubtract the Square of half the given ſumm of the two numbers, and extract the ſquare Root of the Remainder; then adding the ſaid ſquare Root to the ſaid half ſumm of the numbers, the ſumm of this addition ſhall be the greater number; but ſubtracting the ſaid ſquare Root from the ſaid half ſumm of the numbers, the Remainder ſhall be the leſſer number.

QUEST. 8.

There are three numbers in Geometrical proportion continued, and ſuch, that if the difference between the ſumm of the extremes and the mean be multiplyed by the ſumm of the extremes, the Product will be 1120 (or b;) but if the ſaid difference be multiplyed by the ſumm of all the three Proportionals, the Product will be 1456 (or c;) what are the Proportionals ?

RESOLUTION.

1. For the difference of the ſumm of the extremes and mean put	a	a
2. Then, according to the Queſtion, the ſumm of the extremes is	$\frac{1120}{a}$	$\frac{b}{a}$
3. From which ſumm if the difference in the firſt ſtep be ſubtracted, the Remainder will be the mean proportional, to wit, . . .	$\frac{1120}{a} - a$	$\frac{b}{a} - a$
4. Therefore from the two laſt ſteps the ſumm of all three proportionals is	$\frac{2240}{a} - a$	$\frac{2b}{a} - a$

5. But (according to the Queſtion) if the ſumm of all the three proportionals be multiplyed by the difference of the ſumm of the extremes and the mean, the Product muſt be equal to 1456 (or c;) therefore from the firſt and fourth ſteps this following Equation ariſeth, *viz.*

$$2240 - aa = 1456,$$
Or, $$2b - aa = c.$$

6. Which

6. Which Equation being reduced according to the Rules of the twelfth *Chapt.* the value of *a* will be discovered, *viz.*

$$a = 28 = \sqrt{:\overline{2b - c}}.$$

7. Therefore from the sixth and second steps, the summ of the extremes is also known, *viz.*

$$40 = \frac{b}{\sqrt{:2b - c:}} = \text{the summ of the extremes.}$$

8. And from the sixth and third steps, the mean proportional is also given, *viz.*

$$12 = \frac{c - b}{\sqrt{:2b - c:}} = \text{the mean.}$$

9. Lastly, the summ of the extremes of three continual proportionals being given 40, as also the mean 12, the extremes shall also be given severally by the Canon of the fourth *Question* of this *Chapt.* to wit, 4 and 36; therefore the three continual proportionals sought are 4, 12 and 36, which will satisfie the conditions in the Question proposed, as will appear by

The Proof.

I.	$4,\ 12,\ 36$ are $\div\div$; for, $4 \times 36 = 12 \times 12$.
II.	$\overline{4 + 36 - 12}$ into $\overline{36 + 4} = 1120$.
III.	$\overline{4 + 36 - 12}$ into $\overline{4 + 12 + 36} = 1456$.

QUEST. 9.

There are two numbers whose summ is 10 (or *b*,) and the summ of their Cubes is 520 (or *c*;) what are the numbers?

RESOLUTION.

1. For one of the numbers sought put . . .	a	a
2. Then by subtracting that number from the given summ 10 (or *b*,) the other number remains, to wit, ...	$10 - a$	$b - a$
3. The Cube of the former is	aaa	aaa

4. And from the second step the Cube of the latter number is

$$1000 - 300a + 30aa - aaa,$$

Or, $bbb - 3bba + 3baa - aaa.$

4. Therefore the summ of the two Cubes in the third and fourth steps is

$$1000 - 300a + 30aa,$$

Or, $bbb - 3bba + 3baa.$

5. Which summ must be equal to 520 (or *c*) the given summ of the Cubes, whence this Equation ariseth, *viz.* $1000 - 300a + 30aa = 520,$

Or, $bbb - 3bba + 3baa = c.$

6. Which Equation, after due Reduction according to the Rules of the twelfth *Chapt.* will give this Equation; $16 = 10a - aa,$

Or, $\frac{bbb - c}{3b} = ba - aa.$

7. Therefore the last Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of *a*, which are the numbers sought by this Question, will be discovered, *viz.*

$$a = \begin{cases} \frac{1}{2}b + \sqrt{:\overline{\frac{c}{3b} - \frac{bb}{12}}:} = 8. \\ \frac{1}{2}b - \sqrt{:\overline{\frac{c}{3b} - \frac{bb}{12}}:} = 2: \end{cases}$$

8. I say the two numbers sought are 8 and 2; for their summ is 10, and the summ of their Cubes is 520, as was prescribed.

9. Moreover, if the two values of *a* which are exprest by letters in the seventh step be exprest by words, they will give this

CANON.

From the Quotient that ariseth by dividing the given summ of the two Cubes, by the triple of the given summ of their sides, subtract $\frac{1}{12}$ of the Square of the last mentioned summ,

ſumm, and extract the ſquare Root of the Remainder; then adding the ſaid ſquare Root to half the ſaid ſumm of the ſides of the two Cubes, and alſo ſubtracting the ſaid ſquare Root from the ſaid half ſumm, the Summ and Remainder ſhall be the ſides or numbers ſought.

QUEST. 10.

There are two numbers whoſe ſumm is 10 (or b,) and the proportion which their difference beareth to the ſumm of their Squares is as 2 to 29, (or as r to s;) what are the numbers?

RESOLUTION.

1. For the greater number ſought put . . .	a	a
2. Which ſubtracted from the given ſumm 10 (or b) leaves the leſſer number	$10 - a$	$b - a$
3. Therefore the difference of the two numbers is	$2a - 10$	$2a - b$
4. And from the firſt ſtep the Square of the greater number is	aa	aa

5. And from the ſecond ſtep the Square of the leſſer number is

$$100 - 20a + aa,$$
$$\text{Or,}\quad bb - 2ba + aa.$$

6. And from the two laſt ſteps the ſumm of the Squares of the two numbers ſought is

$$100 - 20a + 2aa,$$
$$\text{Or,}\quad bb - 2ba + 2aa.$$

7. Then according to the Queſtion, the difference in the third ſtep muſt be to the ſumm of the Squares in the ſixth ſtep as 2 to 29, (or as r to s;) *viz.*

$$2 \,.\, 29 :: 2a - 10 \,.\, 100 - 20a + 2aa,$$
$$\text{Or,}\quad r \,.\, s :: 2a - b \,.\, bb - 2ba + 2aa.$$

8. Which Analogy may be converted into this following Equation, (according to the Theorem in *Chap.* 1. *Sect.* 13.) *viz.*

$$200 - 40a + 4aa = 58a - 290,$$
$$\text{Or,}\quad rbb - 2rba + 2raa = 2sa - sb.$$

9. Which Equation, after due Reduction according to the Rules in the 12. *Chapt.* will produce this Equation;

$$\tfrac{245}{2} = \tfrac{49}{2}a - aa,$$
$$\text{Or,}\quad \frac{rbb + sb}{2r} = \frac{s + rb}{r}a - aa.$$

10. Therefore by reſolving the Equation in the laſt ſtep according to *Sect.* 10. *Chap.* 15. the two values of a, or the two Roots of that Equation will be made known, *viz.*

$$a = \begin{cases} \tfrac{35}{2} = \dfrac{s}{2r} + \dfrac{b}{2} + \sqrt{:\dfrac{ss}{4rr} - \dfrac{bb}{4}:} \\[2ex] 7 = \dfrac{s}{2r} + \dfrac{b}{2} - \sqrt{:\dfrac{ss}{4rr} - \dfrac{bb}{4}:} \end{cases}$$

11. The leſſer of which two Roots or numbers, to wit 7, is the greater number ſought by this Queſtion; and conſequently, the ſaid 7 being ſubtracted from the given ſumm 10, the Remainder 3 is the leſſer number ſought.

I ſay 7 and 3 will ſolve the Queſtion, for their ſumm is 10; and their difference 4 is to the ſumm of their Squares 58, as 2 to 29; which was preſcribed.

12. *Note.* Although the value of a in the Equation in the ninth ſtep may be either $\frac{35}{2}$ or 7, (for that Equation may be expounded by $\frac{35}{2}$ as well as 7,) yet 7 only, to wit, the leſſer value of a, ſhall be the greater number ſought by this Queſtion. For that the greater value of a, to wit, $\dfrac{s}{2r} + \dfrac{b}{2} + \sqrt{:\dfrac{ss}{4rr} - \dfrac{bb}{4}:}$ can never be equal to either of the two numbers ſought, I prove thus; Firſt, it is manifeſt by each of the values of a expreſt by letters in the tenth ſtep, That if $\dfrac{s}{2r} = \dfrac{b}{2}$, then conſequently $\dfrac{ss}{4rr} = \dfrac{bb}{4}$, and the two values of a are equal one to the other, each

being

being equal to $\frac{s}{2r} + \frac{b}{2}$, that is, b; and therefore in this first case, neither of the two values of a can possibly be equal to either of the two numbers sought; for that which is equal to the summ of two numbers must needs be greater than either of them.

Secondly, If $\frac{s}{2r} \sqsubset \frac{b}{2}$, which is a necessary Determination to make the Question possible, then the greater value of a, that is, $\frac{s}{2r} + \frac{b}{2} + \sqrt{:\frac{ss}{4rr} - \frac{bb}{4}:}$ is manifestly greater than b the given summ of the two numbers sought, and therefore it cannot be equal to either of them. Wherefore the said greater value of a cannot in any case be equal to either of the two numbers sought. Which was to be proved.

But the said lesser value of a is the greater of the two numbers sought, and consequently they are given severally by this following

CANON.

13. From the Quotient that ariseth by dividing the Square of the latter term of the given Reason by the quadruple of the Square of the first term, subtract a quarter of the Square of the given summ of the two numbers sought, and extract the square Root of the Remainder; then subtract that square Root from the summ of the Quotient that ariseth by dividing the latter term of the given Reason by the double of the first, and the half of the given summ of the two numbers, so the Remainder shall be the greater number sought; which subtracted from the said given summ leaves the lesser number.

14. From the premises this following Question may easily be solved, *viz.* The summ of two numbers being given, suppose $\frac{4}{5}$ (or b,) and their difference being equal to the summ of their Squares, to find the numbers.

First, suppose $r = s = 1$, (because the Terms of the Proportion in this Question are equal to one another,) then the two values of a before exprest in the tenth step will be converted into these, *viz.*

$$a = \frac{6}{5} = \frac{1+b}{2} + \sqrt{:\frac{1-bb}{4}},$$

$$a = \frac{3}{5} = \frac{1+b}{2} - \sqrt{:\frac{1-bb}{4}}.$$

The lesser of which values of a, to wit, $\frac{3}{5}$, is the greater of the two numbers sought, and therefore the said $\frac{3}{5}$ being subtracted from $\frac{4}{5}$ the given summ, leaves $\frac{1}{5}$ for the lesser number. I say $\frac{3}{5}$ and $\frac{1}{5}$ will solve the Question, for their difference $\frac{2}{5}$ is equal to the summ of their Squares.

QVEST. II.

There are two numbers, the Product of whose Multiplication is 48 (or p,) and the difference of their Squares is 28 (or d;) what are the numbers?

RESOLUTION.

1. For the greater number put	a	a
2. Then dividing 48 (or p) by a, the Quotient is the lesser number, to wit, ...	$\frac{48}{a}$	$\frac{p}{a}$
3. From the first step the Square of the greater number is	aa	aa
4. And from the second step the Square of the lesser number is	$\frac{2304}{aa}$	$\frac{pp}{aa}$
5. Therefore the difference of the said Squares is	$\frac{aaaa - 2304}{aa}$	$\frac{aaaa - pp}{aa}$

6. Which difference must be equal to the given difference of the Squares, whence this Equation ariseth, *viz.*

$$\frac{aaaa - 2304}{aa} = 28,$$

$$\text{Or,} \quad \frac{aaaa - pp}{aa} = d.$$

7. Which

7. Which Equation, after due Reduction according to the Rules of the twelfth *Chapt.* will produce this;

$$aaaa - 28aa = 2304,$$

Or, $$aaaa - daa = pp.$$

8. Therefore by resolving the last Equation according to the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the greater number sought will be discovered, *viz.*

$$a = 8 = \sqrt{(2)}: \overline{\sqrt{pp + \tfrac{1}{4}dd} + \tfrac{1}{2}d}:$$

Whence the greater number is found 8, by which if the given Product 48 be divided, the Quotient 6 is the lesser number sought.

I say, the numbers 8 and 6 will solve the Question; for the Product of their multiplication is 48, and the difference of their Squares 64 and 36 is 28, as was prescribed.

Moreover, the Equation in the eighth step gives a Canon to find the greater of the two numbers sought, by the help whereof and the given Product the lesser number shall be also given.

CANON.

9. To the Square of the given Product add the Square of half the given difference of the Squares, and extract the square Root of that summ; then to the said square Root add the said half difference, and extract the square Root of this summ, so shall the last square Root be the greater of the two numbers sought; lastly, by the said greater number divide the given Product of the multiplication of both numbers, and the Quotient shall be the lesser number.

QUEST. 12.

There are two numbers the Product of whose multiplication is 48 (or p,) and the summ of their Squares is 100 (or c;) what are the numbers?

RESOLUTION.

1. For one of the numbers sought put . . .	a	a
2. Then dividing 48 (or p) by a, the Quotient will give the other number, to wit, . . .	$\frac{48}{a}$	$\frac{p}{a}$
3. From the first step, the Square of one of the numbers is	aa	aa
4. And from the second step the Square of the other number is	$\frac{2304}{aa}$	$\frac{pp}{aa}$
5. Therefore the summ of the said Squares is	$\frac{aaaa + 2304}{aa}$	$\frac{aaaa + pp}{aa}$

6. Which summ must be equal to the given summ of the Squares, whence this Equation ariseth, *viz.*

$$\frac{aaaa + 2304}{aa} = 100,$$

Or, $$\frac{aaaa + pp}{aa} = c.$$

7. From which Equation, after due Reduction by the Rules in *Chap.* 12, this will arise,

$$2304 = 100aa - aaaa,$$
$$pp = caa - aaaa.$$

8. Which last Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers sought, will be discovered, *viz.*

$$a = \begin{cases} 8 = \sqrt{(2)}: \overline{\tfrac{1}{2}c + \sqrt{\tfrac{1}{4}cc - pp}}: \\ 6 = \sqrt{(2)}: \overline{\tfrac{1}{2}c - \sqrt{\tfrac{1}{4}cc - pp}}: \end{cases}$$

9. I say, 8 and 6 are the numbers required; for the Product of their multiplication is 48, and the summ of their Squares 64 and 36 is 100, as was prescribed. From the last step also ariseth this

CANON.

From the Square of half the given summ of the Squares of the two numbers sought subtract the Square of the given Product of their multiplication, and extract the square Root of the Remainder; then to half the said summ add the said square Root, and from

the said half summ subtract the said square Root; lastly, extract the square Roor of the summ of that Addition, and also of the Remainder of the latter Subtraction, so shall these two square roots be the numbers sought by the Question propos'd.

QUEST. 13.

There are two numbers whose summ is 14 (or b,) and if the summ of their Squares be multiplyed by the summ of their Cubes, the Product is 72800 (or c;) what are the numbers?

RESOLUTION.

1. For one of the numbers sought put . . .	$a + 7$	$a + \frac{1}{2}b$
2. Then, that their summ may be 14 (or b,) the other number must be	$-a + 7$	$-a + \frac{1}{2}b$
3. The Square of the first number is . . .	$aa + 14a + 49$	$aa + ba + \frac{1}{4}bb$
4. The Square of the latter number is . .	$aa - 14a + 49$	$aa - ba + \frac{1}{4}bb$
5. Therefore the summ of their Squares is	$2aa + 98$	$2aa + \frac{1}{2}bb$

6. Again, the Cube of the first number will be

$$aaa + 21aa + 147a + 343,$$
Or, $$aaa + \tfrac{3}{2}baa + \tfrac{3}{4}bba + \tfrac{1}{8}bbb.$$

7. And the Cube of the latter number will be

$$-aaa + 21aa - 147a + 343,$$
Or, $$-aaa + \tfrac{3}{2}baa - \tfrac{3}{4}bba + \tfrac{1}{8}bbb.$$

8. Therefore the summ of the Cubes in the two last steps is

$$42aa + 686, \quad \text{Or,} \quad 3baa + \tfrac{1}{4}bbb.$$

9. Which summ of the Cubes in the last step being multiplied by the summ of the Squares in the fifth step, produceth $84aaaa + 5488aa + 67228$,
Or, $6baaaa + 2bbbaa + \frac{1}{8}bbbbb$.

10. Which Product in the last step must be equal to 72800 (or c) the Product given in the Question, whence this Equation ariseth, *viz.*

$$84aaaa + 5488aa + 67228 = 72800,$$
Or, $$6baaaa + 2bbbaa + \tfrac{1}{8}bbbbb = c.$$

11. And from that Equation, after due Reduction according to the Rules of the twelfth *Chapter*, this will arise; $aaaa + \frac{196}{3}aa = \frac{199}{3}$,

Or, $$aaaa + \tfrac{1}{3}bbaa = \frac{c}{6b} - \tfrac{1}{48}bbbb.$$

12. Which Equation being resolved by the Canon in *Sect.* 6. of *Chap.* 15. the value of a will be discovered, *viz.*

$$a = 1 = \sqrt{(2)}:\overline{\sqrt{\frac{c}{6b} + \tfrac{1}{144}bbbb}.\ - \tfrac{1}{6}bb}:$$

13. Therefore from the twelfth, first and second steps the two numbers sought are made known:

$$7 + 1 = 8 = \tfrac{1}{2}b + \sqrt{(2)}:\overline{\sqrt{\frac{c}{6b} + \tfrac{1}{144}bbbb}.\ - \tfrac{1}{6}bb}:$$

$$7 - 1 = 6 = \tfrac{1}{2}b - \sqrt{(2)}:\overline{\sqrt{\frac{c}{6b} + \tfrac{1}{144}bbbb}.\ - \tfrac{1}{6}bb}:$$

I say the numbers sought are 8 and 6; for their summ is 14, and if 100 the summ of their Squares be multiplyed by 728, the summ of their Cubes, the Product will be 72800, as was prescribed.

Moreover, the thirteenth step gives a Canon to find out the numbers sought.

CANON.

Divide the given Product by six times the given Summ; then to the Quotient add $\frac{1}{144}$ of the Biquadrate of the given summ, and extract the square Root of the summ of that addition; then from the said square Root subtract $\frac{1}{6}$ of the Square of the given summ, and extract the square Root of the Remainder; lastly, add this square Root to half the given summ and subtract it from the said half summ, so shall the Summ and Remainder be the two numbers sought.

QUEST.

QUEST. 14.

There are two numbers the Product of whose multiplication is 20 (or b,) and the summ of their Cubes is 189 (or c;) what are the numbers?

RESOLUTION.

1. For one of the numbers sought put	a	a
2. Then, by dividing the given Product 20 (or b) by a, the other number will be	$\frac{20}{a}$	$\frac{b}{a}$
3. Therefore from the first step, the Cube of the first number is	aaa	aaa
4. And from the second step the Cube of the other number is	$\frac{8000}{aaa}$	$\frac{bbb}{aaa}$
5. Therefore the summ of the said Cubes is	$\frac{aaaaaa + 8000}{aaa}$	$\frac{aaaaaa + bbb}{aaa}$

6. Which summ must be equal to 189 (or c) the summ given in the Question, whence this Equation ariseth, *viz.*

$$\frac{aaaaaa + 8000}{aaa} = 189,$$

Or, $$\frac{aaaaaa + bbb}{aaa} = c.$$

7. Which Equation being reduced according to *Sect.* 2, 3, and 5. of *Chap.* 12. there will arise $8000 = 189aaa - aaaaaa$,
Or, $bbb = caaa - aaaaaa$.

8. And by resolving the Equation in the last step by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers sought by this Question, will be made known, *viz.*

$$a = \begin{cases} 5 = \sqrt{(3)}:\overline{\tfrac{1}{2}c + \sqrt{\tfrac{1}{4}cc - bbb}}: \\ 4 = \sqrt{(3)}:\overline{\tfrac{1}{2}c - \sqrt{\tfrac{1}{4}cc - bbb}}: \end{cases}$$

9. I say, the numbers sought are 5 and 4; for the Product of their multiplication is 20, and the summ of their Cubes 125 and 64 is 189, as was prescribed.

Moreover, from the two values of a exprest by letters in the eighth step, the following Canon ariseth to find out the numbers sought.

CANON.

From the Square of half the given summ subtract the Cube of the given Product, and extract the square Root of the Remainder; then add the said square Root to half the given summ, and also subtract it from the said half summ; lastly, extract the cubick Root of the summ of that addition, and likewise extract the cubick Root of the latter Remainder, so shall these Cubick Roots be the numbers sought.

QUEST. 15.

There are two numbers the Product of whose multiplication is 20 (or b,) and the difference of their Cubes is 61 (or d;) what are the numbers?

RESOLUTION.

1. For the greater of the two numbers sought put	a	a
2. Then, by dividing the given Product 20 (or b) by a, the lesser number will be	$\frac{20}{a}$	$\frac{b}{a}$
3. Therefore from the first step the Cube of the greater number is	aaa	aaa
4. And from the second step the Cube of the lesser number is	$\frac{8000}{aaa}$	$\frac{bbb}{aaa}$
5. Therefore from the two last steps, the difference of the Cubes of the two numbers sought is	$\frac{aaaaaa - 8000}{aaa}$	$\frac{aaaaaa - bbb}{aaa}$

6. Which difference must be equal to 61 (or d) the difference given in the Question, whence this Equation arises, *viz.*

$$\frac{aaaaaa - 8000}{aaa} = 61, \quad \text{Or,} \quad \frac{aaaaaa - bbb}{aaa} = d.$$

7. Which

7. Which Equation, after due Reduction, (according to *Sect.* 2, 3, and 5. of *Chap.* 12.) will give this that follows, *viz.* $aaaaaa - 61aaa = 8000$,
Or, $aaaaaa - daaa = bbb$.

8. Therefore by resolving the Equation in the last step by the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the greater number sought will be made known, *viz.*

$$a = 5 = \sqrt{(3)} : \tfrac{1}{2}d + \sqrt{\tfrac{1}{4}dd + bbb} :$$

9. Whence the greater number sought is found 5, by which if the given Product 20 be divided, the Quotient will give 4 for the lesser number required.

I say, the numbers 5 and 4 will solve the Question proposed; for the Product of their multiplication is 20, and the difference of their Cubes 125 and 64 is 61, as was prescribed.

Moreover, the Equation in the eighth step gives a Canon to find out the greater of the two numbers sought, by the help whereof and the given Product the lesser number is also given.

CANON.

To the Square of half the given difference add the Cube of the given Product, and extract the square Root of the summ of that addition; then add the said square Root to half the given difference and extract the cubick Root of this summ, so shall the said cubick Root be the greater of the two numbers sought; by which greater number if the given Product be divided the Quotient shall be the lesser number sought.

QUEST. 16.

A Merchant having bought certain Clothes, sells them at $17\frac{1}{4}$ *l.* (or b) the Cloth, and then found that by every 100 *l.* (or c) that he had laid out, he gained as many pounds as he paid for one Cloth; what was the first cost of a Cloth?

RESOLUTION.

1. For the first cost of one Cloth put . . .	a	a
2. Which first cost being subtracted from the money for which the Merchant sold one Cloth, there will remain the gain of one Cloth, to wit, ...	$17\frac{1}{4} - a$	$b - a$

3. Then find what was gained in laying out 100 *l.* (or c,) *viz.* say by the Rule of Three,

$$\text{If} \quad a \;.\; 17\tfrac{1}{4} - a \;::\; 100 \;.\; \frac{1725 - 100a}{a},$$

$$\text{Or,} \quad a \;.\; b - a \;::\; c \;.\; \frac{cb - ca}{a}.$$

Whence the gain of 100 *l.* is found $\frac{1725 - 100a}{a}$, or $\frac{cb - ca}{a}$.

4. But according to the Question the gain of 100 *l* (or c) must be equal to the first cost of one Cloth, therefore from the first and third steps this Equation ariseth, *viz.*

$$a = \frac{1725 - 100a}{a}, \quad \text{Or,} \quad a = \frac{cb - ca}{a}.$$

5. Which Equation, after due Reduction (according to *Sect.* 2, and 3. of *Chap.* 12.) will give this that follows, *viz.* $aa + 100a = 1725$,
Or, $aa + ca = cb$.

6. Therefore by resolving the Equation in the last step by the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit, the first cost of a Cloth will be discovered, *viz.*

$$a = 15 = \sqrt{: cb + \tfrac{1}{4}cc :} - \tfrac{1}{2}c$$

I say the first cost of a Cloth was 15 *l.* as will appear by the Proof: For if a Cloth be bought for 15 *l.* and sold for $17\frac{1}{4}$ *l.* the gain is $2\frac{1}{4}$ *l.* Then if 15 *l.* gain $2\frac{1}{4}$ *l.* it will follow that 100 *l.* will gain 15 *l.* which is equal to the first cost of a Cloth; as was prescribed.

Another

Another way of resolving the preceding Queſt. 16.

1. Let the ſame things be given as before, then for the gain of one Cloth put	a	a
2. Which gain being ſubtracted from the money for which one Cloth was ſold, will leave the firſt coſt of a Cloth, to wit, . .	$17\frac{1}{4} - a$	$b - a$

3. Then find what was gained in laying out 100 *l.* (or c,) and ſay by the Rule of Three,

$$\text{If} \quad 17\tfrac{1}{4} - a \;.\; a \;::\; 100 \;.\; \frac{100a}{17\frac{1}{4} - a},$$

$$\text{Or,} \quad b - a \;.\; a \;::\; c \;.\; \frac{ca}{b - a}.$$

Whence the gain of 100 *l.* is found $\frac{100a}{17\frac{1}{4} - a}$, or $\frac{ca}{b - a}$.

4. But, according to the Queſtion, the gain of 100 *l.* (or c) muſt be equal to the firſt coſt of one Cloth; therefore from the ſecond and third ſteps this Equation ariſeth, *viz.*

$$\frac{100a}{17\frac{1}{4} - a} = 17\tfrac{1}{4} - a, \quad \text{Or,} \quad \frac{ca}{b - a} = b - a.$$

5. Which Equation, after due Reduction according to *Sect.* 2, and 3. of *Chap.* 12. will give this that follows, *viz.*

$$\tfrac{269}{2}a - aa = \tfrac{4761}{16}, \quad \text{Or,} \quad ca + 2ba - aa = bb.$$

6. Therefore by reſolving the Equation in the laſt ſtep by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, or the two Roots of that Equation will be made known, *viz.*

$$a = \begin{cases} \frac{529}{4} = \frac{1}{2}c + b + \sqrt{:\overline{\tfrac{1}{4}cc + cb}:} \\ \frac{9}{4} = \frac{1}{2}c + b - \sqrt{:\overline{\tfrac{1}{4}cc + cb}:} \end{cases}$$

The leſſer of which two Roots or numbers, to wit $\frac{9}{4}$ or $2\frac{1}{4}$ *l.* is the gain of a Cloth, which ſubtracted from $17\frac{1}{4}$ *l.* leaves 15 *l.* for the firſt coſt of a Cloth, as before.

Note. Although the value of a in the Equation in the fifth ſtep may be either $\frac{529}{4}$ or $\frac{9}{4}$, (for that Equation may be expounded by $\frac{529}{4}$ as well as $\frac{9}{4}$,) yet $\frac{9}{4}$ only, to wit, the leſſer value of a ſhall be the gain of a Cloth; for $\frac{529}{4}$ is greater than $17\frac{1}{4}$, and conſequently the gain of one Cloth would exceed the money for which one Cloth was ſold. Which abſurdity appears alſo by the greater value of a as 'tis expreſt by Letters in the ſixth ſtep, for $\frac{1}{2}c + b + \sqrt{:\overline{\tfrac{1}{4}cc + cb}}:$ is manifeſtly greater than b.

QUEST. 17.

Each of two Captains, whereof one had a leſſer number of Souldiers in his Company by 40 (or b) than the other, diſtributed equally among the Souldiers of his own Company 1200 (or c) Crowns, whereby it happened that the Souldiers of the leſſer Company had 5 (or d) Crowns a piece more than the Souldiers of the greater Company; the Queſtion is to find the number of Souldiers in each Company, and how many Crowns each Souldier received.

RESOLUTION.

1. For the number of Souldiers in the leſſer Company put	a	a
2. To which adding 40 (or b) the ſumm will give the number of Souldiers in the greater Company, to wit, ...	$a + 40$	$a + b$
3. Then if 1200 (or c) Crowns be equally divided among the Souldiers of the leſſer Company, the Quotient or ſhare of every Souldier will be	$\frac{1200}{a}$	$\frac{c}{a}$
4. Likewiſe, if 1200 (or c) Crowns be equally divided among the Souldiers of the greater Company, the Quotient or ſhare of every Souldier will be	$\frac{1200}{a + 40}$	$\frac{c}{a + b}$
5. To which latter Quotient adding 5 (or d) Crowns, the ſumm is	$\frac{5a + 1400}{a + 40}$	$\frac{da + db + c}{a + b}$

6. But

6. But according to the Question the summ in the last step must be equal to the Quotient in the third step, whence this Equation ariseth, *viz.*

$$\frac{5a + 1400}{a + 40} = \frac{1200}{a}, \quad \text{Or,} \quad \frac{da + db + c}{a + b} = \frac{c}{a}.$$

7. From which Equation after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. this will arise, *viz.* $aa + 40a = 9600$,

$$\text{Or,} \quad aa + ba = \frac{bc}{d}.$$

8. Therefore the Equation in the last step being resolved by the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit, the number of Souldiers in the lesser Company will be discovered, *viz.*

$$a = 80 = \sqrt{:\frac{bc}{d} + \frac{bb}{4}:} - \tfrac{1}{2}b.$$

From the eighth, first, and second steps it is evident that the lesser Company consisted of 80, and the greater 120 Souldiers; which numbers will satisfie the Conditions in the Question. For the difference of the two Companies is 40 Souldiers; also $\frac{1200}{80} = 15$, and $\frac{1200}{120} = 10$; whence it is manifest that the Souldiers of the lesser Company received 15 Crowns a piece, the Souldiers of the greater Company 10 Crowns a piece, and consequently the Souldiers of the lesser Company had 5 Crowns a piece more than the Souldiers of the greater Company, as was prescribed.

QUEST. 18.

Two Merchants sell linnen Cloth in this manner, *viz.* each sells 60 (or b) Ells, and the first Merchant selling 2 (or c) Ells less for one pound than the second, receives for for his 60 Ells 5 (or d) pounds more than the second Merchant for his 60 Ells. The Question is to find how many Ells each Merchant sold for 1 pound?

RESOLUTION.

1. For the number of Ells which the first Merchant sold for 1 *l.* put	a	a
2. To which number of Ells adding 2 (or c,) the summ will be the number of Ells which the latter Merchant sold for 1 *l.* to wit, . .	$a + 2$	$a + c$
3. Then find how much money the first Merchant received for his 60 Ells, *viz.* say by the Rule of Three, If a . 1 :: 60 . $\frac{60}{a}$; Or, a . 1 :: b . $\frac{b}{a}$. whence the first Merchants total money is found	$\frac{60}{a}$	$\frac{b}{a}$
4. Find likewise how much money the lattter Merchant received for his 60 Ells, *viz.* say, If $a + 2$. 1 :: 60 . $\frac{60}{a+2}$; Or, $a + c$. 1 :: b . $\frac{b}{a+c}$. whence the latter Merchants total money is found	$\frac{60}{a + 2}$	$\frac{b}{a + c}$
5. To which latter summ of money adding 5 (or d) pounds, the summ will be . . .	$\frac{5a + 70}{a + 2}$	$\frac{da + dc + b}{a + c}$

6. But according to the Question the summ of money in the last step must be equal to the summ in the third step, whence this Equation ariseth, *viz.*

$$\frac{5a + 70}{a + 2} = \frac{60}{a}, \quad \text{Or,} \quad \frac{da + dc + b}{a + c} = \frac{b}{a}.$$

7. Which

7. Which Equation, after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. will give this that follows, *viz.* $aa + 2a = 24$,

Or, $aa + ca = \frac{bc}{d}$.

8. Which Equation in the last step being resolved by the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit the number of Ells which the first Merchant sold will be made known,

viz. $a = 4 = \sqrt{:\frac{bc}{d} + \frac{cc}{4}:} - \frac{1}{2}c$.

I say the first Merchant sold 4 Ells for 1 pound, and the second 6 Ells for 1 pound, as will appear by the Proof. For if 4 Ells give 1 pound, then 60 Ells will give 15 pounds. Again, if 6 Ells give one pound, then 60 Ells will give 10 pounds. Whence it is manifest that the first Merchant sold his 60 Ells for 5 pounds more than the second sold his 60 Ells, and sold two Ells less for 1 pound than the second Merchant sold for one pound.

QUEST. 19.

Two Societies, whereof one exceeds the other by 4 (or b) men, divide two equal summs of Crowns; the men of the lesser Society have 8 (or c) Crowns a piece more than those of the greater: and the number of Crowns which each Society receives exceeds the number of men of both Societies by 172 (or d.) The Question is, to find the number of Men in each Society, and the number of Crowns which each Society had?

RESOLUTION.

1. For the number of men of the lesser Society put	a	a
2. To which number adding 4 (or b,) the summ will be the number of men of the greater Society, to wit, ...	$a + 4$	$a + b$
3. Then, according to the Question, if 172 (or d) be added to the summ of the men of both Societies, it will give the number of Crowns shared by each Society, to wit,	$2a + 176$	$2a + b + d$
4. Which number of Crowns being divided by (a) the number of men of the lesser Society, the Quotient or share of every man in that Society will be	$\frac{2a + 176}{a}$	$\frac{2a + b + d}{a}$
5. Likewise if the same number of Crowns before exprest in the third step be divided by $a + 4$, (or $a + b$, the number of men of the greater Society,) the Quotient will give the share of every man in this Society, to wit,	$\frac{2a + 176}{a + 4}$	$\frac{2a + b + d}{a + b}$
6. To which Quotient in the last step adding 8 (or c) the summ will be	$\frac{10a + 208}{a + 4}$	$\frac{2a + b + d + ca + cb}{a + b}$

7. But, according to the Question, the summ in the last step must be equal to the Quotient in the fourth step, whence this Equation ariseth, *viz.*

$$\frac{10a + 208}{a + 4} = \frac{2a + 176}{a}, \quad \text{Or,} \quad \frac{2a + b + d + ca + cb}{a + b} = \frac{2a + b + d}{a}.$$

8. From which Equation, after due Reduction according to *Sect.* 2, 3. and 5. of *Chap.* 12. this Equation will arise, *viz.* $aa + 3a = 88$,

Or, $aa + \frac{cb - 2b}{c}a = \frac{bb + bd}{c}$.

9. Therefore by resolving the last Equation according to the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit, the number of men in the lesser Society will be discovered, *viz.*

$$a = 8 = \sqrt{:\frac{cbd + \frac{1}{4}ccbb + bb}{cc}:} - \frac{b}{2} + \frac{b}{c}.$$

10. Lastly, from the ninth, first, second, and third steps, it is manifest that the number of men in the lesser Society was 8, that of the greater 12, and the number of Crowns divided by each Society 192; which numbers will satisfie the conditions in the Question,

Question, as will appear by the Proof: For $\frac{192}{8} = 24$, and $\frac{192}{12} = 16$; whence it is evident that the men of the lesser Society had 8 Crowns a piece more than those of the greater; also 192, the number of Crowns which each Society divided, exceeded 20 the number of men in both Societies by 172, and 12 the number of men in the greater Society exceeded 8 the number of men in the lesser by 4; as was prescribed.

QUEST. 20.

A Grasier having bought certain Oxen for 270 (or b) pounds, finds, that if he had paid that summ for 5 (or c) Oxen fewer, every Ox would have cost him $\frac{3}{4}$ *l.* (or d) more than he paid for an Ox: What was the number of Oxen bought?

RESOLUTION.

1. For the number of Oxen bought put	a	a
2. Then find out the cost of an Ox, and say, If $a \,.\, 270 :: 1 \,.\, \frac{270}{a}$; Or, $a \,.\, b :: 1 \,.\, \frac{b}{a}$. whence the price of an Ox is	$\frac{270}{a}$	$\frac{b}{a}$
3. Subtract 5 (or c) from the number of Oxen bought, and then find what the rest would cost a piece, saying, If $a-5 \,.\, 270 :: 1 \,.\, \frac{270}{a-5}$; Or, $a-c \,.\, b :: 1 \,.\, \frac{b}{a-c}$. Whence the price of an Ox is found	$\frac{270}{a-5}$	$\frac{b}{a-c}$

4. Then according to the Question, the last mentioned price of an Ox must exceed that in the second step by $\frac{3}{4}$ *l.* (or d;) therefore if the former price be subtracted from the latter, the Remainder must be equal to $\frac{3}{4}$ (or d;) whence this Equation ariseth, *viz.*

$$\frac{270}{a-5} - \frac{270}{a} = \frac{3}{4}; \qquad \text{Or,} \quad \frac{b}{a-c} - \frac{b}{a} = d.$$

5. Which Equation, after due Reduction according to the Rules in *Chap.* 12. will give this that follows,

$$aa - 5a = 1800,$$

$$\text{Or,} \quad aa - ca = \frac{bc}{d}.$$

6. Therefore the Equation in the last step being resolved by the Canon in *Sect.* 12. *Chap.* 15. the value of a, to wit the number of Oxen bought will be discovered, *viz.*

$$a = 45 = \sqrt{:\frac{bc}{d} + \frac{cc}{4}:} + \tfrac{1}{2}c.$$

I say the number of Oxen bought was 45, and every Ox cost 6 pounds, as will appear by the Proof: For first, $\frac{270}{45} = 6$; then from 45 Oxen subtracting 5, the remaining 40 Oxen valued at 270 *l.* will yield $6\frac{3}{4}$ *l.* a piece, which exceeds the former price 6 *l.* by $\frac{3}{4}$ *l.* as was prescribed.

QUEST. 21.

A Merchant buyes linnen Clothes of two sorts, *viz.* 90 (or b) Ells of one sort, together with 40 (or c) Ells of a worser sort for 42 (or d) pounds; and he finds that in laying out 1 pound upon each sort he hath $\frac{1}{3}$ (or m) of an Ell more of the worser sort than the other: What was the price of an Ell of each sort?

RESOLUTION.

1. For the number of Ells of the better sort of Cloth which the Merchant bought for 1 *l.* put	a	a
2. Then according to the Quest. the number of Ells of the worser sort bought for 1 *l.* will be	$a + \frac{1}{3}$	$a + m$

3. Find

3. Find the coſt of all the Ells of the worſer ſort, and ſay, If $a+\frac{1}{3} \,.\, 1 :: 40 \,.\, \frac{40}{a+\frac{1}{3}}$; Or, $a+m \,.\, 1 :: c \,.\, \frac{c}{a+m}$. whence the ſaid full Coſt is found . . .	$\frac{40}{a+\frac{1}{3}}$	$\frac{c}{a+m}$
4. Find likewiſe the coſt of all the Ells of the better ſort, and ſay, If $a \,.\, 1 :: 90 \,.\, \frac{90}{a}$; Or, $a \,.\, 1 :: b : \frac{b}{a}$. whence the ſaid full Coſt is	$\frac{90}{a}$	$\frac{b}{a}$
5. Then the two ſumms of money found out in the third and fourth ſteps being added together will give the full coſt of both ſorts of Cloth, to wit, ...	$\frac{130a+30}{aa+\frac{1}{3}a}$	$\frac{ca+ba+bm}{aa+ma}$

6. Which total Coſt expreſt in the laſt ſtep, muſt (according to the Queſtion) be equal to 42 (or d;) whence this Equation ariſeth, *viz.*

$$42 = \frac{130a+30}{aa+\frac{1}{3}a}; \qquad \text{Or,} \quad d = \frac{ca+ba+bm}{aa+ma}.$$

7. Which Equation, after due reduction (according to the Rules in *Chap.* 12.) will give this that follows, *viz.* $aa - \frac{58}{21}a = \frac{5}{7}$,

$$\text{Or,} \quad aa - \frac{c+b-dm}{d}a = \frac{mb}{d}.$$

In which laſt Equation, if inſtead of the known Coefficient $\frac{c+b-dm}{d}$ we take f, that Equation may be expreſt thus;

$$aa - fa = \frac{mb}{d}.$$

8. Therefore by reſolving the laſt Equation according to the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the number of Ells of the better ſort of Cloth which were bought for 1 *l.* will be diſcovered, *viz.*

$$a = 3 = \sqrt{:\frac{mb}{d} + \frac{ff}{4}:} + \tfrac{1}{2}f.$$

Thus it is found that 3 Ells of the better ſort of Cloth did coſt 1 *l.* and conſequently 1 Ell coſt $\frac{1}{3}$ *l.* and 90 Ells 30 *l.* which ſubtracted from 42 *l.* (the full coſt of both ſorts,) leaves 12 *l.* for the full coſt of 40 Ells of the worſer ſort; and conſequently 1 Ell coſt $\frac{3}{10}$ *l.* and at this rate 1 *l.* will buy $3\frac{1}{3}$ Ells, which is more by $\frac{1}{3}$ of an Ell than was bought of the better ſort of Cloth for 1 *l.* Therefore all the conditions in the Queſtion are ſatisfied.

QUEST. 22.

A Merchant having Spices, to wit, 80 ℔ weight (or b) of Mace, and 100 ℔ weight (or c) of Cloves, ſells both quantities for 65 (or d) pounds in money; whereby it happened that he ſold a quantity of Mace for 10 *l.* (or m,) and the like quantity of Cloves with 60 ℔ weight (or n) more of Cloves for 20 *l.* (or r.) The Queſtion is, to find how many ℔ weight of Mace he ſold for 10 *l.*

RESOLUTION.

1. Let the number of ℔ weight of Mace that the Merchant ſold for 10 *l.* be repreſented by	a	a
2. To which number adding 60, the ſumm will give the number of ℔ weight of Cloves that he ſold for 20 *l.* to wit, ...	$a+60$	$a+n$

3. Then find how much money 80 ℔ weight of Mace was sold for, and say, If $a \,.\, 10 :: 80 \,.\, \frac{800}{a}$; Or, $a \,.\, m :: b \,.\, \frac{mb}{a}$. whence the money for which the said 80 ℔ of Mace was sold is	$\frac{800}{a}$	$\frac{mb}{a}$
4. Find likewise how much money 100 ℔ weight of Cloves was sold for, and say, If $a+60 \,.\, 20 :: 100 \,.\, \frac{2000}{a+60}$; Or, $a+n \,.\, r :: c \,.\, \frac{rc}{a+n}$. whence the money for which the said 100 ℔ of Cloves was sold is	$\frac{2000}{a+60}$	$\frac{rc}{a+n}$
5. The summ of both the said summs of money found out in the third and fourth steps is .	$\frac{2800a+48000}{aa+60a}$	$\frac{mba+mbn+rca}{aa+na}$

6. Which summ in the last step must (according to the Question) be equal to 65 *l.* (or d,) hence this Equation ariseth, *viz.*

$$65 = \frac{2800a+48000}{aa+60a}; \qquad \text{Or,} \quad d = \frac{mba+mbn+rca}{aa+na}.$$

7. Which Equation, after due Reduction (according to *Sect.* 12. *Chap.* 2, 3, 5.) will give this following Equation, *viz.* $aa + \frac{220}{13}a = \frac{9600}{13}$,

$$\text{Or,} \quad aa + \frac{dn - mb - rc}{d}a = \frac{mbn}{d}.$$

In which last Equation if we take f instead of the known Coefficient $\frac{dn - mb - rc}{d}$, and g instead of the known number $\frac{mbn}{d}$, that Equation may be exprest thus,

$$aa + fa = g.$$

8. Therefore by resolving the last Equation according to the Canon in *Sect.* 6. *Chap.* 15. the value of a; to wit, the number of ℔ weight of Mace that was sold for 10 *l.* will be made known, *viz.*

$$a = 20 = \sqrt{:g + \tfrac{1}{4}ff:} - \tfrac{1}{2}f.$$

Thus it is found that 20 ℔ weight of Mace was sold for 10 *l.* and consequently 80 ℔ weight for 40 *l.*

Moreover, adding 60 to 20 (before found,) the summ 80 is the number of ℔ weight of Cloves that was sold for 20 *l.* and consequently 100 ℔ of Cloves was sold for 25 *l.* which added to 40 *l.* (the price of 80 ℔ of Mace,) makes 65 *l.* the prescribed summ of money for both quantities of Spices sold.

QUEST. 23.

Two Merchants entred into Partnership; the first brought in a certain summ of pounds which continued in Company 12 (or b) months, and the second put in 30 *l.* (or c) for 17 (or d) moneths; they gained together $18\frac{3}{4}$ *l.* (or m,) whereof the first Merchant had 26 *l.* (or n) for his principal and gain. It is required to find how many pounds the first Merchant brought into the common Stock?

RESOLUTION.

1. For the first Merchants Stock put . . .	a	a
2. Which Stock being multiplyed by the time it continued in Company, produceth . .	$12a$	ba
3. The second Merchants Stock being multiplyed by the time it remained in Company, produceth	510	cd

4. Then

4. Then proceeding with those two Products according to the Rule of Fellowship with Time, find the gain of the first Merchant, and say,

$$\text{If} \quad 12a + 510 \;.\; 18\tfrac{3}{4} :: 12a \;.\; \frac{225a}{12a+510};$$

$$\text{Or,} \quad ba + cd \;.\; m :: ba \;.\; \frac{mba}{ba+cd}.$$

Whence the gain of the first Merchant is found $\frac{225a}{12a+510}$ or $\frac{mba}{ba+cd}$.

5. Which gain added to the first Merchants Stock a, gives for the summ of his Stock and gain,

$$\frac{12aa+735a}{12a+510}; \quad \text{Or,} \quad \frac{baa+cda+mba}{ba+cd}.$$

6. Which summ must be equal to the 26 *l.* (or n) given in the Question, whence this Equation ariseth, *viz.*

$$\frac{12aa+735a}{12a+510} = 26; \quad \text{Or,} \quad \frac{baa+cda+mba}{ba+cd} = n.$$

7. Then by reducing that Equation according to the Rules in *Chap.* 12. there will arise,

$$aa + 35\tfrac{1}{4}a = 1105,$$

$$\text{Or,} \quad aa + \frac{cd+mb-nb}{b}a = \frac{ncd}{b}.$$

8. Which last Equation being resolved by the Canon in *Sect.* 6. of the 15. *Chapt.* the value of a, to wit, the first Merchants Stock will be found 20 pounds, *viz.* If instead of the known Coefficient $\frac{cd+mb-nb}{b}$ we take f, and g instead of the given number $\frac{ncd}{b}$; Then by the said Canon,

$$a = 20 = \sqrt{:g + \tfrac{1}{4}ff:} - \tfrac{1}{2}f.$$

Whence the first Merchants Stock is found 20 *l.* The Proof may be made by the Rule of Fellowship with Time, in manner following.

$$20 \times 12 = 240$$
$$30 \times 17 = 510$$
$$750 \;.\; 18\tfrac{3}{4} :: \begin{cases} 240 \;.\; 6 \\ 510 \;.\; 12\tfrac{3}{4}. \end{cases}$$

QUEST. 24.

Two Merchants entred into Partnership, the first put in a certain number of Pounds for 3 (or b) moneths; the second put in 50 *l.* (or c) more than the first for 5 (or d) moneths: they gained together 140 *l.* (or m,) whereof the first Merchant had such part, that if 60 *l.* (or n) be added to it, the summ will be equal to the Stock wherewith he entred Partnership: What was the Stock and gain of each Merchant?

RESOLUTION.

1. For the Stock of the first Merchant put	a	a
2. To which adding 50 *l.* (or c,) the summ will give the second Merchant's Stock, to wit,	$a+50$	$a+c$
3. Then multiplying the first Merchant's Stock by the time it remained in Company, the Product is	$3a$	ba
4. Likewise by multiplying the second Merchant's Stock by the time it continued in Company, the Product is	$5a+250$	$da+dc$

5. Then proceeding with those two Products according to the Rule of Fellowship with Time, find the first Merchant's Gain, and say,

$$\text{If} \quad 8a+250 \;.\; 140 :: 3a \;.\; \frac{420a}{8a+250};$$

$$\text{Or,} \quad ba+da+dc \;.\; m :: ba \;.\; \frac{mba}{ba+da+dc}.$$

Whence the gain of the first Merchant is found $\frac{420a}{8a+250}$; Or, $\frac{mba}{ba+da+dc}$.

 6. To

6. To which gain add 60 (or n,) so the summ will be
$$\frac{900a + 15000}{8a + 250}; \quad \text{Or,} \quad \frac{mba + nba + nda + ndc}{ba + da + dc}.$$

7. But, according to the Question, the summ in the last step must be equal to (a) the first Merchant's Stock, whence this Equation ariseth;
$$\frac{900a + 15000}{8a + 250} = a = \frac{mba + nba + nda + ndc}{ba + da + dc}.$$

8. Which Equation, after due Reduction according to the Rules in *Chap.* 12. will produce this following Equation, *viz.* $aa - 81\frac{1}{4}a = 1875$,
$$\text{Or,} \quad aa - \frac{mb + nb + nd - dc}{b + d}a = \frac{ndc}{b + d}.$$

9. In which Equation the value of a, to wit, the first Merchant's Stock, will be discovered by the Canon in *Sect.* 8. *Chap.* 15. viz. $a = 100$ *l.* And consequently from the premises the second Merchant's Stock was 150 *l.* the gain of the first 40 *l.* and the gain of the second 100 *l.* All which will be evident by the following Proof wrought by the Rule of Fellowship with Time.

$$100 \times 3 = 300$$
$$150 \times 5 = 750$$
$$1050 \;.\; 140 \;::\; \begin{cases} 300 \;.\; 40 \\ 750 \;.\; 100. \end{cases}$$

QUEST. 25.

A Citizen having bought a House for a certain summ of pounds, sells it for 64 *l.* (or d,) and finds that his loss in 100 pounds (or c) was equal to a fourth part (or m) of the money that he paid for the House. What number of pounds did the Citizen pay for the House?

RESOLUTION.

1. For the number of pounds which the Citizen paid for the house put	a	a
2. Then will the whole loss by sale of the house be	$a - 64$	$a - d$

3. Find how much was lost by 100 *l.* (or c,) and say,
$$\text{If} \quad a \;.\; a - 64 \;::\; 100 \;.\; \frac{100a - 6400}{a};$$
$$\text{Or,} \quad a \;.\; a - d \;::\; c \;.\; \frac{ca - cd}{a}.$$
Whence the loss *per Cent.* is found $\frac{100a - 6400}{a}$; Or, $\frac{ca - cd}{a}$.

4. But according to the Question the loss *per Cent.* was equal to $\frac{1}{4}$ part of the money which the Citizen paid for the House, therefore from the first and third steps this Equation ariseth, *viz.*
$$\frac{100a - 6400}{a} = \frac{a}{4}; \quad \text{Or,} \quad \frac{ca - cd}{a} = ma.$$

5. Which Equation, after due Reduction according to the Rules in *Chap.* 12. will give
$$400a - aa = 25600; \quad \text{Or,} \quad \frac{c}{m}a - aa = \frac{cd}{m}.$$

6. Therefore by resolving the said Equation according to the Canon in *Sect.* 10. *Chap.* 15. both the values of a will be discovered, either of which will solve the Question; which values or numbers are these following, *viz.*
$$a = \begin{cases} 320 = \frac{c}{2m} + \sqrt{:}\, \frac{cc - 4cdm}{4mm}: \\ 80 = \frac{c}{2m} - \sqrt{:}\, \frac{cc - 4cdm}{4mm}: \end{cases}$$

I say either of the numbers 320 and 80 will satisfie the Conditions in the Question, as will be evident by the Proof: For if a House cost 320 *l.* and be sold for 64 *l.* the loss is 256 *l.* and 100 *l.* at that rate of loss will lose 80, which is $\frac{1}{4}$ part of the first Cost 320 *l.*

Again,

Again, if a Houſe coſt 80 *l.* and be ſold for 64 *l.* the loſs is 16 *l.* and 100 *l.* at this rate of loſs will loſe 20 *l.* which is likewiſe $\frac{1}{4}$ part of the firſt Coſt 80 *l.*

QUEST. 26.

Two Merchants entred into Partnerſhip; the ſumm of their Stocks was 165 (or b) pounds: the firſt Merchant's Stock continued in Company 12 (or c) moneths, and the Stock of the ſecond 8 (or d) moneths: they gained a certain ſumm of pounds, which together with their Stocks they divided between themſelves in ſuch manner, that the firſt Merchant received 67 (or f) pounds for his Stock and gain, and the ſecond 126 (or g) pounds for his Stock and gain. It is deſired to find out each Merchant's Stock and Gain.

RESOLUTION.

1. For the firſt Merchant's Stock put . . .	a	a
2. Then, by ſubtracting that Stock (a) from 165 (or b,) there remains the ſecond Merchant's Stock; to wit, ...	$165 - a$	$b - a$
3. And if you ſubtract (a) the firſt Merchant's Stock from 67 (or f) the ſumm of his Stock and Gain, there will remain his Gain only; to wit, ...	$67 - a$	$f - a$
4. Likewiſe, if you ſubtract the ſecond Merchant's Stock (in the ſecond ſtep) from 126 (or g) the ſumm of his Stock and Gain, there will remain his Gain only; to wit,	$a - 39$	$a + g - b$

5. Now according to the nature of the Rule of Fellowſhip with Time, the Gain of the firſt Merchant $67 - a$ muſt be in ſuch proportion to $a - 39$ the Gain of the ſecond, as the Product of the firſt Merchant's Stock a multiplyed by it's time 12 moneths, is to the Product of the ſecond Merchant's Stock $165 - a$ multiplyed by it's time 8 moneths: hence this Analogy, *viz.*

$$67 - a \;.\; a - 39 \;::\; 12a \;.\; 1320 - 8a,$$

That is, $$f - a \;.\; a + g - b \;::\; ca \;.\; db - da.$$

6. Which Analogy, by comparing the Product made by the multiplication of the Means one into the other, to the Product of the Extremes, produceth this Equation, *viz.*

$$12aa - 468a = 8aa - 1856a + 88440,$$

That is, $$caa + cga - cba = daa - dba - dfa + dbf.$$

7. From which Equation after due Reduction this ariſeth, *viz.*

$$aa + 347a = 22110,$$

That is, $$aa + \frac{db + df + cg - cb}{c - d}a = \frac{dbf}{c - d}.$$

8. Wherefore by reſolving the laſt Equation according to the Canon in *Sect. 6. Chap.* 15. the value of a, that is, the number of pounds expreſſing the firſt Merchant's Stock will be found 55; which ſubtracted from 165 *l.* the ſumm of both their Stocks, leaves 110 *l.* for the ſecond Merchant's Stock: then each of their Stocks being ſubtracted from their reſpective Stock and Gain, *viz.* 55 *l.* from 67 *l.* and 110 *l.* from 126 *l.* there remains 12 *l.* for the Gain of the firſt Merchant, and 16 *l.* for the gain of the ſecond; whence the total Gain was 28 *l.* Which numbers will ſolve the Queſtion, as may eaſily be proved by the Rule of Fellowſhip with Time; thus,

$$55 \times 12 = 660$$
$$110 \times 8 = 880$$
$$1540 \;.\; 28 \;::\; \begin{cases} 660 \;.\; 12 \\ 880 \;.\; 16. \end{cases}$$

QUEST. 27.

A certain Foot-man *A* departeth from *London* towards *Lincoln*, and at the ſame time another Foot-man *B* departeth from *Lincoln* towards *London*; each keeping the ſame Road. When they met, *A* ſaith to *B*, I find that I have travelled 20 (or c) miles more than you, and have gone as many miles in $6\frac{2}{3}$ (or d) dayes, as you have gone miles

miles in all hitherto: 'Tis true saith B, I am not so good a Foot-man as you, but I find that at the end of 15 (or f) dayes hence, I shall be at *London*, if I travel as many miles in every one of those 15 dayes, as I have done in every day hitherto. The Question is, to find how many miles those two Cities are distant one from another, and how many miles each Foot-man had travelled when they met one another.

RESOLUTION.

1. For the desired distance between the two Cities put	a	a
2. Then forasmuch as the number of miles each Foot-man had travelled when they met, being added together make the summ (a,) and the difference between those two numbers was 20 (or c,) for A had travelled 20 miles more than B: Therefore (by the Theorem at the end of *Quest.* 1. *Chap.* 14.) the number of miles which A had travelled was .	$\frac{1}{2}a + 10$	$\frac{1}{2}a + \frac{1}{2}c$
3. And (by the same Theorem) the number of miles which B had travelled was . . .	$\frac{1}{2}a - 10$	$\frac{1}{2}a - \frac{1}{2}c$
4. Then say, If in $6\frac{2}{3}$ dayes A had travelled $\frac{1}{2}a - 10$ miles, how many miles did he travel in one day? so by the Rule of Three, you will find	$\frac{\frac{1}{2}a - 10}{6\frac{2}{3}}$	$\frac{\frac{1}{2}a - \frac{1}{2}c}{d}$
5. Say again, If in 15 dayes B must travel $\frac{1}{2}a + 10$ miles, (that is, all the miles which A had travelled,) how many miles must B travel in one day? so you will find . . .	$\frac{\frac{1}{2}a + 10}{15}$	$\frac{\frac{1}{2}a + \frac{1}{2}c}{f}$
6. Say again, If $\frac{\frac{1}{2}a + 10}{15}$ miles were travelled by B in one day, in how many dayes did he travel $\frac{1}{2}a - 10$ miles? so you will find	$\frac{7\frac{1}{2}a - 150}{\frac{1}{2}a + 10}$	$\frac{\frac{1}{2}fa - \frac{1}{2}fc}{\frac{1}{2}a + \frac{1}{2}c}$
7. Say again, If $\frac{\frac{1}{2}a - 10}{6\frac{2}{3}}$ miles were travelled by A in one day, in how many days did he travel $\frac{1}{2}a + 10$ miles? so you will find	$\frac{3\frac{1}{3}a + 66\frac{2}{3}}{\frac{1}{2}a - 10}$	$\frac{\frac{1}{2}da + \frac{1}{2}dc}{\frac{1}{2}a - \frac{1}{2}c}$

8. But the numbers of days found out in the two last steps must be equal to one another; for when A and B met, each had travelled the same number of days, because they began their Journey at one and the same time: Hence this Equation ariseth, *viz.*

$$\frac{3\frac{1}{3}a + 66\frac{2}{3}}{\frac{1}{2}a - 10} = \frac{7\frac{1}{2}a - 150}{\frac{1}{2}a + 10};$$

That is, $$\frac{\frac{1}{2}da + \frac{1}{2}dc}{\frac{1}{2}a - \frac{1}{2}c} = \frac{\frac{1}{2}fa - \frac{1}{2}fc}{\frac{1}{2}a + \frac{1}{2}c}.$$

9. In which Equation, if you double both the Numerators and Denominators, and then reduce the Equation resulting, to a common Denominator, and cast away the common Denominator, the new Numerators being compared to one another will give this following Equation, *viz.*

$$\frac{20}{3}aa + \frac{800}{3}a + \frac{8000}{3} = 15aa - 600a + 6000;$$

That is, $$daa + 2dca + dcc = faa - 2fca + fcc.$$

10. Which last Equation duly reduced gives this that follows, *viz.*

$$104a - aa = 400,$$

That is, $$\frac{2dc + 2fc}{f - d}a - aa = cc.$$

11. Wherefore by resolving the Equation in the last step according to the Canon in *Sect.* 10. *Chap.* 15. the two values of a will be found these, *viz.*

$$a = 100 = \frac{dc + fc + \sqrt{4dfcc}}{f - d}$$

$$a = 4 = \frac{dc + fc - \sqrt{4dfcc}}{f - d}.$$

12. But

12. But although by either of thoſe values of a; to wit, 100 and 4, the Equation in the tenth ſtep may be expounded, yet the greater value only is the deſired number of miles expreſſing the diſtance between the two Cities; for 'tis evident by the Queſtion, that 20 is but part of the number of miles between the two Cities, and therefore 4 the leſſer value of a is much leſs than the ſaid diſtance: Wherefore 100 the greater value of a is the deſired number of miles between the two Cities. And conſequently the ſecond, third, fourth and fifth ſteps being reſolved into numbers, will ſhew, that when the two Foot-men A and B met one another, A had travelled 60 miles, and B 40 miles: Alſo, A travelled 6 miles, and B 4 miles every day; as will eaſily appear by the Proof.

13. But the numbers in this Queſtion muſt not be given at random, for the Denominator of the Fraction $\frac{2dc+2fc}{f-d}$ in the Equation in the tenth ſtep ſhews that the number d muſt be leſs than the number f, otherwiſe the Queſtion is impoſſible; as may eaſily be inferr'd from the literal Equation in the ninth ſtep: for if in that Equation d be ſuppoſed greater than f, then conſequently dcc is greater than fcc, and after due tranſpoſition this Equation will ariſe, *viz.* $dcc-fcc = faa-daa-2dca-2fca$; where if d be greater than f, then the firſt part of the Equation will be a real quantity, that is, greater than nothing, and the latter part leſs than nothing; but to affirm that a quantity greater than nothing is equal to a quantity leſs than nothing is abſurd; the like abſurdity will follow if we ſuppoſe $d = f$.

14. Having ſhew'd that d muſt neceſſarily be leſs than f, I ſhall prove that the leſſer value of a, as it is expreſt by letters in the eleventh ſtep can never be equal to the whole diſtance between the two Cities. For if we ſhould ſuppoſe the leſſer value to be equal to the ſaid diſtance, it muſt neceſſarily be greater than c, which the Queſtion ſhews to be but part of the ſaid diſtance: But from that ſuppoſition, it will follow by undeniable conſequence, that d is greater than f, which is contrary to what hath been before proved. Now to prove the ſaid conſequence;

15. Suppoſe the leſſer value of a to exceed c, viz. $\frac{dc+fc-\sqrt{4dfcc}}{f-d} \sqsubset c$

16. Then by multiplying each part by $f-d$, it follows that . . . . $dc+fc-\sqrt{4dfcc} \sqsubset fc-dc$

17. And by adding $\sqrt{4dfcc}$ to each part, . . . $dc+fc \sqsubset fc-dc+\sqrt{4dfcc}$

18. And by adding dc to each part, ... $2dc+fc \sqsubset fc+\sqrt{4dfcc}$

19. And by ſubtracting fc from each part, . . $2dc \sqsubset \sqrt{4dfcc}$

20. And by ſquaring each part, ... $4ddcc \sqsubset 4dfcc$

21. And by dividing each part by $4dcc$, . . . $d \sqsubset f$

22. Thus from a ſuppoſition that the leſſer value of a in the eleventh ſtep is greater than c, it follows by juſt conſequence that d is greater than f, which is impoſſible, for it hath before been proved that d muſt be leſs than f. And becauſe the Series of Inferences deduced from the ſaid Suppoſition ends in an impoſſibility, therefore that which was ſuppoſed cannot be true; *viz.* The leſſer value of a is not greater than c, and conſequently it cannot be equal to the diſtance between the two Cities. Which was to be proved.

23. Again, by ſuppoſing d to be leſs than f, as it ought to be, to the end the Queſtion may be poſſible, we may prove the leſſer value of a to be leſſer than c, by returning backwards from the 21 ſtep to the 15, in this manner, *viz.*

24. Suppoſe . . . . $d \sqsupset f$.

25. Then by multiplying each part by $4dcc$, . . $4ddcc \sqsupset 4dfcc$

26. And by extracting the ſquare Root out of each part, ... $2dc \sqsupset \sqrt{4dfcc}$

27. And by adding fc to each part, ... $2dc+fc \sqsupset fc+\sqrt{4dfcc}$

28. And by ſubtracting dc from each part, . . $dc+fc \sqsupset fc-dc+\sqrt{4dfcc}$

29. And by ſubtracting $\sqrt{4dfcc}$ from each part, $dc+fc-\sqrt{4dfcc} \sqsupset fc-dc$

30. Wherefore by dividing each part by $f-d$, it is manifeſt that the leſſer value of a is leſs than c, viz. $\frac{dc+fc-\sqrt{4dfcc}}{f-d} \sqsupset c$

Which was to be proved. Wherefore the leſſer value of a cannot poſſibly be equal to the diſtance between the two Cities, for the ſaid diſtance muſt neceſſarily be greater than part of it ſelf.

31. But

31. But it may be objected, That although f be greater than d, yet how doth it appear that $dc + fc$ is greater than $\sqrt{4dfcc}$, to the end that this may be ſubtracted from that, as the leſſer value of a requires, to make it ſelf a poſſible Root of the Equation in the tenth ſtep? In anſwer to this Objection I ſhall in the next place prove that $dc + fc$ is greater than $\sqrt{4dfcc}$.

32. Foraſmuch as theſe quantities are Proportionals, (for the Product of the extremes is equal to the Product of the means,) . . . . $dd \;.\; df :: df \;.\; ff$

33. Therefore (*per* 25 Prop. 5. Elem. *Euclid.*) $dd + ff \sqsubset 2df$

34. And by multiplying all in the laſt ſtep by cc, $ddcc + ffcc \sqsubset 2dfcc$

35. And by adding $2dfcc$ to each part, ... $ddcc + ffcc + 2dfcc \sqsubset 4dfcc$

36. Wherefore by extracting the ſquare Root out of each part in the laſt ſtep, ... $dc + fc \sqsubset \sqrt{4dfcc}$.

Which was to be proved.

Chap. XVII.

Concerning Arithmetical PROGRESSION.

I. *Arithmetical Progreſſion* is, when many numbers (or other quantities of one and the ſame kind) proceed by a common difference or exceſs; as in theſe, 2, 4, 6, 8, 10, 12, 14, *&c.* here 2 is the common difference betwixt 2 and 4, 4 and 6, 6 and 8, 8 and 10, *&c.* So 1, 2, 3, 4, 5, 6, *&c.* are in Arithmetical Progreſſion, 1 being the common difference: Likewiſe 3, 7, 11, 15, 19, *&c.* or 19, 15, 11, 7, and 3, where 4 is the common difference.

II. Arithmetical Progreſſion is either continued, as in the Examples above expreſt, where every two terms that ſtand next to one another, have one common difference; or elſe diſcontinued or interrupted, as in theſe numbers, 3, 5 : 9. 11, where 5 exceeds 3 by 2, and ſo doth 11 exceed 9; but 9 doth not exceed 5 by 2, for the exceſs of 9 above 5 is 4. In like manner 18, 14 : 21, 17, are in Arithmetical Progreſſion diſcontinued.

III. For the better manifeſtation of the following Propoſitions concerning Arithmetical Progreſſion, let there be a rank of numbers in a continued Arithmetical Progreſſion, as, 3, 7, 11, 15, 19, 23, 27, *&c.* which numbers may be repreſented by a, b, c, d, e, f, g, &c. Alſo, let 105 the ſumm of all the terms of the Progreſſion be repreſented by Z; the common exceſs or difference 4 by X; and the number of terms 7 by T: all which are here orderly expreſt underneath.

Quantities in Arithmetical Progreſſion continued:	3	=	a	=	a	
	7	=	b	=	a + X.	
	11	=	c	=	a + 2X.	
	15	=	d	=	a + 3X.	
	19	=	e	=	a + 4X.	
	23	=	f	=	a + 5X.	
	27	=	g	=	a + 6X.	
The Summ of all the Terms is . . .	105	=	Z	=	Z	
The common difference is .	4	=	X	=	X	
The number of Terms is .	7	=	T	=	T.	

IV. Whence it is manifeſt, that if a be put for the firſt and leaſt term of an Arithmetical Progreſſion continued, and X for the common difference, then (according to the Definition in *Sect.* 1.) the ſecond term ſhall be $a + X$, the third $a + 2X$, the fourth $a + 3X$, the fifth $a + 4X$, *&c.* Moreover, according to the Suppoſitions in *Sect.* 3. $a = a$. $b = a + X$. $c = a + 2X$. $d = a + 3X$. $e = a + 4X$, *&c.*

V. Therefore it follows, that the laſt and greateſt term of every Arithmetical Progreſſion continued is compos'd of the firſt (to wit, the leaſt) term, and of the Product of the common difference multiplyed by a number leſs by 1 (or Unity) than the number

number of terms; as g, or $a + 6X$ is compos'd of the first term a and the Product of X multiplied by 6, which is less by 1 than 7 the number of terms.

VI. Therefore the first and last terms, as also the number of terms being severally given, the common difference shall be also given; for if the first (to wit, the smallest) term be subtracted from the last, and the Remainder be divided by a number less by 1 (or Unity) than the number of terms, the Quotient is the common difference, *viz.* $\frac{g - a}{l - 1} = X$.

VII. It is also manifest from *Sect.* 3. That if the first (to wit, the least) term be equal to the common difference, then the last term is equal to the Product of the common difference (or first term) multiplied by the number of terms, *viz.* If $a = X$, then $g = X + 6X = 7X$.

VIII. Therefore in an Arithmetical Progression continued whose first or least term is equal to the common difference, if the last term and the number of terms be severally given, the first term (or the common difference) shall also be given: For if the last term be divided by the number of terms, the Quotient is the first term or common difference; as, if $a = X$, then $g = X + 6X = 7X$; therefore $\frac{7X}{7} = X = a$.

IX. It is also manifest from *Sect.* 7. That when the common difference divideth any term just without any Remainder, then the common difference is the same with the least term in that Progression, and the Quotient is the number of terms; but if any number remain after the Division is finished, then that Remainder is the least term, and the Quotient increased with 1 (or Unity) gives the number of terms (*per* Sect. 4, & 5.) Therefore if any term greater than the least be given, as also the common difference, the least term, as also the number of terms in that Progression shall also be given; as if 27 be some term greater than the least, and 3 the common difference, by dividing 27 by 3, the Quotient 9 is the number of terms, and the least term is equal to the common difference 3; as in this Progression, 3, 6, 9, 12, 15, 18, 21, 24, 27.

But if 27 be given as before, and 4 be prescribed for the common difference, then 27 divided by 4 gives 6 in the Quotient, and there remains 3 for the least term, and 7 (to wit $6 + 1$) is the number of terms; as in this Progression, 3, 7, 11, 15, 19, 23, 27.

X. If three numbers, suppose a, b, c, be in a continued Arithmetical Progression, *viz.* If the excess of c above b be equal to the excess of b above a, the summ of the Extremes, that is, of the first and last Terms shall be equal to the double of the Mean or middle Term; viz. $a + c = 2b$. For,

1. By supposition, ... $\succ c - b = b - a$,
2. Therefore by adding b to each part, it gives $\succ c = 2b - a$;
3. And by adding a to each part of the last Equation $\succ a + c = 2b$.

Which was to be proved.

XI. If four numbers, suppose a, b, c, d, be in Arithmetical Progression whether continued or interrupted, *viz.* If the excess of b above a be equal to the excess of d above c, the summ of the Extremes shall be equal to the summ of the Means, *viz.* $a + d = b + c$. For,

1. By supposition, ... $\succ d - c = b - a$,
2. Therefore by equal addition of a, ... $\succ a + d - c = b$;
3. Therefore by equal addition of c, ... $\succ a + d = b + c$.

Which was to be proved.

XII. If there be as many numbers as you please in a continued Arithmetical Progression, the summ of the Extremes is equal to the summ of any two Means equally distant from the Extremes, and also to the double of the Mean when the number of Terms is odd.

Let a, b, c, d, e, f, be in Arithmetical Progression continued, and increasing from a; I say the summ of the Extremes a and f is equal to the summ of any two Terms equally distant from the Extremes, that is, to the summ of b and e, and to the summ of c and d. For,

1. By supposition, in regard of the continued Progression, ... $\} f - e = b - a$,
2. Therefore by equal addition of e and a to each part, $\succ a + f = b + e$,
3. Again, by supposition . . . . $\succ c - b = e - d$,

Q 4. There-

4. Therefore by equal addition of d and b, to each part $\} \quad c + d = b + e,$

5. Therefore from the ſecond and fourth ſteps (*per* 1. *Axiom.* 1. *Elem.* Euclid.) . . . . . . $\} \quad a + f = c + d = b + e.$

Which was to be proved.

And if more numbers were propos'd, the Demonſtration would not be otherwiſe; therefore the firſt part of the Theorem is manifeſt.

But if the number of terms be odd as in this continued Progreſſion, $a, b, c, d, e, f, g,$ then the ſumm of the extremes a and g is equal to the double of the middle term d, viz. $a + g = 2d$; which I prove thus:

1. By ſuppoſition, in regard of the continued Progreſſion, ... $\} \quad d - c = e - d,$
2. And conſequently by equal addition of c and d, $\} \quad 2d = c + e,$
3. But by what hath been proved concerning the firſt part of the Theorem in this twelfth *Sect.* . . . $\} \quad a + g = c + e,$
4. Therefore from the two laſt ſteps, (*per Axiom.* 1. *Elem.* 1. Euclid.) . . . . $\} \quad a + g = 2d.$

Which was to be demonſtrated. Therefore the Theorem is every way manifeſt.

XIII. In every Arithmetical Progreſſion continued, the ſumm of the extremes multiplied by the number of terms produceth the double of the ſumm of all the terms.

The number of terms is either even or odd; Firſt, let there be an even number of terms, *viz.* ſuppoſe theſe ſix numbers a, b, c, d, e, f to be in Arithmetical Progreſſion continued;

I ſay, ... $6a + 6f = \left\{ \begin{array}{l} 2a + 2b + 2c + 2d, \\ + 2e + 2f. \end{array} \right.$

DEMONSTRATION.

1. It is evident that . . . . $\} \quad 2a + 2f = 2a + 2f,$
2. And by *Sect.* 12. . . . . $\} \quad 2a + 2f = 2b + 2e,$
3. Likewiſe, by the ſame *Sect.* . . . . $\} \quad 2a + 2f = 2c + 2d,$
4. Therefore by adding the three laſt Equation together, $\} \quad 6a + 6f = \left\{ \begin{array}{l} 2a + 2b + 2c, \\ + 2d + 2e + 2f. \end{array} \right.$

Which was to be demonſtrated. And ſo of others when the number of terms is even.

Secondly, let there be an Arithmetical Progreſſion conſiſting of an odd number of terms, ſuppoſe theſe five, $a, b, c, d, e.$

I ſay, ... $5a + 5e = 2a + 2b + 2c + 2d + 2e.$

DEMONSTRATION.

1. It is manifeſt that . . . . $\} \quad 2a + 2e = 2a + 2e,$
2. And by *Sect.* 12. . . . . $\} \quad 2a + 2e = 2b + 2d,$
3. Likewiſe by *Sect.* 12. . . . . $\} \quad a + e = 2c,$
4. Therefore by adding the three laſt Equations together, ... $\} \quad 5a + 5e = 2a + 2b + 2c + 2d + 2e.$

And ſo of others when the number of terms is odd.

XIV. Therefore from the laſt *Sect.* the firſt and laſt terms, as alſo the number of terms in an Arithmetical Progreſſion continued being given, the ſumm of all the terms ſhall be alſo given: For if the ſumm of the firſt and laſt terms be multiplyed by the number of terms the Product is the double ſumm of all the terms, and conſequently the half of that Product is the ſumm it ſelf. For example, If $a, b, c, d, e, f, g,$ be in Arithmetical Progreſſion continued, and T be put for the number of terms, alſo Z for their ſumm (as before;) Then $Ta + Tg + = 2Z$, and conſequently $\frac{1}{2}Ta + \frac{1}{2}Tg = Z.$

XV. Mr. *William Oughtred* in *Probl.* 4. *Chap.* 19. of his incomparable *Clavis Mathemat.* hath very elegantly handled 20 Propoſitions about Arithmetical Progreſſion continued, which (for the more ample Illuſtration of the preceding Rules in this Book,) I ſhall explain in this *Section*, uſing his own Symbols, which are theſe, *viz.*

$$\left.\begin{array}{c} \alpha \\ \omega \\ T \\ X \\ Z \end{array}\right\} \text{Stands for} \left\{\begin{array}{l} \text{The leaſt (or firſt) term.} \\ \text{The greateſt (or laſt) term.} \\ \text{The number of terms.} \\ \text{The common difference of the terms.} \\ \text{The ſumm of all the terms.} \end{array}\right.$$

Any

Any three of these five things being given, the other two shall be also given, by the respective Canons of the following 20 Propositions, which Mr. *Oughtred* states thus:

Given,	Sought,	By Propos.
α, ω, T	Z and X	1 and 2
α, ω, X	T and Z	3 and 4
α, ω, Z	T and X	5 and 6
α, T, X	ω and Z	7 and 8
α, T, Z	ω and X	9 and 10
α, X, Z	ω and T	11 and 12
ω, T, X	α and Z	13 and 14
ω, T, Z	α and X	15 and 16
ω, X, Z	α and T	17 and 18
T, X, Z	α and ω	19 and 20

PROP. I.

1. . . . . . $\left\{ \begin{array}{l} \alpha, \omega, \text{T are given severally;} \\ \text{Z is sought.} \end{array} \right.$

RESOLUTION.

2. By *Sect.* 14. of this *Chapt.* . . . . $\triangleright$ $T\omega + T\alpha = 2Z$.
Which Equation, if exprest by words, gives this

CANON.

Multiply the summ of the first and last terms by the number of terms, the Product shall be the double of the summ of all the terms; and consequently the half of that Product is the required summ of all the terms.

Which Canon may be exemplified by the following (or any other) rank of numbers in Arithmetical Progression continued, *viz.*

3, 7, 11, 15, 19, 23, 27.

PROP. II.

1. . . . . $\left\{ \begin{array}{l} \alpha, \omega, \text{T are given severally;} \\ \text{X is sought.} \end{array} \right.$

RESOLUTION.

2. By *Sect.* 6. of this seventeenth *Chapt.* . . . . . . $\triangleright$ $\frac{\omega - \alpha}{T - 1} = X$.

Which Equation gives this following

CANON.

Divide the excess of the greatest (or last) term above the least, by the number of terms lessened by 1 (or Unity,) and the Quotient is the common difference required.

Which Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progression continued, *viz.*

3, 7, 11, 15, 19, 23, 27.

From the Equation in the second step of *Prop.* 1. and the Equation in the second step of *Prop.* 2. the Canons of all the following 18 Propositions are deduced.

PROP. III.

1. . . . . $\left\{ \begin{array}{l} \alpha, \omega, \text{X are given severally;} \\ \text{T is sought.} \end{array} \right.$

RESOLUTION.

2. The letters put for the things given and sought, without any other letter, are contained in the Equation in the second step of *Prop.* 2. therefore the work here is only to set T alone in that Equation, which may be done thus, *viz.*

3. By the Canon of *Prop.* 2. . . . . $\frac{\omega - \alpha}{T - 1} = X$,

4. Therefore by multiplying each part of that Equation by T — 1, this ariseth, *viz.* . . . . $\omega - \alpha = TX - X$;

5. And by addition of X to each part of the last Equation, this ariseth; . . . . $\omega - \alpha + X = TX$,

6. Therefore each part of the last Equation being divided by X, the number T will be made known, *viz.* $\frac{\omega - \alpha}{X} + 1 = T$.

The last Equation gives this following

CANON.

From the last (to wit, the greatest) term subtract the first, and divide the Remainder by the common difference; then to the Quotient add 1 (or Unity,) so shall the summ be the required number of terms.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. IV.

1. . . . . α, ω, X are given severally; Z is required.

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $T\omega + T\alpha = 2Z$;

3. And by the Canon of *Prop.* 3. . . . . $\frac{\omega - \alpha}{X} + 1 = T$,

4. Now if instead of T in the first part of the Equation in the second step, you multiply into $\omega + \alpha$ that which in the last Equation is found equal to T, the former Equation will be converted into this, *viz.*

$$\frac{\omega\omega - \alpha\alpha}{X} + \omega + \alpha = 2Z.$$

Which in words is this following

CANON.

From the Square of the greatest (or last) term subtract the Square of the least (or first,) then dividing the Remainder by the common difference, and to the Quotient adding the the summ of the first and last terms, the half of the summ of this addition shall be the required Summ of all the terms.

The Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. V.

1. . . . . α, ω, Z, are given severally; T is required.

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $T\omega + T\alpha = 2Z$;

3. Therefore by dividing each part of that Equation by $\omega + \alpha$; this ariseth, *viz.* . . . . $T = \frac{2Z}{\omega + \alpha}$,

Which Equation gives this following

CANON.

Divide the double of the summ of all the terms by the summ of the first and last terms, the Quotient is the number of terms sought; as may be proved by this following (or any other) Rank of numbers in Arithmetical Progression:

3, 7, 11, 15, 19, 23, 27.

PROP.

PROP. VI.

1. . . . . $\left\{ \begin{array}{l} \alpha,\ \omega,\ Z \text{ are given severally;} \\ X \text{ is required.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 4. . . . . $\succ$ $\frac{\omega\omega - \alpha\alpha}{X} + \omega + \alpha = 2Z,$

3. Which Equation multiplied by X produceth, . $\succ$ $\omega\omega - \alpha\alpha + \omega X + \alpha X = 2ZX,$

4. And by subtracting $\omega X + \alpha X$ from each part of the last Equation, this ariseth, *viz.* . . . . $\succ$ $\omega\omega - \alpha\alpha = 2ZX - \omega X - \alpha X;$

5. Therefore by dividing each part of the last Equation by the Coefficients that are drawn into X, you will find, ... $\succ$ $\frac{\omega\omega - \alpha\alpha}{2Z - \omega - \alpha} = X.$

Which last Equation gives this

CANON.

From the Square of the last term subtract the Square of the first (to wit, the least) term; divide the Remainder by the excess whereby the double summ of all the terms exceeds the summ of the first and last terms, so shall the Quotient be the common difference required.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progression:

3, 7, 11, 15, 19, 23, 27.

PROP. VII.

1. . . . . $\left\{ \begin{array}{l} \alpha,\ T,\ X \text{ are given severally;} \\ \omega \text{ is sought.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 2. . . . . $\succ$ $\frac{\omega - \alpha}{T - 1} = X,$

3. Therefore by multiplying each part of the said Equation by $T - 1$, this will be produced, $\succ$ $\omega - \alpha = TX - X,$

4. And by adding α to each part of the last Equation this ariseth, *viz.* . . . . $\succ$ $\omega = TX + \alpha - X.$

Which last Equation gives this

CANON.

To the Product made by the multiplication of the number of terms into the common difference, add the first (to wit, the least) term, and from the summ subtract the said difference, so shall the Remainder be the last term sought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. VIII.

1. . . . . $\left\{ \begin{array}{l} \alpha,\ T,\ X \text{ are given severally;} \\ Z \text{ is sought.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $\succ$ $T\omega + T\alpha = 2Z,$

3. And by the Canon of *Prop.* 7. . . . . $\succ$ $TX + \alpha - X = \omega,$

4. Now to find an Equation that may consist only of the things given and sought in this *Prop.* 8. multiply each part of the Equation in the third step by T, and there will be produced

$$TTX + T\alpha - TX = T\omega.$$

5. Then

5. Then if instead of $T\omega$ in the second step, you take that which in the fourth step is found equal to $T\omega$, the Equation in the second step will be reduced to this, to wit,

$$\overline{TTX + 2T\alpha - TX} = 2Z,$$

That is, $\overline{TX + 2\alpha - X}$ into $T = 2Z$.

Which last Equation gives this

CANON.

6. To the Product of the multiplication of the number of terms by the common difference, add the double of the first (to wit, the least) term, and from the summ of that Addition subtract the common difference; then multiply the Remainder by the number of terms; so shall the Product be the double summ of all the terms, and consequently the half of that Product is the required summ of all the terms.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. IX.

1. . . . . $\left\{\begin{array}{l}\alpha,\ T,\ Z \text{ are given severally;}\\ \omega \text{ is sought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $\succ$ $T\omega + T\alpha = 2Z$,
3. Therefore by equal subtraction of $T\alpha$, . . $\succ$ $T\omega = 2Z - T\alpha$,
4. Therefore by dividing each part of the last Equation by T, this ariseth; . . . . . $\succ$ $\omega = \dfrac{2Z - T\alpha}{T}$.

Which last Equation gives this

CANON.

From the double of the summ of all the terms subtract the Product of the multiplication of the number of terms by the first (to wit, the least) term, and divide the Remainder by the number of terms; so shall the Quotient be the last term sought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. X.

1. . . . . $\left\{\begin{array}{l}\alpha,\ T,\ Z \text{ are given severally;}\\ X \text{ is sought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 8. . . . . $\succ$ $TTX + 2T\alpha - TX = 2Z$,
3. Therefore by equal subtraction of $2T\alpha$ from each part, this will arise; to wit, . . . $\succ$ $TTX - TX = 2Z - 2T\alpha$,
4. And by dividing each part of the last Equation by $TT - T$, the common difference X will be made known, *viz.* . . . . $\succ$ $X = \dfrac{2Z - 2T\alpha}{TT - T}$.

Which last Equation gives this

CANON.

From the double summ of all the terms subtract the double Product made by the multiplication of the number of terms by the least term, and divide the Remainder by the excess of the Square of the number of terms above the number of terms, so shall the Quotient be the common difference sought.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XI.

1. . . . . $\left\{\begin{array}{l}\alpha,\ X,\ Z \text{ are given severally;}\\ \omega \text{ is sought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 4. . . . . $\succ$ $\dfrac{\omega\omega - \alpha\alpha}{X} + \omega + \alpha = 2Z$,

3. There-

3. Therefore by multiplying that Equation by X, this will be produced; to wit, ... $\omega\omega - \alpha\alpha + X\omega + X\alpha = 2ZX$,
4. And by tranſpoſition of $-\alpha\alpha$, this ariſeth; $\omega\omega + X\omega + X\alpha = 2ZX + \alpha\alpha$,
5. And from the laſt Equation by tranſpoſition of $X\alpha$ this ariſeth; . . . . $\omega\omega + X\omega = 2ZX + \alpha\alpha - X\alpha$;
6. Which laſt Equation falling under the firſt of the three Forms in *Sect.* 1. *Chap.* 15. of this Book, the value of ω ſhall be given by the Canon in *Sect.* 6. of the ſame *Chapt.* viz.

$$\omega = \sqrt{:\tfrac{1}{4}XX + 2ZX + \alpha\alpha - X\alpha:} - \tfrac{1}{2}X.$$

Which Equation gives this

CANON.

From the ſumm of theſe three numbers, to wit, the Square of half the common difference; the double Product of the multiplication of the ſumm of all the terms by the common difference; and the Square of the firſt (to wit, the leaſt) term, ſubtract the Product of the firſt term multiplied by the common difference, and extract the ſquare Root of the Remainder; then from the ſaid ſquare Root ſubtract half the common difference, ſo ſhall this laſt Remainder be the laſt and greateſt term ſought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmeticall Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XII.

1. . . . . α, X, Z are given ſeverally; T is ſought.

RESOLUTION.

2. The Canon of *Prop.* 8. gives this Equation, $XTT + 2\alpha T - XT = 2Z$,
3. Where in regard X is drawn into TT (which is the higheſt degree of the quantity ſought,) let every term of the Equation be divided by X, whence this Equation will ariſe; . . . . $TT + \frac{2\alpha T - XT}{X} = \frac{2Z}{X}$.
4. Now it muſt be diſcovered from the things given whether 2α exceeds X, or is leſs, or equal to X. Firſt then ſuppoſe $2\alpha \sqsubset X$, and then the laſt Equation may be expreſt thus;

$$TT + \frac{2\alpha - X}{X}T = \frac{2Z}{X}.$$

5. Which Equation falling under the firſt of the three Forms in *Sect.* 1. *Chap.* 15. the value of T ſhall be given by the Canon in *Sect.* 6. of the ſame *Chapt.* viz.

$$T = \sqrt{:\frac{\alpha\alpha - \alpha X + \frac{1}{4}XX + 2ZX}{XX}:} - \frac{2\alpha - X}{2X}.$$

6. Secondly, if $2\alpha \sqsupset X$, then the Equation in the third ſtep ſhall be expreſt thus;

$$TT - \frac{X - 2\alpha}{X}T = \frac{2Z}{X}.$$

7. Which Equation falling under the ſecond of the three Forms in *Sect.* 1. *Chap.* 15. the value of T ſhall be given by the Canon in *Sect.* 8. of the ſame *Chapt.* viz.

$$T = \sqrt{:\frac{\frac{1}{4}XX - \alpha X + \alpha\alpha + 2ZX}{XX}:} + \frac{X - 2\alpha}{2X}.$$

8. Laſtly, if $2\alpha = X$, then the Equation in the third ſtep will be expreſt thus;

$$TT = \frac{2Z}{X}; \qquad \text{Whence,} \qquad T = \sqrt{\frac{2Z}{X}}.$$

The three Equations in the 5, 7, and 8 ſteps give a threefold Canon to ſolve this 12 *Prop.* viz.

Canon I. *When the double of the leaſt term exceeds the common difference.*

9. To the Square of the exceſs of the leaſt term above half the common difference add the double Product of the multiplication of the ſumm of all the terms by the common difference, divide the ſumm of that Addition by the Square of the common difference and extract the ſquare Root of the Quotient; then from the double of the leaſt term ſubtract the common difference and divide the Remainder by the double of the common difference: laſtly, ſubtracting this Quotient from the ſquare Root before found, the Remainder ſhall be the number of terms ſought.

This

This Canon may be exemplified by the following or the like Series of numbers in Arithmetical Progreſſion continued, where the double of the leaſt term exceeds the common difference of the terms:

3, 5, 7, 9, 11, 13, 15, &c.

Canon II. *When the double of the leaſt term is leſs than the common difference of the terms.*

10. To the Square of the exceſs of half the common difference above the leaſt term, add the double Product of the multiplication of the ſumm of all the terms by the common difference; divide the ſumm of that Addition by the Square of the common difference, and extract the ſquare Root of the Quotient; then from the common difference ſubtract the double of the leaſt term, and divide the Remainder by the double of the common difference; laſtly, adding this Quotient to the ſquare Root before found, the ſumm ſhall be the number of terms ſought.

This Canon may be exemplified by the following or the like Rank of numbers in Arithmetical Progreſſion continued, where the double of the leaſt term is leſs than the common difference:

2, 7, 12, 17, 22, 27, 32, 37.

Canon III. *When the double of the leaſt term is equal to the common difference of the terms.*

11. Divide the double of the ſumm of all the terms by the common difference, ſo ſhall the ſquare Root of the Quotient be the number of terms ſought.

This Canon may be exemplified by the following Rank of numbers in Arithmetical Progreſſion continued, where the double of the leaſt term is equal to the common difference of the terms:

3, 9, 15, 21, 27, 33, 39.

PROP. XIII.

1. . . . . . ω, T, X are given ſeverally; α is ſought.

RESOLUTION.

2. By the Canon of *Prop.* 7. . . . . $TX - X + \alpha = \omega$,
3. Therefore by tranſpoſition of TX — X, this Equation will ariſe, which makes known the value of α; . . $\alpha = \omega + X - TX$.

Which Equation gives this

CANON.

To the laſt (that is, the greateſt) term add the common difference, and from the ſumm ſubtract the Product of the number of terms multiplied by the common difference; ſo ſhall the Remainder be the firſt (or leaſt) term ſought.

This Canon may be exemplified by the following or any other Rank of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XIV.

1. . . . . . ω, T, X are given ſeverally; Z is ſought.

RESOLUTION.

2. By the Canon of *Prop* 1. . . . . $T\omega + T\alpha = 2Z$,
3. And by the Canon of *Prop.* 13. . . . . $\omega + X - TX = \alpha$,
4. Which latter Equation if it be multiplied by T, will produce $T\omega + TX - TTX = T\alpha$,
5. Then if inſtead of $T\alpha$ in the Equation in the ſecond ſtep, you take that which in the fourth ſtep is found equal to $T\alpha$, the Equation in the ſecond ſtep will be converted into this; $2T\omega + TX - TTX = 2Z$,
6. That is, ... $\overline{2\omega + X - TX}$ into $T = 2Z$.

Which Equation gives this

CANON.

To the double of the laſt (to wit, the greateſt) term, add the common difference; from the ſumm ſubtract the Product of the number of terms multiplyed by the common difference:

difference: then multiply the Remainder by the number of terms, the Product shall be the double of the summ of all the terms; and consequently the half of that Product is the required summ of all the terms.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27, 31.

PROP. XV.

1. . . . . $\left\{ \begin{array}{l} \omega,\ T,\ Z \text{ are given severally;} \\ \alpha \text{ is sought.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 9. . . . . $\succ \dfrac{2Z - T\alpha}{T} = \omega,$

3. Therefore multiplying each part of that Equation by T, this will arise; . . . . $\} \ 2Z - T\alpha = T\omega;$

4. And by transposition of $-T\alpha$ in the last Equation this will arise; . . . . $\} \ 2Z = T\omega + T\alpha;$

5. Likewise by transposition of $T\omega$, this Equation ariseth, $\succ 2Z - T\omega = T\alpha,$

6. Therefore each part of the last Equation being divided by T, the value of α will be made known, *viz.* $\} \ \dfrac{2Z}{T} - \omega = \alpha.$

Which Equation gives this

CANON.

Divide the double summ of all the terms by the number of terms, and from the Quotient subtract the last (to wit, the greatest) term; so shall the Remainder be the first and least term sought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XVI.

1. . . . . $\left\{ \begin{array}{l} \omega,\ T,\ Z \text{ are given severally;} \\ X \text{ is sought.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 14. . . . . $\succ \overline{2\omega \mid X - TX} \text{ into } T = 2Z,$

3. That is . . . . $\succ 2T\omega + TX - TTX = 2Z,$

4. Therefore by due transposition this Equation will arise, $\succ 2T\omega - 2Z = TTX - TX,$

5. Therefore by dividing all in the last Equation by $TT - T$, the value of X will be made known, *viz.* $\} \ \dfrac{2T\omega - 2Z}{TT - T} = X.$

Which Equation gives this

CANON.

From the double Product of the multiplication of the number of terms by the greatest term, subtract the double of the summ of all the terms; divide the Remainder by the excess of the Square of the number of terms above the number of terms, so shall the Quotient be the common difference sought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XVII.

1. . . . . $\left\{ \begin{array}{l} \omega,\ X,\ Z \text{ are given severally;} \\ \alpha \text{ is sought.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 6. . . . . $\succ \dfrac{\omega\omega - \alpha\alpha}{2Z - \omega - \alpha} = X;$

3. Therefore each part of that Equation being multiplied by $2Z - \omega - \alpha$, there will arise, ... $\} \ \omega\omega - \alpha\alpha = 2XZ - X\omega - X\alpha;$

4. Whence by equal addition of $X\omega + X\alpha$ you will find, $\succ \omega\omega + X\omega + X\alpha - \alpha\alpha = 2ZX,$

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Now before known quantities can be separated from unknown in the last Equation, we must discover from the things given in the Proposition, Whether $\omega\omega + X\omega$ be equal, greater, or less than $2ZX$? First therefore,

5. Suppose . . . . $\omega\omega + X\omega = 2ZX$,

6. And then by setting $\omega\omega + X\omega$ in the place of $2ZX$ in the Equation in the fourth step, there will arise, ... $\omega\omega + X\omega + X\alpha - \alpha\alpha = \omega\omega + X\omega$,

7. Whence by subtracting $\omega\omega + X\omega$ from each part, and by transposition of $-\alpha\alpha$, this Equation ariseth; . . . . $X\alpha = \alpha\alpha$,

8. Which last Equation being divided by α, gives . . . . . $X = \alpha$.

From the premises ariseth this

CANON I.

9. When the summ of the Square of the last (to wit, the greatest) term and the Product of the multiplication of the said last term by the common difference of the terms is equal to the double of the Product made by the multiplication of the summ and common difference of the terms, then the said difference is equal to the first or least term sought.

This Canon may be exemplified by the following Series of numbers in Arithmetical Progression continued:

2, 4, 6, 8, 10, 12, 14.

10. Secondly, suppose . . . . $\omega\omega + X\omega \sqsubset 2ZX$.

11. Then from the Equation in the fourth step, after due Reduction, there will arise, . . $\alpha\alpha - X\alpha = \omega\omega + X\omega - 2ZX$,

12. In which last Equation all things are known but α, and the said Equation falls under the second of the three Forms in *Sect.* 1. *Chap.* 15. Therefore the value of α, to wit, the first (or least) term sought shall be given by the Canon in *Sect.* 8. of the same *Chapt.* viz.

$$\alpha = \tfrac{1}{2}X + \sqrt{:\omega\omega + X\omega + \tfrac{1}{4}XX - 2ZX:}$$

From the tenth and twelfth steps ariseth

CANON II.

13. If the summ of the Square of the last (to wit, the greatest) term, and the Product of the multiplication of the said last term by the common difference of the terms, exceeds the double of the Product made by the multiplication of the summ and common difference of the terms; then to the summ first mentioned add the Square of half the common difference; from this summ subtract the double Product above mentioned, and extract the square Root of the Remainder: lastly, add the said square Root to half the common difference, so shall the Summ be the first (or least) term sought.

This Canon may be exemplified by the following Progression:

3, 5, 7, 9, 11, 13.

14. Thirdly, suppose . . . . $\omega\omega + X\omega \sqsupset 2ZX$,

15. But in this third case, to the end a possible Equation may arise, this Determination is necessary, *viz.* . . . . $\omega\omega + X\omega + \tfrac{1}{4}XX$, not $\sqsupset 2ZX$,

16. Then from the Equation in the fourth step by transposition of $\omega\omega + X\omega$, this will arise; $X\alpha - \alpha\alpha = 2ZX - \omega\omega - X\omega$.

17. In which last Equation all things are known but α, and the Equation falls under the last of the three Forms in *Sect.* 1. *Chap.* 15. Therefore the two values of α in that Equation shall be given by the Canon in *Sect.* 10. of the same *Chapt.* viz.

$$\alpha = \tfrac{1}{2}X + \sqrt{:\omega\omega + X\omega + \tfrac{1}{4}XX - 2ZX:}$$

Or, $$\alpha = \tfrac{1}{2}X - \sqrt{:\omega\omega + X\omega + \tfrac{1}{4}XX - 2ZX:}$$

18. Whence it is manifest, that if in this third Case it happens that $\omega\omega + X\omega + \tfrac{1}{4}XX = 2ZX$, then $\alpha = \tfrac{1}{2}X$; that is to say, the first (or least) term sought shall be equal to half the given difference of the terms. But if in the said third Case it happens

that

that $aa + Xa + \frac{1}{4}XX \sqsubset 2ZX$, then there will be two unequal Roots or values of a, to wit, thoſe above expreſt, by either of which the Equation in the ſixteenth ſtep may be expounded; yet (as may eaſily be apprehended) only one of thoſe values of a can be ſuch a firſt (or leaſt) term as will agree with the things given in the Propoſition: But which of thoſe two values of a is the leaſt term ſought, you may diſcover by the Proof formed thus, *viz.* Firſt, by the help of one of thoſe unequal values of a found out as above, together with the given laſt (to wit, the greateſt) term and the given common difference of the terms, you may find out (by the Canon of the third *Prop.*) the number of terms, (which muſt alwayes be a whole number,) and then by the ſame value of a, together with the ſaid laſt term and the number of terms you may by the Canon of *Prop.* 1. find out the ſumm of all the terms; then if this ſumm be equal to the ſumm given in the *Propoſ.* propos'd, that value of a by which the Proof was made, is the leaſt term ſought. But if that Proof will not ſucceed, then the other value of a ſhall be the leaſt term ſought; as will be evident by the Proof made as before.

From the five laſt ſteps there will ariſe

CANON III.

19. When the ſumm of the Square of the laſt (to wit, the greateſt) term, and the Product of the multiplication of the ſaid laſt term by the common difference, is leſs than the double of the Product made by the multiplication of the ſumm and common difference of the terms; but the Aggregate of the ſumm firſt mentioned and the Square of half the common difference is not leſs than the ſaid double Product; then from the ſaid Aggregate ſubtract the ſaid double Product and extract the ſquare Root of the Remainder, that done, add the ſaid ſquare Root to half the common difference of the terms, and alſo ſubtract the ſaid ſquare Root from the ſaid half difference, ſo the Summ or elſe the Remainder, (*viz.* ſuch of them, which by the Proof made according to the direction in the preceding eighteenth ſtep will be found to agree with the things given in the Propoſition,) ſhall be the firſt (or leaſt) term ſought.

This Canon may be exemplified by the two following Ranks of numbers in Arithmetical Progreſſion continued:

I.	2, 5, 8, 11, 14, 17.
II.	2, 7, 12, 17, 22, 27.

PROP. XVIII.

1\. . . . . $\left\{ \begin{array}{l} a,\ X,\ Z \text{ are given ſeverally;} \\ T \text{ is required.} \end{array} \right.$

RESOLUTION.

2\. By the Canon of *Prop.* 14. . . . . $\triangleright\ 2aT + XT - XTT = 2Z.$

3\. Therefore dividing every member of the ſaid Equation by X, (becauſe it is drawn into TT the higheſt degree of the number ſought,) this following Equation will ariſe, *viz.*

$$\frac{2aT + XT}{X} - TT = \frac{2Z}{X},$$

That is, $$\frac{2a + X}{X}T - TT = \frac{2Z}{X}.$$

4\. In which all things are known but T, and the ſaid Equation falls under the laſt of the three Forms in *Sect.* 1. *Chap.* 15. Therefore the two values of T will be made known by the Canon in *Sect.* 10. of the ſame *Chapt.* viz.

$$T = \frac{a + \frac{1}{2}X}{X} + \sqrt{:}\ \frac{aa + aX + \frac{1}{4}XX - 2ZX}{XX}:$$

Or, $$T = \frac{a + \frac{1}{2}X}{X} - \sqrt{:}\ \frac{aa + aX + \frac{1}{4}XX - 2ZX}{XX}:$$

5. But although the Equation in the third ſtep may be expounded by either of the two Roots or values of T above expreſt in the fourth ſtep, yet only one of them can be the number of terms ſought; but which of the ſaid numbers, or values of T will ſolve the Propoſition you may diſcover thus: Firſt, If one of the two numbers or values of T before found out be a Fraction or a mixt number, that value cannot be the number of terms ſought; for the number of terms in an Arithmetical Progreſſion is always a whole number. Secondly, If both the values of T happen to be whole numbers, then the true number of terms ſought may be diſcovered by this Proof; *viz.* Firſt, by the help of one of thoſe values of T in whole numbers, together with the given laſt (or greateſt) term, and the given common difference, find out (by the Canon of *Prop.* 13.) the firſt (to wit, the leaſt) term; and then by the ſame number T, together with the firſt and laſt terms, find out (by the Canon of *Prop* 1.) the ſumm of all the terms; laſtly, If the ſumm ſo found out be equal to the ſumm given in the Propoſition propos'd, then that number or value of T by which the Proof was made ſhall be the true number of terms ſought. But if the Proof will not ſucceed to find out a number equal to the ſumm firſt given, then the other value of T is the number of terms ſought; which will be evident by the Proof made therewith in the ſame manner as before.

From the premiſſes there ariſes this

CANON.

6. From the Square of the ſumm of the laſt (to wit, the greateſt) term, and half the common difference, ſubtract the double of the Product of the multiplication of the ſumm of all the terms by the common difference; divide the Remainder by the Square of the ſaid difference, and extract the ſquare Root of the Quotient. That done, add the ſaid ſquare Root to the Quotient which ariſeth by dividing the ſumm of the laſt term and half the common difference by the difference it ſelf, and alſo ſubtract the ſaid ſquare Root from the ſaid Quotient; ſo the Summ, or elſe the Remainder (*viz.* ſuch of them which according to the preceding fifth ſtep will be found to agree with the things given in the *Propoſ.*) ſhall be the number of terms ſought.

This Canon may be exemplified by the three following Progreſſions; in the firſt of which the greater of the two values of T (in the fourth ſtep) is the number of terms ſought; but in each of the two latter Progreſſions the leſſer value of T is the number of terms ſought.

I.	2, 7, 12, 17, 22, 27, 32.
II.	2, 5, 8, 11, 14, 17, 20.
III.	12, 20, 28, 36, 44, 52, 60.

PROP. XIX.

1. . . . . $\left\{ \begin{array}{l} T,\ X,\ Z \text{ are given ſeverally;} \\ \alpha \text{ is ſought.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 10. . . . . $\frac{2Z - 2T\alpha}{TT - T} = X,$

3. Therefore multiplying each part of that Equation by TT − T, this will be produced, to wit, . . $2Z - 2T\alpha = TTX - TX,$

4. In which laſt Equation all things are known but α, whoſe value after due Reduction of that Equation will be found out, *viz.* . . . . $\alpha = \frac{Z}{T} + \frac{1}{2}X - \frac{1}{2}TX,$

Which in words gives this

CANON.

5. Divide the given ſumm of all the terms by the given number of terms, to the Quotient add half the given difference of the terms, and from the ſumm of that addition ſubtract half the Product of the multiplication of the ſaid number of terms by the common difference; ſo ſhall the Remainder be the firſt (to wit, the leaſt) term required.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progreſſion continued:

2, 7, 12, 17, 22, 27, 32.

PROP.

PROP. XX.

1. . . . . . $\left\{ \begin{array}{l} T, X, Z \text{ are given severally;} \\ \omega \text{ is sought.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop* 16. . . . . $\frac{2T\omega - 2Z}{TT - T} = X$,

3. Therefore multiplying each part of that Equation by TT — T, this will be produced, to wit, . . $2T\omega - 2Z = TTX - TX$,

4. In which last Equation all things are known but ω, whose value, after due Reduction of that Equation, will be discovered, *viz.* . . . . $\omega = \frac{Z}{T} + \frac{1}{2}TX - \frac{1}{2}X$.

Which in words gives this

CANON.

5. Divide the given summ of all the terms by the given number of terms; to the Quotient add half the Product of the multiplication of the number of terms by the common difference given, and from the summ of that Addition subtract half the said difference; the Remainder shall be the last (to wit, the greatest) term required.

This Canon may be exemplified by the following or any other Rank of numbers in Arithmetical Progression continued:

2, 5, 8, 11, 14, 17, 20.

Questions to exercise some of the Canons of the preceding Propositions.

Quest. 1. Suppose 40 Stones be so placed in a streight line, that the first is distant from a Basket one yard, the second two, the third three, and the rest in the same excess; now if some Foot-man undertakes to go from the Basket to fetch into it every Stone one after another, how many Yards must he go to perform that work? *Answ.* 1640 Yards.

Forasmuch as the Foot-man must go 2 Yards (to wit, one forwards, and the same backwards,) to fetch the first Stone into the Basket; 4 Yards for the second; 6 for the third, *&c.* here is an Arithmetical Progression continued whose first (or least) term is 2, the common difference of the terms is also 2, and the number of terms is 40; therefore the summ of all the terms, to wit, the number of Yards sought will be found 1640, by the Canon of the preceding eighth *Prop.*

Quest. 2. Two Foot-men, *A* and *B*, depart at the same time from *London* towards *York*, and travel in this manner, *viz.* *A* travelleth 8 (or *c*) Miles every day; *B* travelleth 1 Mile the first day, 2 Miles the second day, 3 Miles the third day, and so forward, travelling every day one Mile more than in the day next preceding: the Question is, to find in how many dayes *B* will overtake *A*? *Answ.* At the end of 15 days, found out by this following

RESOLUTION.

1. For the number of days that *B* had travelled when he overtook *A*, put . . . a

2. Then to find how many Miles *B* had travelled when he overtook *A*, there is an Arithmetical Progression continued wherein the first and least term is 1, (to wit, 1 Mile which *B* travelled the first day,) also the common difference is 1, (for the Question saith that *B* travelled every day 1 Mile more than in the day next preceding,) and the number of terms is *a*, (which we assumed for the number of days that *B* had travelled when he overtook *A*;) therefore the summ of all the terms (or number of Miles that *B* had travelled) will by the Canon of the preceding *Prop.* 8. be found to be . . . $\frac{1}{2}aa + \frac{1}{2}a$

3. And because *A* travelled 8 (or *c*) Miles dayly, and had travelled the same number of dayes as *B* when *B* overtook *A*, therefore 8 (or *c*) multiplied by *a* produceth the number of Miles that *A* had then travelled; to wit, ... ca

4. But

4. But when *B* overtook *A*, each had travelled the ſame number of Miles; therefore the numbers found out in the two laſt ſteps muſt be equal the one to the other, *viz.* . . . . $\frac{1}{2}aa + \frac{1}{2}a = ca$

5. Which Equation after due Reduction gives . . . . . . $a = 2c - 1$

Which in words is this

CANON.

From the double of the number of Miles that *A* travelled dayly, ſubtract 1 (or Unity,) ſo ſhall the Remainder be the number of dayes ſought.

Whence the number of dayes required will be found 15; for the double of 8 is 16, from which ſubtracting 1, the Remainder 15 is the number of dayes ſought; *viz.* *B* will overtake *A* at the end of 15 dayes, as will be evident by

The Proof.

If 15 be the number of terms, and 1 the firſt (or leaſt) term, as alſo the common difference of the terms of an Arithmetical Progreſſion continued; the ſumm of all the terms will (*per* Canon of *Prop.* 8.) be found 120, being the number of Miles which *B* had travelled in 15 dayes, (according to the Progreſſion of 1 Mile the firſt day, 2 Miles the ſecond, 3 Miles the third, *&c.*) Alſo, *A* travelling 8 Miles every day, would in 15 dayes have travelled 120 Miles. Therefore the conditions in the Queſtion are ſatisfied.

Queſt. 3. A Merchant diſcharged a Debt of 1370 *l.* by ſeveral Payments made in this manner, *viz.* the firſt payment was $1\frac{1}{2}$ *l.* the ſecond payment exceeded the firſt by $\frac{1}{6}$ *l.* the third exceeded the ſecond by the ſame exceſs, and the reſt of the payments in like manner. The Queſtion is, to find how many Payments the Merchant made in diſcharging the ſaid Debt? *Anſw.* 120, found out thus:

There is given in the Queſtion $1\frac{1}{2}$, to wit, the firſt and leaſt term of an Arithmetical Progreſſion continued; alſo $\frac{1}{6}$ the difference of the terms, and 1370 the ſumm of all the terms, to find the number of terms, which (by Canon 1 of the foregoing *Prop.* 12. of this *Chapt.*) will be found 120.

Queſt. 4. If a Debt of 1370 *l.* was diſcharged by ſeveral Payments made in ſuch manner, that the ſecond payment exceeded the firſt by $\frac{1}{6}$ *l.* the third the ſecond, the fourth the third, *&c.* in the ſame exceſs, *viz.* every following payment exceeded the next preceding by $\frac{1}{6}$ *l* and that the laſt payment was $21\frac{1}{3}$ *l.* What was the firſt (to wit, the leaſt) Payment, and how many ſeveral Payments did the Debitor make? *Anſw.* The firſt and leaſt Payment was $1\frac{1}{2}$ *l.* (found out by Canon 2. of *Prop.* 17.) and the number of Payments was 120, found out by the Canon of *Prop.* 18.

Queſt. 5. A Foot-man travelled 124 Miles in 8 dayes at this rate, *viz.* The ſecond dayes journey exceeded the firſt by 3 Miles, the third the ſecond by 3 Miles, and ſo forward in that exceſs; How many Miles was his firſt dayes journey, and how many his laſt? *Anſw.* 5, and 26 Miles; found out by the Canons of *Prop.* 19 and 20.

Queſt. 6. A Draper bought 20 Clothes for 20 Crowns a piece, and ſold the firſt Cloth for a certain number of Crowns; the ſecond for two Crowns more then the firſt; the third for two Crowns more than the ſecond; and ſo by increaſing the price of every following Cloth by two Crowns more than the next preceding Cloth, he ſold the laſt Cloth for 41 Crowns. It is deſired to find the number of Crowns for which he ſold the firſt Cloth; and what he gained or loſt by all the Clothes.

This Queſtion implyes an Arithmetical Progreſſion, whoſe number of Terms is 20; the common difference of the Terms is 2; and the laſt Term is 41: therefore by the Canon of *Prop.* 13. of this *Chapt.* the firſt and leaſt Term will be found 3; and then by the Canon of *Prop.* 1. (or by the Canon of *Prop.* 14.) the ſumm of all the Terms will be found 440. Whence 'tis manifeſt that the Draper gained 40 Crowns by the 20 Clothes; for he bought them for 400 Crowns, and ſold them for 440.

Queſt. 7. One diſtributed 456 Pence among a certain number of poor perſons in this manner, *viz.* To the firſt he gave 6 Pence, to the laſt 51 Pence; the number of Pence given to the ſecond exceeded that given to the firſt, the third the ſecond, and ſo forward to the laſt by an equal exceſs. The Queſtion is, to find how many poor perſons there were; and how many Pence every one between the firſt and laſt received?

To ſolve this Queſtion, an Arithmetical Progreſſion muſt be conceived, whoſe firſt Term is 6; the laſt Term is 51; and the ſumm of all the Terms 456: then by the Canon of *Prop.* 5. the number of Terms will be found 16; and by the Canon of *Prop* 6. the common difference of the Terms will be found 3; wherefore there were 16 poor perſons: and if this Arithmetical Progreſſion, to wit, 6, 9, 12, *&c.* be continued to the ſixteenth Term incluſive, it will ſhew the number of Pence which every one of the poor perſons received; and all thoſe 16 Terms or numbers being added together, make the given ſumm 456.

Queſt. 8. A Stationer ſold 7 (or t) Reams of Paper, the particular prices whereof were certain numbers of Shillings in Arithmetical Progreſſion; the price of the ſecond Ream, that is, of that next above the cheapeſt, was 8 (or b) Shillings; and the price of the laſt or deareſt Ream was 23 (or c) Shillings: what was the price of each Ream?

RESOLUTION.

1. For the price of the cheapeſt or firſt Ream put	a	a
2. Then becauſe the price of the ſecond Ream was 8, (or b,) therefore by ſubtracting a from 8, (or b,) there remains the common difference of the Terms of the Progreſſion, *viz*	$8 - a$	$b - a$
3. Then by the help of the leaſt Term, the common difference of the Terms, and the number of Terms, ſeek (by the Canon of *Prop* 7. of this *Chapt.*) the laſt and greateſt Term, which will be found . .	$48 - 5a$	$2a - ta + tb - b$

4. Which greateſt Term laſt found out muſt be equal to 23 (or c,) hence this Equation ariſeth, *viz.*

$$48 - 5a = 23; \qquad \text{Or,} \qquad 2a - ta + tb - b = c.$$

5. From which Equation after due Reduction this ariſeth, *viz.*

$$a = 5 = \frac{tb - b - c}{t - 2}.$$

Which in words is this

CANON.

From the Product of the price of the ſecond Ream of Paper (to wit, of that next above the cheapeſt,) multiplied by the number of Reams, ſubtract the ſumm of the prices of the ſecond and laſt Reams; then divide the Remainder by the exceſs of the number of Reams above 2: ſo ſhall the Quotient be the price of the firſt (or cheapeſt) Ream.

Whence, by the help of the numbers given in the Queſtion, theſe following numbers in Arithmetical Progreſſion will be diſcovered, which ſolve the Queſtion, *viz.* 5, 8, 11, 14, 17, 20, 23.

Queſt. 9. One being asked what were the ſeveral ages of his five (or t) Children, anſwered, that the age of the eldeſt exceeded that of the ſecond by 2 (or x) years; and by the ſame exceſs the ſecond exceeded the third, the third the fourth, the fourth the fifth or youngeſt Childs age; and if the age of the eldeſt Child were multiplied by the age of the youngeſt it would produce 128 (or c) years. It's deſired to find out the age of every one of the five Children.

The numbers ſought by the Queſtion are in Arithmetical Progreſſion.

RESOLUTION.

1. For the age of the youngeſt Child (being the leaſt Term of the Arithmetical Progreſſion in the Queſtion,) put	a	a
2. Then by the help of a, x and t, *viz.* the age of the youngeſt Child, the common difference of their ages, and the number of Children, ſeek (by the Canon of *Prop.* 7. of this *Chapt.*) the age of the eldeſt, that is, the greateſt Term of the Progreſſion, ſo you will find	$a + 8$	$a + tx - x$
3. Therefore the Product of the multiplication of the firſt and laſt Terms of the Progreſſion is . . .	$aa + 8a$	$aa + txa - xa$

4. Which

4. Which Product must be equal to 128 (or c,) the Product given in the Question; hence this Equation, *viz.* $aa + 8a = 128$; Or, $aa + txa - xa = c$.

5. Wherefore, by resolving the last Equation according to the Canon in *Sect.* 6. *Chap.* 15. the value of a, that is, the age of the youngest Child will be discovered, *viz.*

$$a = 8 = \frac{\sqrt{:ttxx - 2txx + xx + 4c:} - tx - x}{2}.$$

Which in words is this

CANON.

From the Product of the number of Children multiplied into the common difference of their ages subtract the said difference; then to the Square of the Remainder add four times the Product of the age of the eldest Child multiplied into the age of the youngest, and extract the square Root of the summ of that Addition: then from the said square Root subtract the Product of the common difference of their Ages multiplied into the excess of the number of Children above Unity; so the half of the Remainder shall be the age of the youngest Child.

Whence these five numbers are discovered, *viz.* 8, 10, 12, 14, 16; which shew the number of years expressing the age of every one of the five Children: for the Product of the first and last numbers is 128, and the common difference is 2, as was required.

Quest. 10. If the summ of 6 (or t) numbers or Terms in Arithmetical Progression be 48 (or z,) and the Product of the common difference multiplied into the least Term be equal to the number of Terms; what are the numbers of that Progression?

RESOLUTION.

1. For the common difference of the Terms put	a	a
2. Then according to the condition in the Question, if the number of Terms be divided by the common difference, the Quotient is the least Term, to wit,	$\frac{6}{a}$	$\frac{t}{a}$
3. Now by the help of the common difference, the least Term, and the number of Terms, seek (by the eighth *Prop.* of this *Chapt.*) the double summ of all the Terms, so you will find	$30a + \frac{72}{a}$	$tta + \frac{2tt}{a} - ta$

4. Which double summ must be equal to twice 48, the summ given in the Question; hence this Equation ariseth, *viz.*

$$30a + \frac{72}{a} = 96;$$

That is, $$tta + \frac{2tt}{a} - ta = 2z.$$

5. Which Equation duly reduced gives

$$\frac{16}{5}a - aa = \frac{12}{5};$$

That is, . . $$\frac{2z}{tt - t}a - aa = \frac{2t}{t - 1}.$$

6. Wherefore by resolving the last Equation according to the Canon in *Sect.* 10. *Chap.* 15. the two values of a will be found these, *viz.*

$$a = 2 = \frac{z + \sqrt{:zz + 2ttt - 2tttt}}{tt - t}:$$

$$a = \frac{6}{5} = \frac{z - \sqrt{:zz + 2ttt - 2tttt}}{tt - t}:$$

7. Each of which values of a, to wit, 2 and $\frac{6}{5}$ may be taken for the common difference sought. Then because 6 is prescribed in the Question for the Product of the least Term multiplied into the common difference, let 6 be divided by the said 2 and $\frac{6}{5}$ severally, and the Quotients 3 and 5 shall be the two least Terms of two Arithmetical Progressions, each of which will solve the Question: And therefore

The six numbers sought may be either these, 3, 5, 7, 9, 11, 13;
Or these, ... 5, $6\frac{1}{5}$, $7\frac{2}{5}$, $8\frac{3}{5}$, $9\frac{4}{5}$, 11.

In each of which Progressions, the number of Terms is 6; the summ of all the Terms is 48; and the common difference multiplied by the least Term produceth the number of Terms. Which was prescribed in the Question.

The end of the First BOOK.

THE ELEMENTS OF THE ALGEBRAICAL ART.

Book II.

CHAP. I.

Concerning the Genesis or production of Powers, from Roots Binomial, Trinomial, &c.

I. I Shall take it for granted, that the Reader of this Second Book of Algebraical Elements is well exercis'd in the First; and therefore without making any repetition of what hath been there explain'd at large, I shall proceed to the handling of new matter in this mysterious Art. First then, Forasmuch as the extraction of Roots is undoubtedly the hardest lesson in Vulgar Arithmetick, and the reason of the Rules delivered in most Treatises of Arithmetick for extracting the Square and Cubick Roots is known but to few practical Arithmeticians, I shall explain what our learned Divine, and famous Mathematician, Mr. *William Oughtred*, hath succinctly delivered upon this subject in the twelfth, thirteenth and fourteenth Chapters of his incomparable *Clavis Mathematicæ*; to which end, in this and the following second Chapters, I shall first shew the Genesis or production of Powers, from Roots binomial, trinomial, *&c.* and then, in the third and fourth Chapters, their Analysis; or the extraction of the Root or Side out of any given Power, whether it be express'd by number or letters.

II. If a Line or number be divided into any two parts, suppose a the greater, and e the lesser, these connected by the sign $+$ or $-$ do constitute a binomial Root, as $a+e$, or $a-e$; the latter of which some call a residual Root, because it imports a Remainder, *viz.* the difference of the two Names or parts of the Root. In like manner these Compound quantities, $a+b+c$; $a-b-c$; and the like, may be called trinomial Roots, because each of them consists of three Names or parts: and $a+b+c+d$ a quadrinomial Root, that is, a Root consisting of four parts; and so of others.

III. From a Root binomial, trinomial, *&c.* Algebraical Powers may be produced in like manner as from a simple Root, *viz.* by a continued multiplication of the Root into it self: As, for example, The binomial Root $a+e$ being multiplied by it self, that is, $a+e$ by $a+e$, produceth $aa+2ae+ee$ the Square of $a+e$. Again, If the Square $aa+2ae+ee$ be multiplied by its Root $a+e$, the Product will be $aaa+3aae+3aee+eee$, which is the Cube of the Root $a+e$; and if the said Cube be multiplied by its Root $a+e$, it will produce the fourth Power; and so you may proceed to find a fifth, sixth, or what Power you please from the binomial Root $a+e$. But for the greater evidence view the following Operation.

Binomial Root, .	$a + e$
	$a + e$
	$aa + ae$
	$\quad + ae + ee$
Square, . . .	$aa + 2ae + ee.$
	$a + e$
	$aaa + 2aae + aee$
	$\quad + aae + 2aee + eee$
Cube, . . .	$aaa + 3aae + 3aee + eee.$
	$a + e$
	$aaaa + 3aaae + 3aaee + aeee$
	$\quad + aaae + 3aaee + 3aeee + eeee$
Biquadrate, . .	$aaaa + 4aaae + 6aaee + 4aeee + eeee.$

After the same manner, if the Residual Root $a - e$ be multiplied by it self, the Product will be $aa - 2ae + ee$ the Square of $a - e$. Again, if the Square $aa - 2ae + ee$ be multiplied by its Root $a - e$, the Product will be $aaa - 3aae + 3aee - eee$, which is the Cube of the Root $a - e$. And so you may proceed to find a fourth, fifth, or what Power yon please from the residual Root $a - e$; view the following Work.

Residual Root, .	$a - e$
	$a - e$
	$aa - ae$
	$\quad - ae + ee$
Square, . . .	$aa - 2ae + ee.$
	$a - e$
	$aaa - 2aae + aee$
	$\quad - aae + 2aee - eee$
Cube, . . .	$aaa - 3aae + 3aee - eee.$
	$a - e$
	$aaaa - 3aaae + 3aaee - aeee$
	$\quad - aaae + 3aaee - 3aeee + eeee$
Biquadrate, . .	$aaaa - 4aaae + 6aaee - 4aeee + eeee.$

By those two Examples it is manifest, that the Powers from the Residual Root $a - e$ differ only in the signs $+$ and $-$ from like Powers formed from the Binomial Root $a + e$; for in every Power of a residual Root, the signs prefixt before the parts or members of the Power are alternately $+$ and $-$; *viz.* the greatest or first member is affirmative, the second negative, the third affirmative, the fourth negative, and so forwards; as you may see in the Cube of $a - e$, where aaa the greatest extreme member is affirmative; the next number in order being $- 3aae$, is negative; the third member $+ 3aee$ is affirmative; and the last (to wit, the least) member $- eee$ is negative. But in every Power produced from a binomial Root whose parts are connected by $+$, as $a + e$, all the members of the Power are affirmative.

IV. If according to the construction in the last preceding Section, a Scale or Rank of Powers be formed from a binomial Root, as from $a + e$; the members of each Power to the tenth inclusive, will be such as you see in the following Table, where the two last Powers are compendiously exprest according to *Cartesius* his way.

A Table

A Table of Powers, produced from the Binomial Root $a + e$.

The Root.	a	e									
(2)	aa	$2ae$	ee								
(3)	aaa	$3aae$	$3aee$	eee							
(4)	$aaaa$	$4aaae$	$6aaee$	$4aeee$	$eeee$						
(5)	$aaaaa$	$5aaaae$	$10aaaee$	$10aaeee$	$5aeeee$	$eeeee$					
(6)	$aaaaaa$	$6aaaaae$	$15aaaaee$	$20aaaeee$	$15aaeeee$	$6aeeeee$	$eeeeee$				
(7)	$aaaaaaa$	$7aaaaaae$	$21aaaaaee$	$35aaaaeee$	$35aaaeeee$	$21aaeeeee$	$7aeeeeee$	$eeeeeee$			
(8)	$aaaaaaaa$	$8aaaaaaae$	$28aaaaaaee$	$56aaaaaeee$	$70aaaaeeee$	$56aaaeeeee$	$28aaeeeeee$	$8aeeeeeee$	$eeeeeeee$		
(9)	a^9	$9a^8e$	$36a^7ee$	$84a^6e^3$	$126a^5e^4$	$126a^4e^5$	$84a^3e^6$	$36aae^7$	$9ae^8$	e^9	
(10)	a^{10}	$10a^9e$	$45a^8ee$	$120a^7e^3$	$210a^6e^4$	$252a^5e^5$	$210a^4e^6$	$120a^3e^7$	$45aae^8$	$10ae^9$	e^{10}

V. By the foregoing Table it is evident, That the Square of $a + e$ consists of $aa + 2ae + ee$; which shews, that if a number be divided into any two parts, the Square of that number shall be equal to the Squares of the parts, and to twice the Product made by the multiplication of the parts one into the other. As, If 12 be divided into 10 and 2, which may be signified by a and e, Then

The Square of 10 is	100	aa
The Product of 10 multiplied by 2 is 20, which doubled makes . . .	40	$2ae$
The Square of 2 is	4	ee
Which three numbers, to wit, 100, 40 and 4 added together, make the Square of 12, *viz.*	144	$= aa + 2ae + ee$.

In like manner, the said Table shews that the Cube or third Power of the binomial Root $a + e$ consists of the Cubes of the names or parts of the Root a and e, together with the triple of the solid Product made by the multiplication of the Square of the greater part a into the lesser part e, and the triple of the solid Product made by the multiplication of the greater part a into the Square of the lesser part e. This may be illustrated by numbers thus; Suppose 12 to be divided into 10 and 2, which may (as before) be represented by a and e; then the Cube of 12, or of $a + e$, will be equal to the summ of these four solid numbers, *viz.*

The Cube of 10 is	1000	*aaa*
The Square of 10 is 100, which multiplied by 2 produceth 200, this tripled makes	600	3*aae*
Again, 10 multiplied by 4 the Square of 2, produceth 40, the triple whereof is	120	3*aee*
The Cube of 2 is	8	*eee*

Which four numbers, *viz.* 1000, 600, 120 and 8, added together make the Cube of 12, (or 12 × 12 × 12;) that is, . . . 1728 = $aaa + 3aae + 3aee + eee$.

After the same manner, the rest of the Powers in the Table might be exprest by words. Whence 'tis evident that this Literal method discovers many properties in Powers, which in Numeral calculations do lie in obscurity.

VI. Moreover, by a bare inspection into the said Table it may be perceived, that the number prefixt to every one of the mean members of every Power produced from the binomial Root $a+e$, is composed of the two numbers prefixt to the next superiour and inferiour members of the next preceding Power: As for example, if you conceive the line upon which $3aae$ is set to be continued forth at length, it will pass between aa, that is, $1aa$, and $2ae$ in the foregoing second Power (or Square;) now I say that the number 3 prefixt to aae is the summ of 1 and 2 the numbers prefixt to aa and ae. Likewise the number 6 prefixt to $aaee$ one of the members of the fourth power, is composed of 3 and 3 the numbers prefixt to aae and aee in the third Power. Again, the number 15 prefixt to $aaaaee$ in the sixth Power, is the summ of 5 and 10 the numbers prefixt to $aaaae$ and $aaaee$ in the fifth Power. Hence a Table may be made to shew what numbers are to be prefixt to the mean members of every Power.

A

2 For the Square.

3 . 3 For the Cube.

4 . 6 . 4 For the fourth Power.

5 . 10 . 10 . 5 For the fifth Power.

6 . 15 . 20 . 15 . 6 For the sixth Power.

7 . 21 . 35 . 35 . 21 . 7 For the seventh Power.

8 . 28 . 56 . 70 . 56 . 28 . 8 For the eighth Power.

9 . 36 . 84 . 126 . 126 . 84 . 36 . 9 For the ninth Power.

10 . 45 . 120 . 210 . 252 . 210 . 120 . 45 . 10 For the tenth Power.

B C

In this Table, the numbers from *A* to *B*, and likewise from *A* to *C*, do proceed from 2 in an Arithmetical Progression having 1 (to wit, Unity) for a common difference; and every one of the mean numbers standing between the same Term of each Progression, is composed of the two numbers which stand next above each mean number respectively: As, 6 which stands between 4 and 4, is the summ of 3 and 3 which stand above and on each side of 6; likewise 10, which is set between 5 and 5, is the summ of 6 and 4 which stand above 10; and so of the rest. So that this Table may be easily continued farther at pleasure.

VII. Any Power of a Binomial or Residual Root exprest by letters, may without a continued multiplication of the Root into it self be easily formed by the following method, which is deduced from the premises, *viz.* Suppose the fifth Power of the binomial

binomial Root $a+e$ be deſired; Firſt, I write all the ſimple Powers of a, deſcending orderly from the fifth Power downwards to the Root a; as $aaaaa$, $aaaa$, aaa, aa and a, as here you ſee in the firſt Columel: then to all thoſe Powers, except the uppermoſt $aaaaa$, I joyn ſuch ſimple Powers of e, that the ſumm of the Indices of both Powers may make 5, *viz.* To $aaaa$ I joyn e; to aaa, ee; to aa, eee; and to a, $eeee$; then I write $eeeee$ underneath, ſo there are ſix diſtinct Members or Terms, every one of which conſiſts of five dimenſions, as you ſee in the ſecond Columel; that done, by the Table in the foregoing *Sect.* 6. I find that the numbers 5, 10, 10 and 5 are to be prefixt before the mean members of the fifth Power; and accordingly I ſet 5 before $aaaae$, 10 before $aaaee$, likewiſe 10 before $aaeee$, and 5 before $aeeee$; laſtly, by prefixing $+$, or ſuppoſing it to be prefixt before every one of the ſaid five members, the fifth Power of the binomial Root $a+e$ is compleated, as you ſee in the third Columel, and in every reſpect agrees with the fifth Power in the Table in the forgoing *Sect.* 4. But if the ſigns $+$ and $-$ be alternately prefixt before the members of the ſaid fifth Power, according to what hath been ſaid at the latter end of *Sect.* 3. it will be the fifth Power of the Reſidual Root $a-e$.

(1)	(2)	(3)
$aaaaa$	$aaaaa$	$aaaaa$
$aaaa$	$aaaae$	$5aaaae$
aaa	$aaaee$	$10aaaee$
aa	$aaeee$	$10aaeee$
a	$aeeee$	$5aeeee$
	$eeeee$	$eeeee$

VIII. Laſtly, from a Root conſiſting of three, four, or any number of parts, the Square, Cube, or any higher Power of the Root may be produced by a continued multiplication of the Root into it ſelf: As, the Trinomial Root $a+b+c$ being multiplied by it ſelf, its Square will be found $aa+2ab+2ac+bb+2bc+cc$; and this Square multiplied again by its Root $a+b+c$ produceth the Cube of the ſame Root, that is, $aaa+3aab+3aac+3abb+6abc+3acc+bbb+3bbc+3bcc+ccc$. After the ſame manner Powers may be produced from a Root conſiſting of four, or any number of parts. And if the conſtitution of Powers expreſt by letters be ſeriouſly conſidered, it will be ſome help to diſcover whether an Algebraick quantity conſiſting of more than three Members or Terms be a perfect Power or not, and alſo give ſome light to diſcover its Root.

CHAP. II.

Concerning the compoſition of Powers in numbers, from a Binomial Root.

Sect. I. *Of the compoſition of a Square, from a number given for the Side or Root.*

1. SUppoſe the Square of the Root 28 be deſired; Firſt write down the Root 28 in ſuch manner that there may be ſpace enough to ſet one figure between 2 and 8, and let a line be drawn under them; as alſo two downright lines, the one next after 2, and the other after 8, to the end the numbers which are to be found out may be orderly placed for Addition: then let the Root 28 be conceived to be divided into theſe two parts 20 and 8, and let a be put for the greater part, and e for the leſſer. Now foraſmuch as the Square of $a+e$ is $aa+2ae+ee$, therefore the Square of 28, or of $20+8$ may be compoſed thus, *viz.* The Square of 20 is 400 (or aa;) the double of 20 is 40 (or $2a$,) which multiplyed by 8 (or e) produceth 320 (that is $2ae$;) and the Square of 8 is 64 (or ee:) laſtly, the ſaid three numbers 400, 320 and 64 being ſet under one another

	2	8	Root propoſed.
$a = 20$	4	00	aa
$e = 8$	3	20	$2ae$
		64	ee
	7	84	Square required.

another, in such order, that units may stand under units, tens under tens, &c. and added together the summ makes 784 the Square of the Root 28; as may easily be proved by multiplying 28 into it self.

2. When the given number or Root whose Square is desired consists of three or more places, as 47803; First, the Square of the two foremost figures towards the left hand, that is, of 47, must be found out in like manner as before in the first Example, so there will be produced 2209 for the Square of 47; as you see in the following Example 2. Secondly, write 47 in a void place and annex a cypher to it, so it makes 470, this number must now be esteemed a, and 8 the next following character of the Root must be taken for e; and then according to these values of a and e, the numbers signified by aa, $2ae$, and ee being added together make 228484 for the Square of 478, (as you see here underneath.) Where observe, that to find the Square of 470, (that is, of a,) you need only annex two cyphers to 2209 which was before found for the Square of 47. Thirdly, annex a cypher to 478 (in a void place,) and it makes 4780 for a new value of a, and the next following character of the Root, to wit, 0, is the new value of e; then according to these values of a and e, the value of $aa + 2ae + ee$ is 22848400, to wit, aa only; for $e = 0$, and consequently $2ae + ee = 0$: so the said 22848400 is found for the Square of 4780. Lastly, by annexing a cypher to 4780 it makes 47800 for a new value of a, and 3 the last figure of the Root is the new value of e: Then according to these values of a and e, the summ of the numbers signified by aa, $2ae$, and ee, makes 2285126809, which is the Square of the said given Root 47803, as may easily be proved by multiplying the said Root by it self. Compare the following Example with the precedent directions.

Example 2. *of* Sect. I.

	4	7	8	0	3	Root proposed.
$a = 40$	16	00				aa
$e = 7$	5	60				$2ae$
		49				ee
$a = 470$	22	09	00			aa
$e = 8$		75	20			$2ae$
			64			ee
$a = 4780$	22	84	84	00		aa
$e = 0$				00		$2ae$
				00		ee
$a = 47800$	22	84	84	00	00	aa
$e = 3$			28	68	00	$2ae$
					09	ee
	22	85	12	68	09	Square required.

Sect. II. *Of the composition of a Cube from a number given for the Side or Root.*

1. Let the Cube of the Root 28 be desired: First, I write the Root 28 in such manner that there may be space enough to set two figures between 2 and 8; then having drawn a line under 28, and down-right lines as before in the Square, I conceive the Root 28 to be divided into 20 and 8, that is, a and e. Now forasmuch as the Cube of $a + e$ is composed of these four members, *viz.* aaa, $3aae$, $3aee$ and eee, (as appears by the Table in *Sect.* 4. *Chap.* 1.)

	2	8	Root proposed.
$a = 20$	8	000	aaa
$e = 8$	9	600	$3aae$
	3	840	$3aee$
		512	eee
	21	952	Cube desired.

therefore the Cube of 20 + 8 (that is, of 28) may be composed thus; *viz.* First, the Cube of 20 is 8000, that is, aaa;) secondly, the triple of the Square of 20 being multi-

multiplied by 8 produceth 9600, (that is, $3aae$;) thirdly, the triple of 20 being multiplied by the Square of 8 produceth 3840, (that is, $3aee$;) fourthly, the Cube of 8 is 512, (that is, eee;) laſtly, the ſaid four numbers 8000, 9600, 3840, 512, being ſet under one another in ſuch order that units may ſtand under units, tens under tens, &c. and added together make 21952 the Cube of the given Root 28.

2. When the given number or Root whoſe Cube is deſired conſiſts of three or more places, as 28503; Firſt, the Cube of the two formoſt figures, that is, of 28, muſt be found out in like manner as before in Example 1. ſo there will be produced 21952. Secondly, write 28 in a void place, and annexing a cypher to it, it makes 280, this number muſt now be eſteemed a, and 5 the next following character of the Root muſt be taken for e; then according to theſe values of a and e, the numbers ſignified by aaa, $3aae$, $3aee$ and eee being added together make 23149125 for the Cube of 285, (as you ſee in Example 2.) where obſerve, that to find the Cube of 280, that is, of a, you need only annex three cyphers to 21952 which was before found for the Cube of 28. Thirdly, annex a cypher to 285 after it is ſet in a ſpare place, and it makes 2850 for a new value of a, and the next following Character of the Root, to wit, 0, is the new value of e: Then according to theſe values of a and e, the value of $aaa + 3aae + 3aee + eee$ is 23149125000, that is, aaa only; for $e = 0$, and conſequently $3aae + 3aee + eee = 0$, ſo the ſaid 23149125000 is found for the Cube of 2850. Laſtly, by annexing a cypher to 2850 it makes 28500 for a new value of a, and 3 the laſt figure of the Root is the new value of e; then according to theſe values of a and e, the ſumm of the numbers ſignified by aaa, $3aae$, $3aee$ and eee makes 23156436019527, which is the Cube of the given Root 28503, as may eaſily be proved by multiplying the ſaid Root into it ſelf cubically. Compare the following Example with the precedent directions.

Example 2. *of* Sect. II.

	2	8	5	0	3	Root propoſed.
$a = 20$	8	000				aaa
$e = 8$	9	600				$3aae$
	3	840				$3aee$
		512				eee
$a = 280$	21	952	000			aaa
$e = 5$	1	176	000			$3aae$
		21	000			$3aee$
			125			eee
$a = 2850$	23	149	125	000		aaa
$e = 0$				000		$3aae$
				000		$3aee$
				000		eee
$a = 28500$	23	149	125	000	000	aaa
$e = 3$		7	310	250	000	$3aae$
				769	500	$3aee$
					27	eee
	23	156	436	019	527	Cube deſired.

Sect.

Sect. III. *Of the composition of a Biquadrate, or the fourth Power, from a number given for the Root.*

1. Let the Root 28 be propofed, and its Biquadrate or fourth Power defired. Firft, I write the Root 28 in fuch manner that there may be fpace enough to fet three figures between 2 and 8; then having drawn a line under 28, and downright lines as in former Examples, I conceive the Root 28 to be divided into 20 and 8, that is, *a* and *e*; now forafmuch as the Biquadrate, or fourth Power produced from the Binomial Root $a + e$ is $aaaa + 4aaae + 6aaee + 4aeee + eeee$, (as appears by the Table in *Sect.* 4. *Chapt.* 1.) therefore the fourth Power of $20 + 8$, (that is, of 28) may be compofed thus, *viz.* Firft, the fourth Power of 20 is 160000, (that is *aaaa*;) fecondly, four times the Cube of 20 being multiplied by 8 produceth 256000, that is, 4*aaae*;) thirdly, fix times the Square of 20 being multiplied by the Square of 8 produceth 153600, (that is, 6*aaee*;) fourthly, four times 20 multiplied by the Cube of 8 produceth 40960, (that is, 4*aeee*;) fifthly, the fourth Power of 8 is 4096, (that is, *eeee*;) laftly, the fumm of all the faid five numbers, to wit, 160000, 256000, 153600, 40960, and 4096 makes 614656, which is the fourth Power of 28 the Root propofed; as will eafily appear by the multiplication of 28 four times into it felf.

	2	8	Root propofed.
$a = 20$	16	0000	*aaaa*
$e = 8$	25	6000	4*aaae*
	15	3600	6*aaee*
	4	0960	4*aeee*
		4096	*eeee*
	61	4656	Biquadrate defired.

2. When the given number or Root whofe fourth Power is defired confifts of three places, as 285; Firft, the fourth Power of the two foremoft figures 28 muft be found out in like manner as in Example 1. of this *Sect.* fo there will be produced 614656 for the fourth Power of 28. Secondly, let 28 be fet in a void place, and annex a cypher to it, fo it makes 280 which muft now be efteemed *a*, and 5 the next following character of the Root muft be taken for *e*; and then according to thefe values of *a* and *e*, the numbers fignified by *aaaa*, 4*aaae*, 6*aaee*, 4*aeee* and *eeee* being added together make 6597500625, which is the fourth Power of the given Root 285, and the work will ftand as you fee in the following Example 2. After the fame manner the work is to be continued when the given Root confifts of more than three places, as is manifeft by the following Example 3.

Example 2. *of* Sect. III.

	2	8	5	Root propofed.
$a = 20$	16	0000		*aaaa*
$e = 8$	25	6000		4*aaae*
	15	3600		6*aaee*
	4	0960		4*aeee*
		4096		*eeee*
$a = 280$	61	4656	0000	*aaaa*
$e = 5$	4	3904	0000	4*aaae*
		1176	0000	6*aaee*
		14	0000	4*aeee*
			625	*eeee*
	65	9750	0625	Biquadrate required.

Example

Example 3. *of* Sect. III.

	2	8	0	5	Root proposed.
$a = 20$	16	0000			*aaaa*
$e = 8$	25	6000			4*aaae*
	15	3600			6*aaee*
	4	0960			4*aeee*
		4096			*eeee*
$a = 280$	61	4656	0000		*aaaa*
$e = 0$			0000		4*aaae*
			0000		6*aaee*
			0000		4*aeee*
			0000		*eeee*
$a = 2800$	61	4656	0000	0000	*aaaa*
$e = 5$		4390	4000	0000	4*aaae*
		11	7600	0000	6*aaee*
			140	0000	4*aeee*
				625	*eeee*
	61	9058	1740	0625	Biquadrate desired.

Sect. IV. *Of the composition of the fifth Power from a number given for its Root.*

1. Let the Root 28 be proposed, and its fifth Power desired: First, let the Root 28 be written in such manner that there may be space enough to set four figures between 2 and 8; then having drawn a line under 28, and down-right lines as in the Examples of the precedent Sections, let 28 be conceived to be divided into 20 and 8, that is, a and e; now forasmuch as the fifth Power produced from the Binomial Root $a + e$ is $aaaaa + 5aaaae + 10aaaee + 10aaeee + 5aeeee + eeeee$, (as is manifest by the Table in *Sect.* 4. *Chap.* 1.) Therefore the fifth Power of $20 + 8$, (that is, of 28) may be composed thus; First, the fifth Power of 20 is 3200000, (that is, *aaaaa*;) secondly, five times the fourth Power of 20 being multiplied by 8 produceth 6400000, (that is, 5*aaaae*;) thirdly, ten times the Cube of 20 being multiplied by the Square of 8 produceth 5120000, (that is, 10*aaaee*;) fourthly, ten times the Square of 20 multiplied by the Cube of 8 produceth 2048000, (that is, 10*aaeee*;) fifthly, five times 20 multiplied by the fourth Power of 8 produceth 409600, (that is, 5*aeeee*;) sixthly, the fifth power of 8 is 32768, (that is, *eeeee*,) lastly, the Summ of all those six numbers, *viz.* 3200000, 6400000, 5120000, 2048000, 409600 and 32768 makes 17210368, which is the fifth Power of 28 the Root proposed; as will easily appear by multiplying 28 five times into self.

	2	8	
$a = 20$	32	00000	*aaaaa*
$e = 8$	64	00000	5*aaaae*
	51	20000	10*aaaee*
	20	48000	10*aaeee*
	4	09600	5*aeeee*
		32768	*eeeee*
	172	10368	

2. When the given number or Root whose fifth Power is desired consists of three places, as 285; First, the fifth Power of the two foremost figures 28 must be found out in like manner as in Example 1. of this *Sect.* so there will be produced 17210368 for the fifth Power of 28. Secondly, let 28 be set in a void place and annex a cypher to it, so it makes 280, which must now be esteemed a, and 5 the next following character of the Root must be taken for e; then according to these values of a and e, the numbers signified by *aaaaa*, 5*aaaae*, 10*aaaee*, 10*aaeee*, 5*aeeee* and *eeeee* being added together make 1880287678125, which is the fifth Power of the given Root 285, and the work will stand as you see in the following Example 2. Nor will the Operation be more difficult, (though more laborious,) to find the fifth Power of a number (or Root) consisting of four or more places.

Example 2. *of* Sect. IV.

	2	8	5	Root propofed.
$a = 20$	32	00000		*aaaaa*
$e = 8$	64	00000		5*aaaae*
	51	20000		10*aaaee*
	20	48000		10*aaeee*
	4	09600		5*aeeee*
		32768		*eeeee*
$a = 280$	172	10368	00000	*aaaaa*
$e = 5$	15	36640	00000	5*aaaae*
		54880	00000	10*aaaee*
		980	00000	10*aaeee*
		8	75000	5*aeeee*
			3125	*eeeee*
	188	02876	78125	Fifth Power defired.

By the precedent Rules and Examples of this Chapter, the ingenious Reader will eafily apprehend, how to compofe the fixth, feventh, or any higher Power, from a Root given in number and confidered as a Binomial $a + e$, as before hath been directed. The main bufinefs confifting in a right underftanding of the numbers fignified by a and e, and in finding out the numbers anfwering to the members of the defired Power of $a + e$, according to the Table in *Sect.* 4. of the precedent *Chapt.* 1.

Chap. III.

Concerning the refolution of Powers expreft by numbers: or, The extraction of all kinds of Roots out of Powers given in numbers.

Sect. I. *Of the extraction of the Square Root out of a number given.*

1. LET it be obferved in general, That the Refolution of every Power given in number confifts in a regular Subtraction of thofe numbers which are fuppofed to be added together in the compofition of each Power refpectively, according to the Rules of the laft preceding Chapter, wherein I prefuppofe the Reader to be well exercis'd. And for the more ready extraction of any Root, it will be convenient to have in a readinefs the refpective Powers of the nine fingle figures; as, if the fquare Root be defired, then the Squares of 1, 2, 3, 4, 5, 6, 7, 8, 9 will be ufeful, which Roots and Squares are expreft in the following Tabulet.

ROOTS.	1	2	3	4	5	6	7	8	9
SQUARES.	1	4	9	16	25	36	49	64	81

2. When a whole number is propofed and its Square-root defired, the number propofed muft be prepared for Extraction, by diftributing it into parts or members after this manner; *viz.* Firft fet a point over the firft or Units place of the given number, then paffing over the fecond place fet another point over the third, alfo paffing over the fourth place fet another point over the fifth; and in that order, if there be more places in the given number, points are to be fet, fo that between every two points which ftand next to one another there will be one place without any point over it. As, for example, if the fquare Root of 119025 be defired, I fet points as here you fee, whereby the faid number is diftributed into three members, to wit, 11, 90, 25. In like manner if the fquare Root of 785 be defired, the

$\dot{1}1\dot{9}0\dot{2}5$

the points will ſtand as here you ſee, whereby the ſaid 784 is diſtributed into two members 7 and 84. The points ſet as aforeſaid ſhew the number of places that will be found in the Root; for if there be two points, there will be two places in the Root; if three points, then the Root will conſiſt of three places, &c. The points alſo ſhew what member of the number given belongs to the finding out of every ſingle Character of the Root ſought, as is evident by the Rules in *Sect.* 1. of the precedent *Chapt.* 2. Theſe things being premiſed as preparatory to the Extraction of the ſquare Root, I ſhall proceed to Examples.

$\dot{7}\,8\dot{4}$

Example 1.

3. Let it be required to extract the ſquare Root of 784. By the preceding Rule 2. it is evident that the deſired Root conſiſts of two places. *viz.* of ſome number of Tens under 100, and of ſome number of Unities under 10; which two numbers, (agreeable to the compoſition of a Square in *Sect.* 1. of the precedent *Chapt.* 2.) may be repreſented by a and e, ſo that a and e ſignifies the Root ſought; and conſequently the Square of $a+e$, that is, $aa+2ae+ee$ is equal to the propoſed number 784. Now to find out the number of Tens, (that is, a,) in the Root; (after a crooked line is drawn on the right hand of the given number, that the Root, like the Quotient in Diviſion, may be ſet next after the ſaid crooked line, as alſo a down-right line next after each of the points, as here you ſee,) the firſt work in the Extraction is alwayes to ſubtract the greateſt ſquare whole number contained in the firſt member towards the left hand, from the ſaid member, and to write the Root of the ſaid ſquare number in the Quotient for the firſt ſingle figure of the deſired Root: ſo 4 being the greateſt Square contained in the firſt member 7, I ſubſcribe 4 under 7, and ſet 2 the Root of the ſaid 4 in the Quotient; then after a line is drawn under 4, I ſubtract 4 from 7, or, 400 from 784, and there remains the *Reſolvend* 384, that is, that part of the given number 784 which is yet to be reſolved. Now obſerve, that the ſaid 2 in the Quotient, in reſpect of the next following unknown character of the Root, is really 20, which is the number ſignified by a in the Compoſition; and the Square of 20, to wit, 400, is aa, which being the firſt number found in the Compoſition, is the firſt number to be ſubtracted in the Reſolution. Obſerve alſo, that the next ſingle Character of the Root, whether it happen to be a figure or a cypher, is called e, which is yet unknown.

$\dot{7}$	$8\dot{4}$	(2
4		
3	84	

4. Then I proceed to find the value of e, that is, a ſingle Character with this condition, that the ſumm of the numbers ſignified by $2ae$ and ee may not exceed the *Reſolvend* 384; for from this number that ſumm muſt be ſubtracted. Now becauſe (for the reaſon aforeſaid) a is 20, therefore $2a$ is 40, which muſt be eſteemed a *Diviſor*, and ſet under the *Reſolvend*; then I divide the ſaid *Reſolvend* 384 by 40, and find the Quotient 9 for the number e, provided it will anſwer the condition before-mentioned; and therefore I make tryal (in a waſt Paper) to ſee whether 9 will ſatisfie the ſaid condition or not, in this manner, *viz.* If e be 9, and $2a$ 40; then conſequently $2ae$ is 360, and ee is 81; therefore $2ae+ee=441$, this ought to be ſubtracted from the *Reſolvend* 384; but 441 exceeds 384, and therefore cannot be ſubtracted from it, ſo as to leave a real Remainder, whence I conclude, that e muſt be leſs than 9: and therefore I make tryal with 8 in like manner as before with 9, *viz.* If $e=8$, and $2a=40$, then conſequently $2ae=320$, and $ee=64$, therefore $2ae+ee=384$, which may be ſubtracted from the *Reſolvend* 384; wherefore I conclude that e, (that is, the figure which muſt follow 2 in the Quotient,) is 8, which I ſet in the Quotient: then I ſubſcribe 320 and 64 (before found) under the *Reſolvend* 384, (in ſuch order that Units may ſtand under Units, and Tens under Tens,) and adding the ſaid 320 and 64 together, the ſumm is 384, (which ſome Authors call the *Gnomon*, others, the *Ablatitium*,) which ſubtracted from the *Reſolvend* 384 leaves 0; ſo the whole Extraction is finiſh'd,

	$\dot{7}$	$8\dot{4}$	(28
Subtract	4	00	aa
	3	84	*Reſolvend.*
$a=20$		40	$2a$. *Diviſor.*
$e=8$	3	20	$2ae$
		64	ee
Subtract	3	84	*Ablatitium.*
		00	

and

and the ſquare Root of the given number 784 is found 28, which is the true Root ſought; for 28 multiplied by 28 produceth 784.

NOTE 1.

The firſt Operation in the Extraction of the ſquare Root, is alwayes to ſubtract the greateſt ſquare whole number (that is, *aa*,) contained in the firſt member (towards the left hand) of the given number, from the ſaid member, and to ſet the Root of the ſaid Square in the Quotient, (as hath been ſhewn in the third ſtep,) which Root is the firſt figure of the Root ſought. This work is no more repeated in the whole Extraction; but the work in the fourth ſtep is to be renewed for the finding out of every following Character in the Root.

NOTE 2.

After the firſt figure of the Root ſought is known, and ſet in the Quotient, let it be written in a void place and multiplied by 10, (by annexing to the ſaid firſt figure a cypher towards the right hand,) then is the Product to be taken for the value of *a*, in order to the finding out of the firſt *Diviſor*. Alſo, when the firſt and ſecond Characters of the Root are ſet in the Quotient, and there be yet another to come forth, then the number conſiſting of thoſe two Characters with a Cypher annexed to them is to be taken for a new value of *a*, in order to the finding out of the ſecond *Diviſor*. Likewiſe, when the firſt, ſecond and third Characters of the Root are ſet in the Quotient, and there be yet another to come forth, then the number conſiſting of thoſe three Characters with a Cypher annexed to them, is to be taken for a new value of *a*; and ſo forwards, when there be more Characters in the Root. The reaſon of which work is manifeſt from the Compoſition of Powers in the precedent *Chap.* 2.

But the letter *e* repreſents every ſingle unknown figure or cypher next following that part of the Root which is already diſcovered and ſet in the Quotient. This Note concerning the eſtimation of *a* and *e* is to be obſerved not only in the Extraction of the Square-root, but of any Root whatever.

NOTE 3.

After the number ſignified by *a* is found out by Note 2. the *Diviſor*, which ſhews how to begin the tryal in ſearching out the unknown ſingle Character repreſented by *e*, is conſequently known: for in the Reſolution of every Power produced from the Binomial Root $a + e$, the *Diviſor* conſiſts of ſuch Powers of *a* as are multiplied into the Powers of *e*; and becauſe the Square-root of $a + e$ is $aa + 2ae + ee$, therefore in the extraction of the Square-root the *Diviſor* is $2a$; ſo that when the number *a* is known, the *Diviſor* $2a$ is conſequently known.

NOTE 4.

When the *Diviſor* is found out by Note 3. as alſo the *Ablatitium*, (that is, the number to be ſubtracted,) which in the extraction of the Square-root is compos'd of $2ae$ and ee, the two numbers ſignified by $2ae$ and ee muſt each of them be ſet in ſuch order under the preſent *Reſolvend*, (that is, the number remaining to be reſolved,) that Units may ſtand under Units, Tens under Tens, &c. to the end that the *Ablatitium* may be rightly compoſed and ſubtracted from the preſent *Reſolvend*.

NOTE 5.

When the *Diviſor* is not contained once in the particular or preſent *Reſolvend*, a cypher (to wit, 0,) muſt be ſet in the Quotient; and then the *Reſolvend* muſt be augmented with the next member (towards the right hand) of the Power propoſed, for a new particular *Reſolvend*: Alſo a new *Diviſor* muſt be found out by Note 3, and the like is to be done as often as the *Diviſor* is not contained once in the particular *Reſolvend*. The practice of theſe Notes will be ſhewn in the following Example.

Example

Example 2.

5. If the ſquare Root of 2285126809 be deſired, it will be found 47803 by the precedent Rules, and the work will ſtand as here you ſee underneath.

	22	85	12	68	09	(47803. *Root.*
Subtract	16					*aa*
	6	85				*Reſolvend.*
$a = 40$		80				2*a.* *Diviſor.*
$e = 7$	5	60				2*ae*
		49				*ee*
Subtract	6	09				*Ablatitium.*
		76	12			*Reſolvend.*
$a = 470$		9	40			2*a.* *Diviſor.*
$e = 8$		75	20			2*ae*
			64			*ee*
Subtract		75	84			*Ablatitium.*
			28	68		*Reſolvend.*
$a = 4780$			95	60		2*a.* *Diviſor.*
$e = 0$			28	68	09	*Reſolvend.*
$a = 47800$			9	56	00	2*a.* *Diviſor.*
$e = 3$			28	68	00	2*ae*
					9	*ee*
Subtract			28	68	09	*Ablatitium.*
			00	00	00	

Explication of Example 2.

The firſt figure of the Root is 4, (by the foregoing Note 1.) whoſe Square 16 ſubtracted from 22 the firſt member towards the left hand of the number propoſed leaves 6, to which the ſecond member 85 being annexed, there ariſeth 685 for the next *Reſolvend*: Or to cauſe the ſame effect, ſuppoſe 0 to be annexed to 4 the firſt figure of the Root, and it makes 40, (that is, *a*,) whoſe Square 1600 (or *aa*) ſubtracted from 2285 the two firſt Members of the number firſt propoſed, leaves (as before) the *Reſolvend* 685.

Then, the firſt figure of the Root being found 4, the value of *a* is 40, (by Note 2.) which doubled gives 80 for a Diviſor to the *Reſolvend* 685, (by Note 3.) and then by dividing and making tryal as is directed in the precedent fourth ſtep, the number *e* will be found 7 for the ſecond figure of the Root, and conſequently the numbers ſignified by 2*ae* and *ee* are 560 and 49 ;theſe being ſet orderly and added together (according to Note 4.) make the *Ablatitium* 609, which ſubtracted from the ſaid *Reſolvend* 685, there remains 76, to which annexing 12 the third member of the number firſt propoſed, it makes 7612 for a new *Reſolvend.*

Again, the two formoſt figures of the Root being found 47, the new value of *a* is 470, (by Note 2.) which doubled gives 940 for a Diviſor to the ſaid *Reſolvend* 7612, (by Note 3.) then by dividing and making tryal as is directed in the fourth ſtep, the value of *e* is found 8 for the third figure of the Root; whence the numbers ſignified by 2*ae* and *ee* are 7520 and 64; theſe being ſet orderly and added together (according to Note 4.) make the *Ablatitium* 7584, which ſubtracted from the *Reſolvend* 7612 before-mentioned, leaves 28, to which annexing 68 the fourth member of the number firſt propoſed, it makes 2868 for a new *Reſolvend.*

Again, the three formoſt figures of the Root being 478, the value of *a* is 4780, (by Note 2.) which doubled gives 9560 for a Diviſor to the ſaid *Reſolvend* 2868,(by Note 3.) then by dividing as aforeſaid the value of *e* is found 0; therefore, (according to Note 5.) I ſet 0 in the Quotient, and becauſe in this caſe the *Ablatitium* is alſo 0, the *Reſolvend* 2868 from which the ſaid *Ablatitium* ought to be ſubtracted remains the ſame without alteration; therefore by annexing 09 the laſt member of the number firſt propoſed,

to

to the ſaid 2868 it makes 286809 for a new (and the laſt) *Reſolvend:* laſtly, by proceeding as before, the laſt figure of the Root will be found 3; ſo that the Square-root ſought is 47803; for this multiplied by it ſelf produceth 2285126809, the number whoſe Square-root was deſired.

The premiſes may ſuffice to ſhew a perfect Method of extracting the ſquare Root of a whole number having an exact Square Root, which I have explain'd at large, that the reaſon and certainty of the Rules might be apparent: But this Method may be contracted into more practical and compendious Rules, as I have ſhewn in the 32. *Chapt.* of Mr. *Wingate*'s common Arithmetick.

6. But when a whole number hath not a Square root exactly expreſſible by any rational or true number, then to approach infinitely near the exact Root, firſt, payrs of Cyphers, as 00, 0000, 000000, or 00000000, *&c.* are to be annexed to the number given; then eſteeming the number given with the cyphers annexed to be one whole number, let its ſquare Root be extracted according to the precedent (or other practical) Rules; that done, look how many points were ſet over the number firſt given, for ſo many of the formoſt places in the Quotient are to be taken for the Integers in the Root, and the reſt following thoſe Integers expreſs the fractional part of the Root in decimal parts: As, for example, if the ſquare Root of 12 be deſired; I annex ſix cyphers to 12, thus, 12.000000, and then the ſquare Root of $\dot{1}\dot{2}.0\dot{0}0\dot{0}0\dot{0}$ being extracted, it will be found 3.464, that is, $3\frac{464}{1000}$: but becauſe after the Extraction is finiſh'd there happens to be a Remainder, I conclude, that $3\frac{464}{1000}$ is leſs than the true Root, but $3\frac{465}{1000}$ is greater than it. So that by annexing three pairs of cyphers, you will not miſs $\frac{1}{1000}$ part of an Unit of the ttue Root, and by annexing eight cyphers you will not want $\frac{1}{10000}$ part; and in that order you may approach as near as you pleaſe when you cannot obtain the exact ſquare Root of a whole number given.

7. The ſquare Root of a vulgar Fraction is found out thus, *viz.* Firſt, if the Fraction be not in its leaſt terms, let it be reduced to the leaſt terms; then extract the ſquare Root of the Numerator for a new Numerator, and the ſquare Root of the Denominator for a new Denominator, ſo ſhall this new Fraction be the ſquare Root of the Fraction propoſed. As, for example, the ſquare Root of $\frac{9}{16}$ is $\frac{3}{4}$; likewiſe, the ſquare Root of $\frac{1}{4}$ is $\frac{1}{2}$.

But when either the Numerator or Denominator of a vulgar Fraction hath not a perfect ſquare Root, then to find the ſquare Root of that Fraction very near, firſt reduce the Fraction to a decimal Fraction whoſe Numerator may conſiſt of an even number of places, *viz.* of two, four, or ſix places, *&c.* then extract the ſquare Root of that decimal as if it were a whole number, and the Root that comes forth ſhall be a decimal Fraction expreſſing nearly the ſquare Root of the Fraction propoſed: As, for example, if the ſquare Root of $\frac{13}{16}$ be deſired, I firſt reduce it to this decimal Fraction .81250000; (for, as 16 . 13 :: 100000000 . 81250000,) then by extracting the ſquare Root of .81250000 as if it were a whole number, I find .9013, that is $\frac{9013}{10000}$, which is near the ſquare Root of $\frac{13}{16}$, for it wants not $\frac{1}{10000}$ part of an Unit of the exact ſquare Root of $\frac{13}{16}$.

8. Laſtly, if the ſquare Root of a mixt number be deſired, firſt reduce it to an improper Fraction, and then extract the ſquare Root of that improper Fraction as before; but if it hath not an exact ſquare Root, then reduce the fractional part of the mixt number firſt propoſed to a decimal Fraction of an even number of places, and after this decimal is annexed to the Integers of the mixt number, extract the ſquare Root out of the whole, then ſo many points as were ſet over the Integers, ſo many of the formoſt places in the Quotient are to be taken for the Integers in the Root, and the reſt expreſs the fractional part of the Root in decimal parts: As, for example, the ſquare Root of $34\frac{33}{64}$, that is, of $\frac{2209}{64}$, will be found $\frac{47}{8}$, or $5\frac{7}{8}$; and the ſquare Root of $7\frac{2}{3}$, that is, of 7.666666, *&c.* is 2.768, *&c.* that is, $2\frac{768}{1000}$, *&c.*

Sect.

Sect. II. *Of the extraction of the Cubick Root out of a number given.*

1. For the more ready extraction of the Cubick Root of a number given, the following Tabulet will be useful, which shews at first sight the Cubick Root of any cubical whole number less than 1000.

ROOTS.	1	2	3	4	5	6	7	8	9
CUBES.	1	8	27	64	125	216	343	512	729

2. When a whole number is proposed and its cubick Root desired, the number given must be prepared for Extraction, by distributing it into parts or members after this manner; *viz.* First, a point is to be set over the Units place of the given number; then passing over the second and third places towards the left hand, another point is to be set over the fourth place; also passing over the fifth and sixth places another point is to be set over the seventh place: and in that order as many points are to be set as the number propos'd will admit, and consequently between every two adjacent points there will be two places without points. So if the cubick Root of 1331 be desired, after points are set as is above directed, the said 1331 will be distributed into two members, to wit, 1 and 331. In like manner if the cubick Root of 21952 be required, the points will stand as you see in the Example, and the said 21952 will be distributed into two members 21 and 952; likewise this number 941192 being pointed in the same order will be distributed into the two members 941 and 192; and this number 23156436019527 into these five members, 23, 156, 436, 019, 527. The points shew the number of places that will be found in the Root; for so many points as there be, so many places will the Root consist of; they likewise shew what member of the number propos'd belongs to the extraction of every single Character of the Root sought.

$$\begin{array}{r} \dot{1}33\dot{1} \\ 2\dot{1}95\dot{2} \\ 9\dot{4}119\dot{2} \\ 2\dot{3}15\dot{6}43\dot{6}01\dot{9}52\dot{7} \end{array}$$

3. The given number whose cubick Root is desired may be conceived to be produced from the cubical multiplication of the Binomial Root $a + e$, and then the said number will be compos'd of these four members or solid numbers, *viz.* aaa, $3aae$, $3aee$ and eee, (as appears by the third Power in the Table in *Sect.* 4. *Chap* 1.) Now because the Resolution of a Cubick number, *viz.* the extraction of the cubick Root, is deducible from the steps of the Composition of a Cubick number from its Root, (for such numbers as are added in the Composition are to be subtracted in the Resolution,) respect must be had to *Sect.* 2. *Chap.* 2. of this Book.

Example 1.

4. Let it be required to extract the cubick Root of 21952. By the precedent second Rule it is evident that the desired Root consists of two places, *viz.* of some number of Tens under 100, and of some number of Unities under 10, which two numbers, (agreeable to the Composition of a Cube in *Sect* 2. of the precedent *Chap.* 2.) may be represented by a and e, so that $a + e$ signifies the Root sought; and consequently the Cube of $a + e$, that is, $aaa + 3aae + 3aee + eee$ is equal to the given number 21952. Now to find out the number of Tens, (that is, a) in the Root, (after a crooked line is drawn on the right hand of the given number, that the Root, like the Quotient in Division, may be set next after the said crooked line, as also a down-right line next after each of the points, as here you see.) The first work in the Extraction is alwayes to subtract the greatest Cubick whole number contained in the first member towards the left hand, from the said member, and to write the Root of the said Cube-number in the Quotient for the first single figure of the desired cubick Root: So 8 being the greatest Cube contained in the first member 21, I subscribe 8 under 21, and set 2 the cubick Root of the said 8 in the Quotient, then after a line is drawn under 8, I subtract 8 from 21, or, 8000 from 21952, and there remains the *Resolvend* 13952, that is, that part of the proposed number 21952 which is yet to be resolved. Now observe, that the said 2 in the Quotient,

$$\begin{array}{r|l} 21 & 952\;(2 \\ 8 & \\ \hline 13 & 952 \end{array}$$

in

in respect of the next following unknown character of the Root, is really 20, which is the number signified by a in the Composition, and the Cube of 20, to wit, 8000, is aaa, which being the first number found in the Composition, is first to be subtracted in the Resolution. Observe also, that the next single character of the Root whether it happen to be a figure or a cypher is called e, which is yet unknown.

5. Then I proceed to find the value of e, that is, a single Character with this condition, that the summ of the numbers signified by $3aae$, $3aee$ and eee may not exceed the remaining *Resolvend* 13952, for from this number that summ must be subtracted. Now because (for the reason aforesaid) a is 20, therefore $3aa = 1200$, and $3a = 60$; then subscribing the said 1200 and 60 under the *Resolvend* 13952, (in such order that Units may stand under Units, and Tens under Tens, &c.) and adding them together, the summ is 1260, which must be esteemed a *Divisor*, and set under the *Resolvend*. Then by supposing I were to divide the said *Resolvend* 13952 by 1260, I find the Quotient exceeds 9, but e alwayes represents a single figure or a cypher, and therefore it cannot exceed 9; wherefore I make tryal with 9 (in a void place) to see whether it will answer the before-mentioned condition to which e is subject, in this manner, *viz.* Forasmuch as it was before found that $3aa = 1200$, and $3a = 60$, it will follow, if we suppose $e = 9$, that $3aae = 10800$, also $3aee = 4860$, and $eee = 729$; therefore $3aae + 3aee + eee = 16389$: this ought to be subtracted from the *Resolvend* 13952, but 16389 exceeds 13952, and therefore cannot be really subtracted from it, whence I conclude that e must be less than 9; and therefore I make tryal with 8 in like manner as before with 9; *viz.* Having before found that $3aa = 1200$, and $3a = 60$, it will follow, if we suppose $e = 8$, that $3aae = 9600$, also $3aee = 3840$, and $eee = 512$; therefore $3aae + 3aee + eee = 13952$, which may be subtracted from the *Resolvend* 13952; wherefore I conclude that e, (that is, the figure which must follow 2 in the Quotient,) is 8, which I set in the Quotient: then I subscribe the three numbers before found, to wit, 9600, 3840 and 512 under the *Resolvend* 13952, (in such order that Units may stand under Units, Tens under Tens, &c.) and adding together the said three numbers so subscribed, their summ makes 13952, (the *Ablatitium*,) which subtracted from the *Resolvend* 13952 leaves 0. So the Extraction is finish'd, and 28 is found to be the cubick Root of the proposed number 21952; for 28 multiplied into it self cubically, *viz.* $28 \times 28 \times 28$ produceth 21952.

	21	952	(28
Subtract	8		aaa
	13	952	*Resolvend.*
$a = 20$	1	200	$3aa$
		60	$3a$
	1	260	*Divisor.*
$e = 8$	9	600	$3aae$
	3	840	$3aee$
		512	eee
	13	952	*Ablatitium.*
		000	

NOTE 1.

The first Operation in the extraction of the cubick Root, is alwayes to subtract the greatest Cubick whole number, (that is, aaa) contained in the first member (towards the left hand) of the given number, from the said member, and to set the Root of the said Cube-number in the Quotient; which Root is the first figure of the Root sought, as hath been shewn in the fourth step. This work is no more repeated in the whole Extraction, but the work in the fifth step is to be renewed for the finding out of every following Character in the Root.

NOTE 2.

The number signified by a is to be found out by Note 2. in *Sect.* 1. of this *Chapt.* and then the *Divisor* for the finding of the unknown single Character represented by e is consequently known: For in the Resolution of every Power produced from the Binomial Root $a + e$, the *Divisor* consists of such Powers of a as are multiplied into the Powers of e; and because the Cube of $a + e$ is $aaa + 3aae + 3aee + eee$, therefore in the extraction of the cubick Root, the *Divisor* is compos'd of $3aa$ and $3a$; so that when the number a is known, the *Divisor* $3aa + 3a$ is consequently known.

NOTE

NOTE 3.

When the *Divisor* is found out by the precedent Note 2. as also the *Ablatitium*, which in the extraction of the cubick Root is compos'd of $3aae$, $3aee$ and eee; the numbers signified by the said $3aae$, $3aee$ and eee must each of them be set in such order under the particular or present *Resolvend*, that Units may stand under Units, Tens under Tens, &c. to the end the *Ablatitium* may be rightly composed and subtracted from the *Resolvend*.

NOTE 4.

When the *Divisor* is not contained once in the particular or present *Resolvend*, a cypher (to wit, 0,) must be set in the Quotient; and then the *Resolvend* must be augmented with the next member (towards the right hand) of the Power proposed, for a new particular *Resolvend*: Also, a new *Divisor* must be found out by Note 2. of this *Sect.* and the like is to be done as often as the *Divisor* is less than the *Resolvend*.

The practice of these Notes will be shewn in the following Example.

Example 2.

6. If the Cubick Root of 23156436019527 be desired, it will be found 28503 by the precedent Rules; and the work will stand as here you see underneath.

	23	156	436	019	527	(28503. *Root.*
Subtract	8					*aaa*
	15	156				*Resolvend.*
$a = 20$	1	200				$3aa$
		60				$3a$
	1	260				*Divisor.*
$e = 8$	9	600				$3aae$
	3	840				$3aee$
		512				eee
Subtract	13	952				*Ablatitium.*
	1	204	436			*Resolvend.*
$a = 280$		235	200			$3aa$
			840			$3a$
		236	040			*Divisor.*
$e = 5$	1	176	000			$3aae$
		21	000			$3aee$
			125			eee
Subtract	1	197	125			*Ablatitium.*
	0	007	311	019		*Resolvend.*
$a = 2850$		24	367	500		$3aa$
$e = 0$			8	550		$3a$
		24	376	050		*Divisor.*
		7	311	019	527	*Resolvend.*
$a = 28500$		2	436	750	000	$3aa$
				85	500	$3a$
		2	436	835	500	*Divisor.*
$e = 3$		7	310	250	000	$3aae$
				769	500	$3aee$
					27	eee
Subtract		7	311	019	527	*Ablatitium.*
		0	000	000	000	

Explication of Example 2.

The first figure of the Root is 2, (by Note 1.) whose Cube 8 subtracted from 23 the first membet of the number propos'd leaves 15, to which the second member 156 being

annexed, there ariſeth 15156 for the next *Reſolvend:* Or, to cauſe the ſame effect, ſuppoſe 0 to be annexed to 2 the firſt figure of the Root and it makes 20, (that is, *a*,) whoſe Cube 8000 (or *aaa*) ſubtracted from 23156 the two formoſt members of the number firſt propos'd, leaves (as before) the *Reſolvend* 15156.

Then, the firſt figure of the Root being found 2, the value of *a* is 20, and the *Diviſor* is 1260, (by Note 2.) and then by dividing and making tryal as is directed in the foregoing fifth ſtep, the number *e* will be found 8 for the ſecond figure of the Root, and conſequently the numbers ſignified by 3*aae*, 3*aee* and *eee*, are 9600, 3840 and 512; theſe being ſet orderly and added together (according to Note 3.) make the *Ablatitium* 13952, which ſubtracted from the *Reſolvend* 15156 leaves 1204, to which annexing 436 the third member of the number firſt propoſed, it makes 1204436 for a new *Reſolvend*. The reſt of the Operation in Example 2. being but a repetition of what hath been directed for finding out the ſecond figure of the Root, I ſhall leave it to the Learners practice.

The precedent Rules and Notes in this *Sect.* 2. for extracting the cubick Root of a whole number having an exact cubick Root are expreſt at large, that the Reaſon of the work might be apparent; but this method may be contracted into more practical and compendious Rules, as I have ſhewn in the 33. *Chapt.* of Mr. *Wingate*'s common Arithmetick.

7. But when a whole number hath not a cubick Root exactly expreſſible by any rational or true number, then to approach infinitely near the exact Root, firſt, ternaries of cyphers, *viz* three, or ſix, or nine, or twelve, &c. cyphers are to be annexed to the whole number given; then eſteeming the number given with the cyphers annexed to be one whole number, let its cubick Root be extracted by the precedent (or other practical) Rules: that done, look how many points were ſet over the number firſt given, for ſo many of the formoſt places in the Quotient are to be taken for the Integers in the Root, and the reſt following thoſe Integers expreſs the fractional part of the Root in decimal parts: As, for example, if the cubick Root of 8302348 be deſired, I annex ſix cyphers to 8302348, thus, 8302348.000000, and then the cubick Root of 8302348.000000 being extracted, it will be found 202.48, that is, $202\frac{48}{100}$; but becauſe after the extraction is finiſh'd there happens to be a Remainder, I conclude that $202\frac{48}{100}$ is leſs than the true cubick Root ſought, but $202\frac{49}{100}$ is greater than it; ſo that by annexing ſix cyphers you will not miſs $\frac{1}{100}$ part of an Unit of the true Root, and by annexing nine cyphers you will not want $\frac{1}{1000}$ part; and in that order you may approach as near as you pleaſe when you cannot obtain the exact cubick Root of a whole number given.

8. The cubick Root of a vulgar Fraction is found out thus, *viz.* Firſt, if the Fraction be not in its leaſt terms, let it be reduced to the leaſt terms; then extract the cubick Root of the Numerator for a new Numerator, and the cubick Root of the Denominator for a new Denominator, ſo ſhall this new Fraction be the cubick Root of the Fraction propoſed; as, for example, the cubick Root of $\frac{8}{27}$ is $\frac{2}{3}$, and the cubick Root of $\frac{1}{8}$ is $\frac{1}{2}$.

9. But when either the Numerator or Denominator of a vulgar Fraction hath not a perfect cubick Root, then to find the cubick Root of that Fraction very near, firſt reduce the Fraction to a decimal Fraction whoſe Numerator may conſiſt of ternaries of places, *viz.* either of three, ſix, nine, or twelve, &c. places, and then extract the cubick Root of that decimal as if it were a whole number, and the Root that comes forth ſhall be a decimal Fraction expreſſing nearly the cubick Root of the vulgar Fraction propoſed: As, for example, if the cubick Root of $\frac{2}{3}$ be deſired, I firſt reduce it to this decimal Fraction, .666666666666, and then by extracting the cubick Root of the ſaid decimal as if it were a whole number, I find .8735, that is, $\frac{8735}{10000}$; which is near the cubick Root of $\frac{2}{3}$, for it wants not $\frac{1}{10000}$ part of an Unit of the exact cubick Root of $\frac{2}{3}$.

10. Laſtly, if the cubick Root of a mixt number, that is, of a whole number with a Fraction in its leaſt terms, be deſired; firſt reduce it to an improper Fraction, and then extract the cubick Root of that improper Fraction in like manner as before in the eighth ſtep; but if it hath not an exact cubick Root, then reduce the fractional part of the mixt number firſt propoſed to a decimal Fraction whoſe Numerator may conſiſt of ternaries of places, and after this decimal is annexed to the Integers of the mixt number, extract the cubick Root out of the whole, then ſo many points as were ſet over the Integers, ſo many of the formoſt places in the Quotient are to be taken for the Integers in the Root, and the reſt expreſs the fractional part of the Root in decimal parts: As, for example, the

cubick

cubick Root of $12\frac{19}{27}$, that is, of $\frac{343}{27}$, will be found $\frac{7}{3}$ or $2\frac{1}{3}$; and the cubick Root of $2\frac{3}{8}$, that is, of 3.375000000, *&c.* will be found 1.334, *&c.* that is, $1\frac{334}{1000}$, *&c.*

Sect. III. *Of the extraction of the Biquadratick Root out of a number given.*

1. The briefest way to extract the Root of a Biquadratick number, that is, of a number produced by the multiplication of some number or Root four times into it self, is first to extract the square Root of the number proposed, and then to extract the square Root of that Root; as, for example, if the Root of the Biquadratick number, or fourth Power 256 be desired; First, the square Root of 256 being extracted is 16; and then the square Root of 16 is 4, which is the Root of the fourth Power 256: for $4 \times 4 \times 4 \times 4$ produceth 256. But my purpose being to explain the general Method for the extracting of all kinds of Roots, I shall upon that Foundation shew how to extract the Root of a Biquadratick number.

2. For the more ready extraction of the biquadratick Root, the following Tabulet will be useful, which shews at first sight the Root of any Biquadratick whole number under 10000.

Roots.	1	2	3	4	5	6	7	8	9
Fourth Powers.	1	16	81	256	625	1296	2401	4096	6561

3. When a whole number is proposed, and it is desired to extract the Biquadratick Root of that number, set points over the given number in this manner, *viz.* first set a point over the Units place, then passing over the three next places towards the left hand set another point over the fifth place, and in that order as many points are to be set as the given number will admit, that there may be three places between every two adjacent points. So if the biquadratick Root of 614656 be desired, after points are set as is above directed, the said 614656 will be distributed into two members, to wit, 61 and 4656: in like manner this number 6597500625 being pointed in the same order will be distributed into these three members, 65, 9750, and 0625. The points shew the number of places that will be found in the Root, as also what member of the number propos'd belongs to the extraction of every single Character of the Root sought.

$$\dot{6}1465\dot{6}$$
$$6\dot{5}975\dot{0}062\dot{5}$$

4. The given number whose Biquadratick Root is desired may be conceived to be produced from the multiplication of the Binomial Root $a+e$ four times into it self, and then the said number will be composed of these five members or numbers, *viz.* $aaaa$, $4aaae$, $6aaee$, $4aeee$, $eeee$, (as is manifest by the fourth Power in the Table in *Sect.* 4. *Chapt.* 1. of this Book.) Now because the Resolution of a Biquadratick number, *viz.* the extraction of the Biquadratick Root is deducible from the steps of the Composition of a Biquadratick number from its Root, (for such numbers as are added in the Composition are to be subtracted in the Resolution,) respect must be had to *Sect.* 3. *Chap.* 2. of this Book.

Example.

5. Let it be required to extract the Biquadratick Root of 614656. After the number given is prepared by punctations as before is directed, I seek in the Tabulet in the precedent second step of this *Sect.* 3. for the greatest Biquadratick whole number contained in 61 the first member (towards the left hand) of the number proposed, and finding it to be 16, I subscribe 16 under 61, and write 2 the Root of the said fourth Power 16 in the Quotient, for the first figure of the Root sought; then after a line is drawn under 16, I subtract 16 from 61, or 160000 from 614656, and there remains to be resolved 454656.

$$\begin{array}{r|l|l} 61 & 4656 & (2 \\ 16 & & \\ \hline 45 & 4656 & \end{array}$$

The *Divisor* for the finding out of e, that is, every Character which is to follow the first figure of the Root, is always in the extraction of the Biquadratick Root composed of these numbers, *viz.* $4aaa$, $6aa$, and $4a$, for these are all the Powers of a that are drawn into the Powers of e in the fourth Power of $a + e$; (as is evident by the Table in *Sect.* 4. *Chap.* 1.) and because the first figure of the Root is found 2, and consequently (by Note 2. in *Sect.* 1. of this *Chapt.*) the number signified by a is 20, therefore the summ of the numbers signified by $4aaa$, $6aa$ and $4a$ is 34480, which is the *Divisor*; then supposing I were to divide the *Resolvend* 454656 by the *Divisor* 34480, I find the Quotient exceeds 9, but in regard e alwayes represents either a single figure or a cypher it cannot exceed 9; and therefore I make trial (in a waste paper) with 9, to see whether it will constitute an *Ablatitium* that doth not exceed the *Resolvend* 454656, *viz.* I suppose $e = 9$; then because a was before found 20, the *Ablatitium* which in the extraction of the Biquadratick Root is alwayes composed of $4aaae$, $6aaee$, $4aeee$ and $eeee$, will exceed the *Resolvend*, from which it ought to be subtracted: But if $e = 8$, then the *Ablatitium* will be equal to the *Resolvend*, and consequently that being subtracted from this, there will remain 0, wherefore I set 8 in the Quotient, and conclude that the Biquadratick Root of the given number 614656 is 28; for $28 \times 28 \times 28 \times 28$ produceth 614656.

	61	4656	(28. *Root.*
Subtract	16		$aaaa$
	45	4656	*Resolvend.*
$a = 20$	3	2000	$4aaa$
		2400	$6aa$
		80	$4a$
	3	4480	*Divisor.*
$e = 8$	25	6000	$4aaae$
	15	3600	$6aaee$
	4	0960	$4aeee$
		4096	$eeee$
Subtract	45	4656	*Ablatitium.*
	0	0000	

Sect. IV. *Of the extraction of the Root of the fifth Power given in number.*

1. For the more ready extraction of the Root of any fifth Power given in number, this Tabulet will be useful, which shews at first sight the fifth Powers of every single figure, and consequently any fifth Power in number under 100000 being given, its Root is hereby discovered.

Roots.	5th *Powers.*
1	1
2	32
3	243
4	1024
5	3125
6	7776
7	16807
8	32768
9	59049

2. When a whole number is given for a fifth Power and its Root desired, that is, such a number which being multiplied five times into it self will produce the given number, it must be prepared for extraction by punctations in this manner; *viz.* First let a point be set over the Units place of the given number, then passing over the four next places towards the left hand, set another point over the sixth place; and in that order as many points are to be set as the given number will admit, that there may be four places between every two adjacent points. So if the Root of the fifth Power 17210368 be desired, after points are set as is above directed, the said 17210368 will be distributed into two members, to wit, 172 and 10368: in like manner this number 1880287678125 will be distributed into these three members, 188, 02876 and 78125.

$\dot{1}7\dot{2}1036\dot{8}$

$\dot{1}88\dot{0}287\dot{6}7812\dot{5}$

The points (as before hath been said) shew the number of places that will be found in the Root, as also what member of the number given belongs to the extraction of every single character of the Root sought.

3. Every

3. Every number confidered as a fifth Power may be conceived to be produced from the multiplication of the Binomial Root $a+e$ five times into it felf, and then the faid number will be compofed of thefe fix members or numbers, *viz* *aaaaa*, 5*aaaae*, 10*aaaee*, 10*aaeee*, 5*aeeee* and *eeeee*; (as is manifeft by the fifth Power in the Table in *Sect.* 4. *Chap.* 1. of this Book.) Now becaufe the Refolution of the fifth Power, *viz* the extraction of $\surd(5)$ out of a given number, is deducible from the fteps of the Compofition of a fifth Power from its Root given in number; (for fuch numbers as are added in the Compofition are to be fubtracted in the Refolution,) the Learner muft be exercis'd in *Sect.* 4. *Chap.* 2. of this Book.

Example.

Let it be required to extract $\surd(5)$ out of 17210368, *viz.* to find a Root or number which being multiplied five times into it felf will produce 17210368: After the given number is prepared by punctations as before is directed, I feek in the Tabulet in the firft ftep of this Section 4. for the greateft fifth Power contained in 172 the firft member (towards the left hand) of the given number, and finding it to be 32, I fubfcribe 32 under 172, and write 2 the Root of the faid fifth Power 32 in the Quotient for the firft figure of the Root fought; then after having drawn a line under 32, I fubtract 32 from 172, or, 3200000 from 17210368, and there remains to be refolved 14010368.

172	10368	(2
32		
140	10368	

Then to difcover the *Divifor*, which fhews how to begin the tryal in the finding out of *e*, that is, every Character (whether it be a figure or cypher) which is to follow the firft figure of the Root, I take fuch Powers of *a* as are multiplied into the Powers of *e* in the fifth Power produced from $a+e$, viz. 5*aaaa*, 10*aaa*, 10*aa* and 5*a*, fo the fumm of thefe four numbers make the *Divifor*: and becaufe the firft figure of the Root is found 2, and confequently (by Note 2. in *Sect.* 1. of this *Chapt.*) the number fignified by *a* is 20, therefore the fumm of the numbers fignified by 5*aaaa*, 10*aaa*, 10*aa* and 5*a* is 884100, which is the *Divifor*; then fuppofing I were to divide the *Refolvend* 14010368 by the *Divifor* 884100, I find the Quotient exceeds 9, but in regard *e* alwayes reprefents a fingle figure or a cypher it cannot exceed 9; therefore I make tryal (in a void place) with 9, to fee whether it will conftitute an *Ablatitium* that doth not exceed the *Refolvend* 14010368, *viz.* I fuppofe $e=9$, then becaufe *a* was found 20, the *Ablatitium* 5*aaaae* + 10*aaaee* + 10*aaeee* + 5*aeeee* exceeds the *Refolvend* from which it ought to be fubtracted; But if $e=8$, then the *Ablatitium* will be equal to the *Refolvend*, and confequently that being fubtracted from this, there will remain 0, wherefore I fet 8 in the Quotient; fo 28 is found to be the $\surd(5)$ of the given number 17210368; for $28 \times 28 \times 28 \times 28 \times 28$ produceth 17210368. Compare the following work with the precedent Rules of *Sect.* 4.

	172	10368	(28. *Root.*
	32	00000	*aaaaa*
	140	10368	*Refolvend.*
$a=20$	8	00000	5*aaaa*
		80000	10*aaa*
		4000	10*aa*
		100	5*a*
	8	84100	*Divifor.*
$e=8$	64	00000	5*aaaae*
	51	20000	10*aaaee*
	20	48000	10*aaeee*
	4	09600	5*aeeee*
		32768	*eeeee*
	140	10368	*Ablatitium.*
	000	00000	

By the precedent Rules and Examples of this *Chapt.* the ingenious Reader will eafily perceive how to extend this general method to the extraction of the Roots of all kinds of

of Powers in numbers, *viz.* of the ſixth, ſeventh, eighth, *&c.* Powers; as alſo to find out the Roots infinitely near of ſuch Powers as have not Roots exactly expreſſible by any rational or true number.

CHAP. IV.

Concerning the extraction of Roots out of Powers expreſt by Letters.

I. IN a ſeries or Scale of Powers produced from a Root, ſuppoſe from a, as in this ſeries, a, aa, aaa, $aaaa$, $aaaaa$, a^6, a^7, a^8, &c. thoſe Powers only whoſe Indices are even numbers are Squares; as aa, $aaaa$, a^6, a^8, &c. (whoſe Indices are 2, 4, 6, 8, *&c.*) are Squares: and thoſe Powers only whoſe Indices are diviſible by 3, are Cubes, as aaa, $aaaaaa$, a^9, &c. (whoſe Indices are 3, 6, 9, *&c.*) are Cubes. Therefore every Power whoſe Index is a Prime number greater than 3, as $aaaaa$, a^7, a^{11}, &c.) (whoſe Indices are 5, 7, 11, *&c.*) is neither a Square nor a Cube. But every Power whoſe Index is diviſible by 6, as a^6, a^{12}, a^{18}, &c. is both a Square and a Cube, becauſe the Index is diviſible both by 2 and by 3.

II. If a Simple quantity be expreſt by the ſame letter repeated an even number of times, the ſquare Root thereof is eaſily extracted; for the Root muſt be ſuch that its Index may be the half of the Index of the Quantity propoſed: As, $\sqrt{aa}$, (that is, the ſquare Root of aa,) is a; for 1, the Index of the Root a, is the half of 2 the Index of the Square aa: in like manner $\sqrt{aaaa}$ is aa, whoſe Index 2 is the half of 4 the Index of the Square $aaaa$: again, $\sqrt{aaaaaa}$ is aaa, whoſe Index 3 is the half of 6 the Index of the Square a^6.

III. And with the like facility you may extract the Cubick Root of a Simple quantity which is expreſt by one and the ſame letter repeated ſuch a number of times as is diviſible by 3; for the Cubick Root muſt be ſuch that its Index may be $\frac{1}{3}$ of the Index of the Cube propoſed: as, $\sqrt{(3)}aaa$, (that is, the cubick Root of the Quantity aaa,) is a, whoſe Index 1 is $\frac{1}{3}$ of 3 the Index of aaa: in like manner $\sqrt{(3)}a^6$ is aa, whoſe Index 2 is $\frac{1}{3}$ of 6 the Index of the Cube a^6.

IV. If the Index of a ſimple Power expreſt by the ſame letter be ſome Prime number greater than 3, as 5, 7, 11, *&c.* then neither $\sqrt{(2)}$, nor $\sqrt{(3)}$, nor any other Root except that denoted by ſuch Index or Prime number can be exactly extracted out of the ſaid Power: ſo no Root can be exactly extracted out of $aaaaa$ or a^5, but $\sqrt{(5)}$, which is a; nor any Root out of a^7 but $\sqrt{(7)}$ which is alſo a. But when the Root cannot be exactly extracted, the ſign of the Root is to be prefixt to the Quantity; as to expreſs the ſquare Root of $aaaaa$ or a^5, I write $\sqrt{aaaaa}$ or $\sqrt{a^5}$: likewiſe I expreſs the cubick Root of a^5, thus, $\sqrt{(3)}a^5$; and $\sqrt{(4)}$ of a^7, thus, $\sqrt{(4)}a^7$; and ſo of others.

V. When ſome Power of an unknown ſimple Root a is found equal to ſome known number, and the Index of that unknown Power is not a Prime number, then the value of the Root a in number may oftentimes be diſcovered by two or more extractions more eaſily than by one ſingle extraction of a Root out of the ſaid unknown number. As, for Example;

If there be propoſed or found out . . . . $aaaaaa = 729$

Then to find out the value of a, you need not extract the $\sqrt{(6)}$ of 729 by the general method before delivered in *Chap.* 3. but firſt by that method extract the ſquare Root of 729, and then by *Sect.* 2. of this *Chapt.* the ſquare Root of $aaaaaa$, ſo thoſe two Roots compared give this Equation, *viz.* . . . . $aaa = 27$

Laſtly, by extracting the cubick Root of each part of the laſt Equation, the value of a the Root ſought is diſcovered, *viz.* . . . . $a = 3$

Or

Or thus,

First, by extracting the cubick Root of each part of the Equation proposed, there ariseth . . . . $aa = 9$

And then by extracting the square Root of each part of the last Equation, the same value of the Root a is found out as before, to wit, . $a = 3$

In like manner, if . . . . $a^9 = 19683$

First, by extracting the cubick Root, it gives . . . . $a^3 = 27$

And again, by extracting the Cubick root of that Root the Root a is made known, *viz.* . . . . $a = 3$

VI. When two or more Squares, Cubes, or other Powers exprest by different letters be multiplied one into another, then if the Root of each Power, *viz.* the square Root if they be Squares, or the cubick Root if they be Cubes, &c. be extracted, the Product made by the multiplication of these Roots one into another shall be a like Root of the Power or Product first given: As, for example, $\sqrt{aabb}$ is ab, which is the Product of the square Roots of aa and bb; likewise, $\sqrt{(3)}aaabbb$ is ab, which is the Product of the cubick Roots of aaa and bbb.

Again, $\sqrt{aabbcc}$ is abc, which is the Product of the square Roots of aa, bb and cc; in like manner, $\sqrt{(3)}27aaabbb$ is $3ab$, which is the Product of the cubick Roots of 27, aaa and bbb; and $\sqrt{16aabbcc}$ is $4abc$, which is the Product of the square Roots of 16, aa, bb and cc. The like is to be understood of others.

But if the square Root of $5aabb$ be desired, because 5 is not a Square, the said Root is to be exprest either thus, $\sqrt{5aabb}$; or thus, $\sqrt{5} \times ab$; or thus, $ab\sqrt{5}$. In like manner, to denote the square Root of $aaabbb$ I write $\sqrt{a^3b^3}$: and to signifie the cubick Root of $aabb$, I write $\sqrt{(3)}aabb$; but the cubick Root of $3aaabbb$ may be written either thus, $\sqrt{(3)}3a^3b^3$; or thus, $\sqrt{(3)}3 \times ab$; or thus, $ab\sqrt{(3)}3$.

Concerning the extraction of Roots out of Compound quantities exprest by Letters.

VII. Before the Learner enters upon the Extraction of Roots out of Compound Squares, Cubes or other Powers exprest by letters, he ought to be well exercis'd in the eighth and ninth Chapters of my first Book of *Algebraical Elements*; as also in the foregoing first, second and third Chapters of this Book, and in the precedent Rules of this Chapter; all which well understood will render the following Rules and Examples of this Chapter very plain and easie.

VIII. *Rules for the extraction of Square Roots out of Compound Quantities exprest by Letters.*

Rule 1. Set the particular members of the compound Algebraick quantity whose square Root is required, in such order, that one of the simple Squares may stand outermost towards the left hand, and next after the same such other member or members wherein you find the same letter or letters as are in the said simple Square; then the square Root of the said simple Square is to be set in the Quotient for the first member of the compound Root sought, and the Square it self is the first Quantity to be subtracted from the compound Quantity proposed. This is the first work, which is no more to be repeated in the whole Extraction.

Rule 2. Double the Root before set in the Quotient for the first *Divisor*; likewise, to find every following *Divisor*, double every simple Quantity that stands in the Quotient, and take the summ of the Products for the *Divisor*.

Rule 3. When the *Divisor* is found out, divide only the first simple Quantity (towards the left hand) in the *Resolvend*, by the first simple Quantity in the *Divisor*, and set that which comes forth next after the member or members of the Root sought that was before found out.

Rule 4. After the first simple Square is subtracted (according to Rule 1.) then every following *Ablatitium*, that is, the summ of the Quantities to be subtracted from the respective *Resolvend*, must be composed of these two Products, *viz.* the Product made by the multiplication of the whole *Divisor* by that particular Quantity which was last set in the Quotient, and the Square of the same simple Quantity.

The practice of these Rules will be apparent in the following Examples.

Exam-

Example 1.

Let it be required to extract the ſquare Root of $aa + 2ab + bb$.

Firſt, I extract the ſquare Root of aa, and it is a, which I ſet in the Quotient; then multiplying a by it ſelf, I ſet the Product aa under, and ſubtract it from the quantity firſt propoſed, and there remains $2ab + bb$. This is the firſt work which anſwers to Rule 1. and is no more to be repeated.

The Square,	$aa + 2ab + bb$	$(a + b.$ The Root.
Subtract	aa	
Remainder,	$+ 2ab + bb$	
Diviſor,	$+ 2a)$	
Subtract	$+ 2ab + bb$	
Remainder,	0 0	

Secondly, the Diviſor (according to Rule 2.) is $2a$, which I ſet under $2ab$.

Thirdly, I divide $+ 2ab$ by the Diviſor $+ 2a$, and the Quotient is $+ b$, which I ſet next after a, (the particular Root before found out,) acccording to Rule 3.

Fourthly, I multiply the Diviſor $+ 2a$ by $+ b$, (that was laſt ſet in the Quotient,) and the Product is $+ 2ab$, to which adding $+ bb$, (the Square of $+ b$,) the ſumm is $+ 2ab + bb$, which (according to Rule 4.) I ſet under and ſubtract from the Reſolvend $+ 2ab + bb$, and there remains 0: ſo the Extraction being finiſh'd, the Root ſought is found $a + b$; for if it be multiplied by it ſelf it produceth $aa + 2ab + bb$ the quantity firſt propoſed.

Note. By what I have ſaid in the eighth and ninth Chapters of my firſt Book of *Algebraical Elements*, 'tis eaſie to diſcover at firſt ſight whether a Compound Algebraick Quantity conſiſting of three Terms be a perfect Square or not, and if a Square what its Root is. Nevertheleſs, in this firſt Example I have expreſt the work at large according to the four Rules before given, that the like Operation may the more eaſily be perceived in the following Examples.

Example 2.

If the ſquare Root of $aa - 2ab + 2ac - 2bc + bb + cc$ be deſired, it will be found $a - b + c$, by the precedent Rules, and the work ſtands as here you ſee underneath.

The Square,	$aa - 2ab + 2ac - 2bc + bb + cc$	$(a - b + c.$ The Root
Subtract	aa	
Remainder,	$- 2ab + 2ac - 2bc + bb + cc$	
Diviſor,	$+ 2a)$	
Subtract	$- 2ab + bb$	
Remainder,	$+ 2ac - 2bc + cc$	
Diviſor,	$+ 2a - 2b)$	
Subtract	$+ 2ac - 2bc + cc$	
Remainder	0 0 0	

Example 3.

In like manner the ſquare Root of $64aabb + 32abc - 144ab + 4cc - 36c + 81$ will be found $8ab + 2c - 9$; as is manifeſt by the following Operation.

The Square,	$64aabb + 32abc - 144ab + 4cc - 36c + 81$	$(8ab + 2c - 9$
Subtract	$64aabb$	
Remainder,	$+ 32abc - 144ab + 4cc - 36c + 81$	
Diviſor,	$+ 16ab)$	
Subtract	$+ 32abc \quad + 4cc$	
Remainder,	$- 144ab \quad - 36c + 81$	
Diviſor,	$+ 16ab \quad + 4c)$	
Subtract	$- 144ab \quad - 36c + 81$	
Remainder,	0 0 0	

Example 4.

Example 4.

Again, the ſquare Root of $dddd + 2dddb + 3ddbb + 2dbbb + bbbb$ will be found $dd + db + bb$, and the Extraction ſtands thus;

The Square,	$d^4 + 2d^3b + 3d^2b^2 + 2db^3 + b^4$	($dd + db + bb.$
Subtract	d^4	
Remainder,	$+ 2d^3b + 3d^2b^2 + 2db^3 + b^4$	
Diviſor,	$+ 2d^2$)	
Subtract	$+ 2d^3b + d^2b^2$	
Remainder,	$+ 2d^2b^2 + 2db^3 + b^4$	
Diviſor,	$+ 2d^2 + 2db$)	
Subtract	$+ 2d^2b^2 + 2db^3 + b^4$	
Remainder,	0 0 0	

IX. *Rules for the extraction of Cubick Roots out of Compound Quantities expreſt by Letters.*

Rule 1. Set the particular members or parts of the Compound Algebraick Quantity whoſe cubick Root is required, in ſuch order, that one of the ſimple Cubes may ſtand outermoſt towards the left hand, and next after the ſame ſuch other members wherein you find the ſame letter or letters as are in the ſaid ſimple Cube; then the cubick Root of the ſaid ſimple Cube is to be ſet in the Quotient for the firſt member of the Root ſought, and the ſimple Cube it ſelf is the firſt Quantity to be ſubtracted from the compound Quantity propoſed. This is the firſt work, and no more to be repeated in the whole Extraction.

Rule 2. The firſt Diviſor muſt be compoſed of the triple of the Square of the Root before ſet in the Quotient, (which triple Square I call the firſt part of the Diviſor,) and the triple of the ſame Root, (which triple Root I call the latter part of the Diviſor;) likewiſe, every following Diviſor muſt be compoſed of the triple of the Square of the ſumm of all the ſimple Quantities or parts of the Root already found out and ſet in the Quotient, and of the triple of the ſame ſumm.

Rule 3. When the Diviſor is found out, divide only the firſt ſimple Quantity (towards the left hand) in the *Reſolvend*, by the firſt ſimple Quantity in the Diviſor, and ſet that which comes forth in the Quotient, next after the member or members of the Root ſought before found out.

Rule 4. After the firſt ſimple Cube is ſubtracted, (according to Rule 1.) then every following *Ablatitium*, that is, the ſumm of the quantities to be ſubtracted from the *Reſolvend*, muſt be compoſed of theſe three Products, *viz.* Firſt, the Product made by the multiplication of the firſt part of the Diviſor, (to wit, the triple Square mentioned in Rule 2.) by the ſimple Quantity laſt ſet in the Quotient; ſecondly, the Product made by the multiplication of the latter part of the Diviſor (to wit, the triple Root or ſumm mentioned in Rule 2.) by the Square of the ſame ſimple Quantity; and thirdly, the Cube of the ſaid ſimple Quantity laſt ſet in the Quotient.

The practice of theſe Rules will appear in the following Examples.

Example 1.

Let it be required to extract the Cubick Root out of $aaa + 3aae + 3aee + eee$.

Firſt, beginning at the left hand, I extract the cubick Root of aaa, and it is a, which I ſet in the Quotient, then multiplying the ſaid Root a cubically it makes aaa, which I ſubtract from the Compound quantity firſt propoſed for Extraction, and there remains to be reſolved $+ 3aae + 3aee + eee$. This is the firſt work, which anſwers to Rule 1. and is no more to be repeated in the whole Extraction.

The Cube,	$aaa + 3aae + 3aee + eee$	($a + e$. The Root.
Subtract	aaa	
Remainder,	$+ 3aae + 3aee + eee$	
Diviſor,	$+ 3aa + 3a$)	
Subtract	$+ 3aae + 3aee + eee$	
Remainder,	0 0 0	

Secondly, I ſeek a Diviſor thus, *viz.* to $+3aa$, which is the triple of aa the Square of the Root a, I add $+3a$ the triple of the ſaid Root a, and the ſumm $3aa+3a$ is the Diviſor, which I ſet underneath the remaining *Reſolvend*, according to Rule 2.

Thirdly, according to Rule 3. I divide $+3aae$ by $+3aa$, and it gives $+e$, which I ſet in the Quotient next after a.

Fourthly, to find out the *Ablatitium* (or quantity next to be ſubtracted) I make a threefold Multiplication, *viz.* Firſt, I multiply $+3aa$ (the firſt part of the Diviſor) by $+e$ the Root laſt ſet in the Quotient, and the Product is $+3aae$; ſecondly, I multiply $+3a$ the latter part of the Diviſor by $+ee$ the Square of the ſaid Root e, and the Product is $+3aee$; thirdly, I multiply the ſaid Root e cubically, and the Product is eee; laſtly, I ſubtract the ſumm of the ſaid three Products from the *Reſolvend*, and there remains 0. So the Extraction is finiſh'd, and $a+e$ is the true Cubick Root ſought; for if it be multiplied cubically, it will produce $aaa+3aae+3aee+eee$ firſt propoſed.

Example 2.

In like manner, the cubick Root extracted out of $125aaa+225aae+135aee+27eee$ is $5a+3e$, and the work ſtands thus:

The Cube,	$125aaa+225aae+135aee+27eee$	$(5a+3e.$ Root.
Subtract	$125aaa$	
Remainder,	$+225aae+135aee+27eee$	
Diviſor,	$+75aa+15a)$	
Subtract	$+225aae+135aee+27eee$	
Remainder,	$0 \quad 0 \quad 0$	

Example 3.

So the cubick Root of $27a^6-54a^5+171a^4-188a^3+285aa-150a+125$ will be found $3aa-2a+5$, and the Operation ſtands thus:

		The Root.
Cube,	$27a^6-54a^5+171a^4-188a^3+285aa-150a+125$	$(3aa-2a+5.$
Subtr.	$27a^6$	
Rem.	$-54a^5+171a^4-188a^3+285aa-150a+125$	
Diviſor,	$+27a^4+9a^2)$	
Subtr.	$-54a^5+36a^4-8a^3$	
Rem.	$+135a^4-180a^3+285aa-150a+125$	
Diviſor,	$\left\{\begin{array}{l} +27a^4-36a^3+12aa \\ \qquad\qquad\quad +9aa-6a \end{array}\right.$	
Add theſe	$\left\{\begin{array}{l} +135a^4-180a^3+60aa \\ \qquad\qquad\quad +225aa-150a \\ \qquad\qquad\qquad\qquad +125 \end{array}\right.$	
Subtract	$+135a^4-180a^3+285aa-150a+125$	
	$0 \quad 0 \quad 0 \quad 0 \quad 0$	

If there be occaſion to extract the Root of the fourth, fifth, or other higher Compound Power, the Diviſors and Ablatitious quantities may be drawn out of the Table in *Sect.* 4. *Chap* 1. of this Book.

X. *Concerning the extraction of Roots out of Algebraical Fractions.*

1. Foraſmuch as in the extraction of Roots out of Fractions, the Root of the Numerator and Denominator being ſeverally extracted gives the Root ſought; therefore if the ſquare Root of $\frac{aabb}{cc}$ be to be extracted, I write $\frac{ab}{c}$ for the Root ſought; for the ſquare Root of the Numerator $aabb$ is ab, and the ſquare Root of the Denominator cc is c.

In

In like manner if the square Root of $\frac{aaaa - 2aabb + bbbb}{aa + 4ab + 4bb}$ be desired; by extracting the square Root out of the Numerator and Denominator, there ariseth $\frac{aa - bb}{a + 2b}$ for the Root sought.

And for the same reason the cubick Root of this Fraction, ... $\frac{27a^6 - 54a^5 + 171a^4 - 188a^3 + 285aa - 150a + 125}{aaa - 9aa + 27a - 27}$ will be $\frac{3aa - 2a + 5}{a - 3}$; which is found by extracting the cubick Root out of the Numerator and Denominator of the Fraction proposed.

2. But if the Root sought cannot be extracted out of the Numerator and Denominator, then the radical sign $\surd$ with the Index of the Power, if it exceed a Square, is to be prefixt to the Fraction; as, to denote the square Root of $\frac{ccxx}{4bb} - ac$, that is, of $\frac{ccxx - 4abbc}{4bb}$, I write $\surd\frac{ccxx - 4abbc}{4bb}$, or, (because the square Root of the Denominator is $2b$,) the square Root of the quantity proposed may be exprest thus, $\frac{\surd\overline{ccxx - 4abbc}}{2b}$; likewise, the cubick Root of $\frac{a^3b^3}{aa + bb}$ may be designed either thus, $\surd(3)\frac{a^3b^3}{aa + bb}$, or (because the Numerator is a Cube) thus, $\frac{ab}{\surd(3)\overline{aa + bb}}$. The like is to be understood in expressing the irrational Roots of higher Powers.

Chap. V.

Concerning Geometrical Proportion.

I. THE *Difference* of two numbers is found out by Subtraction, but the *Ratio*, *Reason* or *Habitude* of one number to another is discovered by dividing the Antecedent (or first number) by the Consequent, (or second number;) for the the Quotient denominates the *Ratio*, Reason, or (as some call it) the Proportion, which the Antecedent hath to the Consequent: As if 6 be compared to 2, then $\frac{6}{2}$, that is $\frac{3}{1}$, or 3, shews that 6 hath triple Reason to 2; *viz.* 6 contains 2 thrice, or 6 is in proportion to 2 as 3 to 1: but if 2 be compared to 6, then $\frac{2}{6}$ or $\frac{1}{3}$ shews that 2 hath subtriple Reason to 6; *viz.* 2 is $\frac{1}{3}$ part of 6, or 2 is in proportion to 6 as 1 to 3. In like manner if the quantity a be compared to the quantity b, then $\frac{a}{b}$ expresseth the *Ratio* or Reason of a to b; and $\frac{b}{a}$ shews the Reason of b to a.

Note, that the Reason of two numbers or quantities ought to be exprest by the smallest Terms or Quantities than can possibly be found to express that Reason: So the Denominator of the Reason of 16 to 12 is $\frac{4}{3}$, where 16 and 12 are first reduced to the smallest Terms 4 and 3, (by dividing the said 16 and 12 severally by their greatest common Divisor 4,) and then dividing the Antecedent 4 by the Consequent 3; the Quotient $\frac{4}{3}$ expresseth the Reason or Proportion of 16 to 12; *viz.* 16 is to 12 as 4 to 3. In like manner the Reason of bb to ba, or of bbb to bba is $\frac{b}{a}$.

II. Quantities which proceed by equal Differences are said to be in a continued Arithmetical Progression, (as hath been shewn in *Chapt.* 17. *Book* 1. of my *Algebraical Elements*;) but quantities which proceed by equal Reasons, (or Proportions,) are said to be in a continued Geometrical Progression or Proportion: So these number 2, 6, 18,

54, 162 are continually proportional, becaufe the Reafon (or Proportion) of the firft to the fecond is equal to the Reafon of the fecond to the third, alfo of the third to the fourth, and fo forward; *viz.* $\frac{2}{6}$ (or $\frac{1}{3}$,) $= \frac{6}{18} = \frac{18}{54} = \frac{54}{162}$; or backward, $\frac{162}{54} = \frac{54}{18} = \frac{18}{6} = \frac{6}{2}$ (or 3.) In like manner if thefe quantities a, b, c, d, e be fuch, that $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e}$; or backwards, if $\frac{e}{d} = \frac{d}{c} = \frac{c}{b} = \frac{b}{a}$, then thofe quantities are continually proportional; *viz.* as the firft is in proportion to the fecond, fo is the fecond to the third, the third to the fourth, *&c.*

But if there be four fuch quantities that the Reafon (or Proportion) of the firft to the fecond, is equal to the Reafon of the third to the fourth; but the Reafon of the fecond to the third, is not equal to the Reafon of the firft to the fecond, then thofe quantities are faid to be in Geometrical Proportion difcontinued or interrupted; fuch are thefe four numbers, $2 . 6 :: 12 . 36$; for $\frac{2}{6}$ (or $\frac{1}{3}$) $= \frac{12}{36}$, but $\frac{6}{12}$ (or $\frac{1}{2}$) is not equal to $\frac{2}{6}$ or $\frac{1}{3}$. In like manner if a, b, c, d be fuch quantities that $\frac{a}{b} = \frac{c}{d}$, but $\frac{b}{c}$ is not equal to $\frac{a}{b}$, (or $\frac{c}{d}$;) then are a, b, c, d difcontinual Proportionals.

III. If three quantities be Proportionals, the Product made by the mutual multiplication of the Extremes is equal to the Square of the Mean; as,

If there be propofed	$\begin{cases} 18, 6, 2 \ \div\div \\ a, b, c \ \div\div \end{cases}$
Then this Equation enfueth, ...	$ac = bb = 36$
For fince by fuppofition	$a . b :: b . c$
It follows (by *Sect.* 1, and 2.) that	$\frac{a}{b} = \frac{b}{c} = 3$
Whence by multiplying each part by c, ...	$\frac{ac}{b} = b = 6$
And by multiplying each part of the laft Equation by b, it produceth	$ac = bb = 36$

Which was to be proved.

IV. If four quantities be Proportionals, whether they be continual or difcontinual, the Product made by the mutual multiplication of the extremes is equal to the Product of the means; and confequently if the Product of the means be divided by either of the extremes, the Quotient is the other extreme. As, for example,

Let four difcontinual Proportionals be propofed . . .	$\begin{cases} d . c :: b . a \\ 12 . 4 :: 15 . 5 \end{cases}$
Then by the foregoing *Sect.* 2.	$\frac{d}{c} = \frac{b}{a} = 3$
And by multiplying each part of that Equation by a, this is produced, *viz.*	$\frac{da}{c} = b = 15$
And by multiplying each part of the laft Equation by c, the firft part of the Propofition is manifeft, *viz.* . . .	$da = cb = 60$
And, by dividing each part by d, there arifeth . . .	$a = \frac{cb}{d} = 5$

Which laft Equation being compared with the four Proportionals firft propofed, doth fhew, that if three quantities d, c, b be given, to find fuch a fourth as fhall have the fame proportion to b as c hath to d, then the Product of the fecond and third terms, to wit, cb, being divided by the firft term d will give the fourth Proportional fought, which is the very Operation in the Rule of Three direct.

V. If three quantities a, b, c be Proportionals, and the firft and fecond, to wit, a and b be given feverally, the third is alfo given; for by *Sect.* 3. of this *Chapt.* $ac = bb$, whence by dividing each part by a there arifeth $c = \frac{bb}{a}$, which fhews, that if the Square of the mean or fecond term be divided by the firft, the Quotient is the third Proportional; hence a, b, and $\frac{bb}{a}$ are continual Proportionals. In like manner if three quantities in continual proportion be given feverally, and a fourth Proportional be defired, the

the Square of the third term divided by the second gives the fourth: as if there be given these three, $a, b, \frac{bb}{a}$ ∺; then by dividing the Square of $\frac{bb}{a}$, to wit, $\frac{bbbb}{aa}$ by b, the Quotient $\frac{bbb}{aa}$ shall be the fourth continual proportional: hence $a, b, \frac{bb}{a}, \frac{bbb}{aa}$ are continual proportionals. Likewise if the Square of the fourth continual proportional be divided by the third, the Quotient will be the fifth; so to those four continual proportionals, this fifth will be found, to wit, $\frac{bbbb}{aaa}$; and so forwards infinitely. Therefore,

VI. If numbers, how many soever, be continually proportionals, and the least term be esteemed the first, that next greater than the least the second, and so forwards; then the second term is produced by the multiplication of the first into the Reason of the second term to the first, the third term is produced by the multiplication of the first into the Square of the same Reason, the fourth term is produced by the multiplication of the first into the Cube of the same Reason; and in like manner every following term is produced by the multiplication of the first into such a Power of the Reason of the second term to the first as hath fewer dimensions by one than the number of terms hath unities: as in these following six continual proportionals, to wit,

$$a\ ,\ b\ ,\ \frac{bb}{a}\ ,\ \frac{bbb}{aa}\ ,\ \frac{bbbb}{aaa}\ ,\ \frac{bbbbb}{aaaa}\ ∺$$

$$2\ ,\ 6\ ,\ 18\ ,\ 54\ ,\ 162\ ,\ 486\ ∺$$

Supposing a to be the first and least term, the second term b is equal to the Product of the first term a into $\frac{b}{a}$, to wit, the Reason of the second term to the first; also the third term $\frac{bb}{a}$ is produced by the multiplication of the first term a into the Square of the same Reason, that is into $\frac{bb}{aa}$; and the fourth term $\frac{bbb}{aa}$ is produced by the multiplication of the first term a into the Cube of the same Reason, that is, into $\frac{bbb}{aaa}$; and the fifth term $\frac{bbbb}{aaa}$ is produced by the multiplication of the first term a into the fourth Power of the same Reason, that is into $\frac{bbbb}{aaaa}$: and so forwards.

But if the greatest term be esteemed the first, that next less than the greatest the second, and so downwards; then the second term is equal to the Quotient that ariseth by dividing the first (or greatest) term by the Reason of the first to the second; the third is equal to the Quotient that ariseth by dividing the first term by the Square of the same Reason; the fourth term is equal to the Quotient that ariseth by dividing the first term by the Cube of the same Reason; and in like manner every term beneath the greatest is equal to the Quotient that ariseth by dividing the first (or greatest term) by such a Power of the Reason of the greatest to the greatest but one, (or second term,) as hath fewer dimensions by one than the number of terms: as in these following six continual proportionals, to wit,

$$\frac{bbbbb}{aaaa}\ ,\ \frac{bbbb}{aaa}\ ,\ \frac{bbb}{aa}\ ,\ \frac{bb}{a}\ ,\ b\ ,\ a\ ∺$$

$$486\ ,\ 162\ ,\ 54\ ,\ 18\ ,\ 6\ ,\ 2\ ∺$$

If we suppose $\frac{bbbbb}{aaaa}$ to be the first and greatest term, then the second term $\frac{bbbb}{aaa}$ is equal to the Quotient of the first term $\frac{bbbbb}{aaaa}$ divided by $\frac{b}{a}$, to wit, by the Reason of the first term to the second; also the third term $\frac{bbb}{aa}$ is equal to the Quotient of the first term $\frac{bbbbb}{aaaa}$ divided by $\frac{bb}{aa}$, that is, by the Square of the Reason $\frac{b}{a}$; and the fourth

fourth term $\frac{bb}{a}$ is equal to the Quotient of the firſt term $\frac{bbbbb}{aaaa}$ divided by $\frac{bbb}{aaa}$ the Cube of the ſame Reaſon: and ſo of the reſt.

VII. From the laſt preceding Section it follows, that if in a Series or Rank of numbers which are in continual proportion, the firſt term, the ſecond term and the number of terms be given ſeverally, the laſt term ſhall be alſo given by this Rule; *viz.* Firſt, (according to the *Note* in *Sect.* 1. of this *Chapt.*) find out the ſmalleſt numbers that may ſhew the Reaſon of the greater of the two given terms to the leſs; then eſteeming the ſaid Reaſon as a Root, find ſuch a Power thereof whoſe Index may be equal to the given multitude of terms leſs by unity, which Power multiplied by the firſt term, when the firſt term is leſs than the ſecond, gives the laſt, to wit, the greateſt term. But when the firſt term is greater than the ſecond, then the firſt term divided by the ſaid Power gives the laſt term; as if there be given a and b the firſt and ſecond of ſix numbers in continual proportion, and that b is greater than a; then the Reaſon of b to a is $\frac{b}{a}$, (by *Sect.* 1. of this *Chapt.*) and the fifth Power of $\frac{b}{a}$ is $\frac{bbbbb}{aaaaa}$, this multiplied by the firſt term a produceth $\frac{bbbbb}{aaaa}$ which is the ſixth Proportional ſought, (as is evident by *Sect.* 6.) but if the firſt term a be greater than the ſecond term b, then the Reaſon of a to b is $\frac{a}{b}$, whoſe fifth Power is $\frac{aaaaa}{bbbbb}$, by which if you divide the firſt term a, the Quotient is the ſixth term $\frac{bbbbb}{aaaa}$.

This Rule may be exemplified by theſe four following Ranks of numbers in continual proportion.

2,	6,	18,	54,	162,	468	$\div\!\div$
3072,	768,	192,	48,	12,	3	$\div\!\div$
2,	3,	$\frac{9}{2}$,	$\frac{27}{4}$,	$\frac{81}{8}$,	$\frac{243}{16}$	$\div\!\div$
$\frac{1024}{81}$,	$\frac{256}{27}$,	$\frac{64}{9}$,	$\frac{16}{3}$,	4,	3	$\div\!\div$

VIII. If there be given two Integers expreſſing a Reaſon in the leaſt terms, and it be deſired to find out a given multitude of continual Proportionals in the ſame Reaſon, and that all the terms may be Integers; Firſt, to thoſe two Integers, or firſt and ſecond Proportionals given, find out (by *Sect.* 5. or 6. of this *Chapt.*) ſo many Proportionals as with thoſe given may make the deſired multitude; then multiply every term by the Denominator of the laſt term, ſo ſhall the Products be continual Proportionals in Integers in the ſame Reaſon as the two terms firſt given. As, for example, if a and b be given, and it be deſired to find three Proportionals in Integers in the Reaſon of a to b; firſt, to a and b I find a third Proportional, which (by *Sect.* 5.) is $\frac{bb}{a}$, then a, b, $\frac{bb}{a}$ being multiplied ſeverally by the Denominator a, the Products aa, ab, bb are Proportionals expreſt by Integers, and in the Reaſon of a to b, as was deſired.

Hence if $a = 2$, and $b = 3$; then aa, ab and bb will give 4, 6 and 9, which are continual Proportionals in Integers in the given Reaſon of 2 to 3.

So if four continual Proportionals in the Reaſon of a to b be deſired; firſt (by *Sect.* 5. or 6.) theſe will be found continual Proportionals, to wit, a, b, $\frac{bb}{a}$, $\frac{bbb}{aa}$, which multiplied ſeverally by aa, (the Denominator of the laſt term,) will produce aaa, aab, abb, bbb, which are four continual Proportionals in Integers in the given Reaſon of a to b. Hence if $a = 2$, and $b = 3$; then aaa, aab, abb and bbb will give 8, 12, 18 and 27, which are continual Proportionals in Integers in the given Reaſon of 2 to 3.

In like manner theſe five quantities $aaaa$, $aaab$, $aabb$, $abbb$ and $bbbb$ will be found continual Proportionals in the Reaſon of a to b; ſo that if $a = 2$, and $b = 3$, then thoſe five Proportionals will give theſe five, to wit, 16, 24, 36, 54 and 81 $\div\!\div$ in the Reaſon of 2 to 3: after the ſame manner you may proceed infinitely.

IX. If

IX. If there be quantities in continual proportion, how many ſoever, the Product made by the multiplication of the extremes is equal to the Product of any two means equally diſtant from the extremes; and alſo to the Square of the mean term when the number of terms is odd: as, for example, If a, b, c, d, e, f be continual Proportionals, I ſay the Product of the extremes a and f, to wit, af is equal to the Product of any two terms equally diſtant from the extremes, *viz.* to the Product cd and to the Product be; For,

1. By ſuppoſition, (and by *Sect.* 1, and 2.) . . . . . . $\frac{a}{b} = \frac{e}{f}$
2. Therefore by multiplying each part by f, it produceth . . $\frac{af}{b} = e$
3. And by multiplying each part of the laſt Equation by b, it gives $af = be$
4. Again, by ſuppoſition . . . . $\frac{b}{c} = \frac{d}{e}$
5. Therefore (by multiplying in like manner as before,) . . . . $cd = be$
6. Therefore from the third and fifth Equations (*per* 1. *Axiom.* 1. *Elem.* Euclid.) . . . . $af = cd = be$

Which was to be proved. And if more continual Proportionals even in multitude were propoſed, the Demonſtration would not be otherwiſe.

But if the multitude of terms be odd, as in theſe ſeven quantities which we may ſuppoſe to be continually proportional, $a, b, c, d, e, f, g \div\div$; then the Product made by the multiplication of the two extremes a and g is equal to the Square of the middle term d; viz. $ag = dd$. For,

1. By ſuppoſition, (and by *Sect.* 1, and 2.) . . . . . . $\frac{c}{d} = \frac{d}{e}$
2. Therefore by multiplying each part of that Equation by d, it makes . . . . $c = \frac{dd}{e}$
3. And by multiplying each part of the laſt Equation by e, it produceth . . . . $ce = dd$
4. And by what hath been already proved in the firſt part of this Propoſition, ... $ce = ag$
5. Therefore from the two laſt Equations (*per* 1. *Ax.* 1. *Elem.* Eucl.) $ag = dd$

Which was to be proved. Therefore the Propoſition is every way manifeſt. But for further illuſtration;

Let there be propoſed theſe ſix continual Proportionals in numbers, to wit, . . $2, 6, 18, 54, 162, 486 \div\div$

Then according to the firſt part of the Propoſition, ... $2 \times 486 = 6 \times 162 = 18 \times 54 = 972$

Again, let there be propoſed theſe ſeven continual Proportionals, to wit, . . . $2, 6, 18, 54, 162, 486, 1458$

Then according to the latter part of the Propoſition, ... $2 \times 1458 = 54 \times 54 = 2916.$

X. If four quantities be Proportionals, $a . b :: c . d$, they ſhall be alſo alternly, and inverſly, and compoſedly, and dividedly, and converſly, Proportionals, *viz.*

If	$a . b :: c . d$ $6 . 4 :: 12 . 8$	
Then alternly,	$a . c :: b . d$ $6 . 12 :: 4 . 8$	*per* 16. *prop.* 5. *Elem.* Eucl.
And inverſly,	$c . a :: d . b$ $12 . 6 :: 8 . 4$	*per Cor.* of *prop.* 4. *Elem.* 5.
And compoſedly,	$a+b . b :: c+d . d$ $10 . 4 :: 20 . 8$	*per* 18. *Prop.* 5. *Elem.*
And dividedly,	$a-b . b :: c-d . d$ $2 . 4 :: 4 . 8$	*per* 17. *prop.* 5. *Elem.*
And converſly,	$a . a+b :: c . c+d$ $6 . 10 :: 12 . 20$	*per Cor.* of *prop.* 19. *Elem.* 5.

But

But that the Learner may the better perceive the meaning and use of these ways of arguing about Proportionals, I shall apply some of them to the Resolution of this following

QUEST.

The difference (b) between the greater extreme and mean of three quantities continually proportional being given, as also the difference (c) between the mean and lesser extreme, to find the Proportionals; but the first difference must be greater than the latter.

RESOLUTION.

1. For the mean Proportional sought put	a
2. To which adding the given difference (b) the summ is the greater extreme, to wit, ...	$a+b$
3. But if from the mean (a,) the given difference (c) be subtracted, the Remainder is the lesser extreme, to wit,	$a-c$
4. Then (according to the Question) these three quantities $a+b$, a, and $a-c$ must be in continual proportion, *viz.*	$a+b\ .\ a :: a\ .\ a-c$
5. Therefore by division of Reason, ...	$b\ .\ a :: c\ .\ a-c$
6. And alternately, (or by permutation,)	$b\ .\ c :: a\ .\ a-c$
7. And by division of Reason, ...	$b-c\ .\ c :: c\ .\ a-c$
8. Wherefore by conversion of Reason, ...	$b-c\ .\ b :: c\ .\ a$

Which last Analogy if it be exprest by words gives this

CANON.

As the difference between the two given Differences is to either of them, so is the other to the mean Proportional sought.

Therefore if $36=b$, and $12=c$; the Canon will discover 18 for the mean Proportional sought, (to wit, a in the Resolution,) which increased with 36, and lessened by 12, gives 54 and 6 for the extremes. Therefore the three Proportionals sought are manifestly 54, 18 and 6.

Note. If the Analogy in the fourth step of the Resolution be converted into an Equation, by comparing the Product made by the mutual multiplication of the extremes to the Product of the means, that Equation after due Reduction will give the same Canon as above; so that the argumentation in the four last steps of the Resolution is not of necessity, but only to shew how without the help of any Equation, the number sought may sometimes be made the fourth Term of an Analogy whose three first Terms are known, whence by the Rule of Three the number sought is also known. Which ways of inferring one Analogy out of another are more proper when the nature of a Question will admit the same, than the common way of proceeding by Equations; especially in the Resolution of Geometrical Problems, where every step ought to be exprest in the most simple Terms, to the end the Composition of the Problem may the more easily be formed by the steps of the Resolution, but in a retrograde or backward order, as I shall hereafter shew in the Fourth Book of my *Algebraical Elements.*

XI. If Proportionals be multiplied or divided by Proportionals, the Products also or Quotients shall be Proportionals; as,

If these four proportional numbers, to wit, ...	$a\ .\ b :: ca\ .\ cb.$ $2\ .\ 4 :: 3\times2\ .\ 3\times4$
be multiplied by these four proportional numbers, ...	$d\ .\ f :: gd\ .\ gf.$ $5\ .\ 6 :: 7\times5\ .\ 7\times6$
there will be produced these four proportional numbers, to wit, . . .	$ad\ .\ bf :: cgad\ .\ cgbf.$ $2\times5\ .\ 4\times6 :: 3\times7\times2\times5\ .\ 3\times7\times4\times6$

Whereby the first part of the Proposition is manifest.

And if these four proportional numbers, to wit, ...	$ad\ .\ bf :: cgad\ .\ cgbf$
be divided by these four Proportionals, to wit, ...	$d\ .\ f :: gd\ .\ gf$
the Quotients will be these four Proportionals, to wit, ...	$a\ .\ b :: ca\ .\ cb$

Whereby the latter part of the Proposition is manifest.

Hence

Hence it may easily be proved, that the Squares, Cubes, fourth Powers, fifth Powers, &c. of proportional numbers shall be also Proportionals; as,

If . . . . $a \;.\; b :: ca \;.\; cb$

Then their Squares also shall be Proportionals, *viz.* $aa \;.\; bb :: ccaa \;.\; ccbb$

And the Cubes of the first four Proportionals shall also be Proportionals, *viz.* . . . . $aaa \;.\; bbb :: cccaaa \;.\; cccbbb$

And so of higher Powers.

XII. In every Series or Rank of Quantities continually proportional, all the mean Terms between the first and the last are both Antecedents and Consequents of Reasons; as,

If . . . . a, b, c, d, e, f ∺

That is, . . . $a \;.\; b :: b \;.\; c :: c \;.\; d :: d \;.\; e :: e \;.\; f$

It is evident that every Term except the last (f) is an Antecedent of a Reason, and every Term except the first (a) is a Consequent; wherefore if (s) be put for the summ of all the Terms in the Series, then $s-f$ shall be the summ of all the Antecedents, and $s-a$ the summ of all the Consequents; Therefore,

From the premises (*per* 12. *prop.* 5. *Elem.* Euclid.) this Analogy ariseth, *viz.* . . . . $a \;.\; b :: s-f \;.\; s-a$

Whence by comparing the Product of the extremes to the Product of the means . . . . $as-aa = bs-bf$

Therefore, by due Transposition in that Equation, when b is greater than a, ... $bf-aa = bs-as$

And by dividing each part of the last Equation by $b-a$, there ariseth . . . . $\frac{bf-aa}{b-a} = s$

But if a exceed b, then there will arise . . . . $\frac{aa-bf}{a-b} = s$

Which two last Equations give a Canon to find the summ of all the Terms of a Geometrical Progression, the first, second and last Terms being severally given.

CANON.

Divide the difference between the Square of the first Term and the Product made by the multiplication of the second Term into the last, by the difference of the first and second Terms; so the Quotient shall be the summ of all the Terms of the Geometrical Progression proposed.

Examples in numbers.

Let the values of these, . . . a, b, c, d, e, f ∺
be exprest by these numbers, . $32, 48, 72, 108, 162, 243$ ∺

Then by the Canon, ... $\frac{bf-aa}{b-a} = 665$ the Summ of all.

But if the values of the same Proportionals, ... a, b, c, d, e, f ∺
be expounded by these numbers, . . $243, 162, 108, 72, 48, 32$ ∺

Then by the Canon, ... $\frac{aa-bf}{a-b} = 665$ the Summ of all.

XIII. If what hath been said in the eighth *Sect.* of this *Chapt.* be compared with the Table in *Sect.* 4. *Chap.* 1. of this Book, it will be manifest that if we cast away the numbers of multitude which are prefix'd to all the mean Terms or Members belonging to any Compound Power produced from a Binomial Root, suppose from $a+e$, then all the Members or Simple quantities whereof the said Compound Power is composed are in continual proportion: As, for example, the Members whereof the Square of $a+e$ is composed are aa, $2ae$ and ee; now if 2 which is prefix'd to ae be cast away, then aa, ae and ee are continual Proportionals, (as is evident by the preceding eighth *Sect.* of this *Chapt.*)

Again, it appears by the said Table, that the Members whereof the Cube of $a+e$ is composed are aaa, $3aae$, $3aee$ and eee; here if 3 and 3 which are prefix'd to the mean Terms be cast away, then these four quantities aaa, aae, aee and eee will be in continual proportion.

Likewife, forafmuch as the fourth Power of $a+e$ is compofed of thefe Members, *aaaa*, 4*aaae*, 6*aaee*, 4*aeee* and *eeee*; by cafting away the numbers of multitude 4, 6 and 4, thefe five quantities *aaaa*, *aaae*, *aaee*, *aeee* and *eeee* fhall be continual Proportionals: and fo of higher Powers infinitely.

XIV. Forafmuch as by the laft preceding *Sect.* thefe quantities are in continual proportion, to wit, ... $\div\, aa \,;\, ae \,;\, ee$

Therefore their fquare Roots alfo fhall be in continual proportion, (*per* 22. *prop.* 6. *Elem.* Euclid.) to wit, . . . $\div\, a \,,\, \sqrt{ae} \,,\, e$

Hence, if a mean Proportional between any two given numbers *a* and *e* be defired, it fhall be $\sqrt{ae}$; as, if $a=12$, and $e=3$, then $ae=36$, and $\sqrt{ae}$ or $\sqrt{36}$, that is, 6, is a mean Proportional between 12 and 3; for as 12 is to 6, fo is 6 to 3.

Again, forafmuch as thefe quantities are in continual proportion, to wit, ... $\div\, aaa \,,\, aae \,,\, aee \,,\, eee$

Therefore their cubick Roots alfo fhall be continual Proportionals, (*per* 37. *prop.* 11. *Elem.* Euclid.) to wit, $\div\, a,\, \sqrt{(3)}aae,\, \sqrt{(3)}aee,\, e$

Hence, if two mean Proportionals between any two given numbers *a* the greater and *e* the leffer be defired, then $\sqrt{(3)}aae$ fhall be the greater mean, and $\sqrt{(3)}aee$ the leffer; as if $a=54$, and $e=2$, then $aae=5832$, and $\sqrt{(3)}aae=\sqrt{(3)}5832$; therefore $\sqrt{(3)}5832$, that is, 18 is the greater mean fought; alfo $aee=216$, and therefore $\sqrt{(3)}216$, that is, 6, is the leffer mean: fo that 18 and 6 are the two defired mean Proportionals between 54 and 2; for 54, 18, 6 and 2 are in continual proportion. But when one mean next to either of the extremes is found out, the other mean may be found out by *Sect.* 5. of this *Chapt.* without extracting any Root.

After the fame manner, by the help of the faid Table in *Sect.* 4. *Chap.* 1. of this Book, continued to higher Powers if need be, you may find out as many mean proportional numbers as fhall be defired between any two given numbers: As, if you would find five mean proportional numbers between 1458 (or *a*,) and 2 (or *e*;) look into the faid Table for the fixth Power, (to wit a Power whofe Index exceeds by unity the number of means fought,) and you will find *aaaaaa*, 6*aaaaae*, 15*aaaaee*, 20*aaaeee*, 15*aaeeee*, 6*aeeeee* and *eeeeee*; then cafting away 6, 15, 20, 15 and 6 which are prefix'd to the mean terms, and extracting $\sqrt{(6)}$ out of every one of thofe fix terms after the faid numbers prefix'd are caft away, there will arife *a*, $\sqrt{(6)}aaaaae$, $\sqrt{(6)}aaaaee$, $\sqrt{(6)}aaaeee$, $\sqrt{(6)}aaeeee$, $\sqrt{(6)}aeeeee$ and $e \div$; now to find the five mean proportional numbers anfwering to thofe five proportional Roots expreft by letters which fall between *a* and *e*, it will be convenient to find the fmalleft mean firft, *viz.* forafmuch as *a* was put for 1458, and *e* for 2, therefore $aeeeee=46656$, and $\sqrt{(6)}aeeeee=\sqrt{(6)}46656$, that is, 6, fhall be the leaft mean fought: then 2 being the firft Proportional, or leffer extreme, and 6 the fecond, the third will (by *Sect.* 5. of this *Chapt*) be found 18, the fourth 54, the fifth 162, the fixth 486, and the feventh, to wit, the greater extreme, was firft given 1458: fo that between 2 and 1458, five mean Proportionals are found out as was defired; and the feven continual Proportionals are thefe, to wit, 2, 6, 18, 54, 162, 486 and 1458.

Many other admirable properties adherent to numbers in Geometrical Proportion continued, are deducible from the faid Table of Powers in *Sect.* 4. *Chap.* 1. of this Book, as will partly appear by the Theorems in the following fixth Chapter, which I find difperfed in feveral Algebraical Treatifes.

CHAP.

Chap. VI.

Various Theorems about Quantities in Continual proportion.

Theorem 1.

If three numbers be Proportionals, the Solid number made by the continual multiplication of all the three is equal to the Cube of the mean.

Let three Proportionals be exposed in Integers, accerding to *Sect.* 8, or 13. of the preceding *Chap.* 5. . . . . $\left\{\begin{array}{ccccc} aa, & ae, & ee & \div\div \\ 9, & 6, & 4 & \div\div \end{array}\right.$

Thence it is evident, that *aaaeee* the Product made by the multiplication of all the three Proportionals one into another, is equal to the Cube of the mean *ae*, as is affirmed by the Theorem.

Theor. 2.

If three numbers be Proportionals, the Product made by the multiplication of the Square of the first by the third, is equal to the Product of the Square of the second by the first:

As in these three, ... $\left\{\begin{array}{ccccc} aa, & ae, & ee & \div\div \\ 9, & 6, & 4 & \div\div \end{array}\right.$

It is evident that $aaaa \times ee = aaee \times aa = aaaaee$.

Theor. 3.

If three numbers be Proportionals, the Square of the summ of the extremes is equal to both the Squares of the extremes, together with twice the Square of the mean:

As in these three, ... $\left\{\begin{array}{ccccc} aa, & ae, & ee & \div\div \\ 9, & 6, & 4 & \div\div \end{array}\right.$

The Square of $aa + ee$ is $aaaa + 2aaee + eeee$, which is manifestly equal to the Squares of *aa* and *ee*, together with twice the Square of *ae*.

Theor. 4.

If three numbers be Proportionals, the Product of the lesser extreme multiplied by the difference of the extremes, is equal to the difference of the Squares of the mean and lesser extreme:

As in these three, ... $\left\{\begin{array}{ccccc} aa, & ae, & ee & \div\div \\ 9, & 6, & 4 & \div\div \end{array}\right.$

It is evident that $ee \times \overline{aa - ee} = aaee - eeee$.

Theor. 5.

If three numbers be Proportionals, the Product of the greater extreme multiplied by the difference of the extremes, is equal to the difference of the Squares of the greater extreme and the mean:

As in these three, ... $\left\{\begin{array}{ccccc} aa, & ae, & ee & \div\div \\ 9, & 6, & 4 & \div\div \end{array}\right.$

It is evident that $aa \times \overline{aa - ee} = aaaa - aaee$.

Theor. 6.

If three numbers be Proportionals, the difference of the Squares of the extremes is equal to the Square of the difference of the extremes, together with twice the difference of the Squares of the mean and lesser extreme:

As in these three, ... $\left\{\begin{array}{ccccc} aa, & ae, & ee & \div\div \\ 9, & 6, & 4 & \div\div \end{array}\right.$

1. The difference of the Squares of the extremes is . } $aaaa - eeee$
2. The Square of $aa - ee$ (the difference of the extremes) is . . . . } $aaaa - 2aaee + eeee$
3. The double of the difference of the Squares of the mean and lesser extreme is . . . . } $+ 2aaee - 2eeee$

Now the summ of the two latter of those three Quantities is manifestly equal to the first, as the Theorem affirms.

Theor. 7.

If three numbers be Proportionals, the difference of the Squares of the greater extreme and the mean is equal to the Square of the difference of the extremes, and to the difference of the Squares of the mean and the lesser extreme:

As in these three, ... $\left\{\begin{array}{l} aa \;,\; ae \;,\; ee \;\div\!\div \\ 9 \;,\; 6 \;,\; 4 \end{array}\right.$

1. The difference of the Squares of the greater extreme and the mean is . . . . $aaaa - aaee$
2. The Square of $aa - ee$ (the difference of the extremes) is . . . . $aaaa - 2aaee + eeee$
3. The difference of the Squares of the mean and lesser extreme is . . . . $+ aaee - eeee$

Now the summ of the two latter of those three Quantities is manifestly equal to the first, as the Theorem affirms.

Theor. 8.

If three numbers be Proportionals, then as the first is to the third, so is the Square of the first to the Square of the second; and so is the Square of the second to the Square of the third:

As in these three, ... $\left\{\begin{array}{l} aa \;,\; ae \;,\; ee \;\div\!\div \\ 9 \;,\; 6 \;,\; 4 \;\div\!\div \end{array}\right.$

1. It is evident that . . . . $aa \;.\; ee \;::\; aa \;.\; ee$
2. Therefore by drawing aa as a common Factor into the two latter terms of that Analogy, this ariseth, ... $aa \;.\; ee \;::\; aaaa \;.\; aaee$
3. And by drawing ee as a common Factor into the two latter terms of the first Analogy, this ariseth, ... $aa \;.\; ee \;::\; aaee \;.\; eeee$

By which two last Analogies the truth of the Theorem is manifest.

Theor. 9.

If three numbers be Proportionals, then as the first is to the second, (or as the second is to the third,) so is the difference of the first and second, to the difference of the second and third:

As in these three, ... $\left\{\begin{array}{l} aa \;,\; ae \;,\; ee \;\div\!\div \\ 9 \;,\; 6 \;,\; 4 \end{array}\right.$

1. It is evident (as before hath been shewn in *Theor.* 4.) that . . . . $ee \times \overline{aa - ee} = aaee - eeee$
2. And by Multiplication it will appear that $\overline{ae + ee} \times \overline{ae - ee} = aaee - eeee$
3. Therefore from the two last Equations, (*per* 1. *Ax.* 1. *Elem.* Euclid.) . . . . $ee \times \overline{aa - ee} = \overline{ae + ee} \times \overline{ae - ee}$
4. Therefore, by resolving the last Equation into Proportionals, ... $aa - ee \;.\; ae - ee \;::\; ae + ee \;.\; ee$
5. Therefore by Division of Reason, . . . $aa - ae \;.\; ae - ee \;::\; ae \;.\; ee$

Which was to be demonstrated.

Theor. 10.

If four numbers be continually proportional, the summ of the means is a mean Proportional between the summ of the first and second and the summ of the third and fourth.

Let four continual Proportionals be exposed in Integers, to wit, $\left\{\begin{array}{l} aaa \;,\; aae \;,\; aee \;,\; eee \;\div\!\div \\ 8 \;,\; 4 \;,\; 2 \;,\; 1 \;\div\!\div \end{array}\right.$

Then according to the import of the Theorem, it must be proved that these three Quantities are Proportionals, *viz.*

$$aaa + aae \;.\; aae + aee \;.\; aee + eee \;\div\!\div$$

But that they are Proportionals it will be evident by Multiplication, for the Product of the extremes is equal to the Square of the mean: therefore the truth of the Theorem is manifest.

Theor. 11.

Theor. 11.

If four numbers be continual Proportionals, the ſumm of all is to the ſumm of the means, as the ſumm of the firſt and third to the ſecond:

As in theſe four, ... $\left\{\begin{matrix} aaa, & aae, & aee, & eee & ∺ \\ 8, & 4, & 2, & 1 & ∺ \end{matrix}\right.$

1. The ſumm of all four is . . . . > $aaa + aae + aee + eee$
2. The ſumm of the means is . . . . . . > $aae + aee$
3. The ſumm of the firſt and third is . . > $aaa + aee$
4. And the ſecond is . . . . > aae

I ſay thoſe four quantities are Proportionals, in ſuch order as they are above written; for it will appear by multiplication, that the Product of the extremes is equal to the Product of the means: therefore the Theorem is manifeſt.

Theor. 12.

If four numbers be in continual proportion, the ſumm of all is to the ſumm of the means, as the ſumm of the Squares of the means is to the Product of the means or extremes:

As in theſe four, ... $\left\{\begin{matrix} aaa, & aae, & aee, & eee & ∺ \\ 8, & 4, & 2, & 1 & ∺ \end{matrix}\right.$

1. The ſumm of all is . . . . > $a^3 + a^2e + ae^2 + e^3$
2. The ſumm of the means is . . . . . . > $a^2e + ae^2$
3. The ſumm of the Squares of the means is > $a^4e^2 + a^2e^4$
4. The Product of the means or extremes is > a^3e^3

I ſay thoſe four quantities are Proportionals, in ſuch order as they are above written; for it will appear by multiplication, that the Product of the extremes is equal to the Product of the means: therefore the Theorem is manifeſt.

Theor. 13.

If four numbers be continual Proportionals, the ſumm of the Squares of the means is a mean Proportional between the ſumm of the Squares of the firſt and ſecond, and the ſumm of the Squares of the third and fourth:

As in theſe four, ... $\left\{\begin{matrix} aaa, & aae, & aee, & eee & ∺ \\ 8, & 4, & 2, & 1 & ∺ \end{matrix}\right.$

1. The ſumm of the Squares of the firſt and ſecond is . . . . } $a^6 + a^4e^2$
2. The ſumm of the Squares of the means is > $a^4e^2 + a^2e^4$
3. The ſumm of the Squares of the third and fourth is . . . . } $a^2e^4 + e^6$

I ſay thoſe three quantities are Proportionals, in ſuch order as they are above written; for it will appear by multiplication that the Square of the mean (or ſecond quantity) is equal to the Product of the extremes: therefore the Theorem is manifeſt.

Theor. 14.

If four numbers be continual Proportionals, the Square of the ſumm of the means is equal to the Square of their difference, together with four times the Product of the extremes or means:

As in theſe four, ... $\left\{\begin{matrix} aaa, & aae, & aee, & eee & ∺ \\ 8, & 4, & 2, & 1 & ∺ \end{matrix}\right.$

1. The Square of $a^2e + ae^2$ (the ſumm of the means) is . . . . } $a^4e^2 + 2a^3e^3 + a^2e^4$
2. The Square of $a^2e - ae^2$ (the difference of the means) is . . . . } $a^4e^2 - 2a^3e^3 + a^2e^4$
3. The quadruple of the Product of the extremes or means) is . . . . } $+ 4a^3e^3$

Now it is evident that the firſt of thoſe three Quantities is equal to the ſumm of the ſecond and third: therefore the Theorem is manifeſt.

Theor. 15.

Theor. 15.

If four numbers be continual Proportionals, the ſumm of their Squares ſhall be to the ſumm of the Products of the firſt into the ſecond, and the third into the fourth; as the ſumm of all the four Proportionals to the ſumm of the means:

As in theſe four, ... $\begin{cases} aaa, & aae, & aee, & eee \;\div\div \\ 8, & 4, & 2, & 1 \;\div\div \end{cases}$

1. The ſumm of the Squares of the four Proportionals is . . . . $a^6 + a^4e^2 + a^2e^4 + e^6$
2. The ſumm of the Products of the firſt into the ſecond, and the third into the fourth is $a^5e + ae^5$
3. The ſumm of all the four Proportionals is $a^3 + a^2e + ae^2 + e^3$
4. The ſumm of the means is . . . . . $a^2e + ae^2$

I ſay thoſe four quantities are Proportionals in ſuch order as they are above ſeated, for it will appear by multiplication that the Product of the extremes is equal to the Product of the means: therefore the Theorem is manifeſt.

Theor. 16.

If from the Square of the ſumm of four numbers in continual proportion the ſumm of their Squares be ſubtracted, and from half the Remainder there be alſo ſubtracted the Square of the ſumm of the two means, this latter Remainder ſhall be the ſumm of the Products of the firſt Proportional into the ſecond, and of the third into the fourth, and ſhall be to the ſumm of the Squares of thoſe four Proportionals, as the ſumm of the two means is to the ſumm of all the Proportionals:

As in theſe four, ... $\begin{cases} aaa, & aae, & aee, & eee \;\div\div \\ 8, & 4, & 2, & 1 \;\div\div \end{cases}$

1. The Square of the ſumm of the four Proportionals will by multiplication be found
$$a^6 + 2a^5e + 3a^4e^2 + 4a^3e^3 + 3a^2e^4 + 2ae^5 + e^6.$$
2. The ſumm of the Squares of the four Proportionals is
$$a^6 + a^4e^2 + a^2e^4 + e^6.$$
3. Which ſumm of the Squares being ſubtracted from the ſaid Square of the ſumm, the half of the Remainder will be
$$+ a^5e + a^4e^2 + 2a^3e^3 + a^2e^4 + ae^5.$$
4. The Square of the ſumm of the two means, to wit, of $a^2e + ae^2$ is
$$+ a^4e^2 + 2a^3e^3 + a^2e^4.$$
5. Which laſt mentioned Square being ſubtracted from the half Remainder in the third ſtep, there will remain the ſumm of the Products of the firſt Proportional into the ſecond, and of the third into the fourth, to wit,
$$+ a^5e + ae^5.$$
6. Now according to the import and meaning of the Theorem it remains to prove, that the Remainder in the laſt ſtep is to the ſumm of the Squares in the ſecond ſtep, as the ſumm of the two mean Proportionals is to the ſumm of all four, *viz.* that

Theſe four quantities are Proportionals, $\begin{cases} + a^5e + ae^5. \\ + a^6 + a^4e^2 + a^2e^4 + e^6 :: \\ + a^2e + ae^2. \\ + a^3 + a^2e + ae^2 + e^3. \end{cases}$

7. But that they are Proportionals will be evident by multiplication; for the Product of the extremes is equal to the Product of the means, each Product being
$$a^8e + a^7e^2 + a^6e^3 + a^5e^4 + a^4e^5 + a^3e^6 + a^2e^7 + ae^8.$$

Therefore the Theorem is manifeſt.

Theor. 17.

If four numbers be continual Proportionals, the ſumm of all their Squares ſhall be to the ſumm of the Squares of the means; as the ſumm of the Products of the firſt into the ſecond and the third into the fourth, to the Product of the means or extremes.

This is inferr'd from *Theor.* 12, and 15. by exchange of equal Reaſons.

Theor. 18.

If four numbers be continual Proportionals, the ſumm of the Squares of the extremes ſhall be to the ſumm of the Squares of the means; as the exceſs whereby the ſumm of the

the Products of the firſt into the ſecond and third into the fourth exceeds the Product of the means is to the Product of the means or extremes.

This is inferr'd from *Theor.* 17. by Diviſion of Reaſon.

Theor. 19.

If four numbers be continual Proportionals, the ſumm of the firſt and third ſhall be to the ſecond; as the ſumm of the Squares of the means, is to the Product of the means or extremes.

This is deduced from *Theor.* 11, and 12. by exchange of equal Reaſons.

Theor. 20.

If four numbers be continual Proportionals, the ſumm of all their Squares ſhall be to the ſumm of the Products of the firſt into the ſecond, and the third into the fourth; as the ſumm of the firſt and third is to the ſecond.

This is deduced from *Theor.* 17, and 19. by exchange of equal Reaſons.

Theor. 21.

If four numbers be continual Proportionals, the ſumm of the Cubes of the means is equal to the Product made by the multiplication of the ſumm of the extremes into the Product of the means or extremes:

As in theſe four, ... $\left\{ \begin{matrix} aaa, & aae, & aee, & eee \\ 8, & 4, & 2, & 1 \end{matrix} \right.$ ÷÷

1. The ſumm of the Cubes of the means is > $a^6e^3 + a^3e^6$
2. The ſumm of the extremes is . . . . . > $a^3 + e^3$
3. The Product of the means or extremes is > a^3e^3

Now it is evident that the firſt of thoſe three Quantities is equal to the Product of the ſecond Quantity multiplied by the third, as is affirmed by the Theorem.

Theor. 22.

If four numbers be continual Proportionals, the Cube of the ſumm of the extremes is equal to the Cubes of the extremes, together with the triple ſumm of the Cubes of the means:

As in theſe four, ... $\left\{ \begin{matrix} aaa, & aae, & aee, & eee \\ 8, & 4, & 2, & 1 \end{matrix} \right.$ ÷÷

1. The Cube of $a^3 + e^3$ (the ſumm of the extremes) is . . . . } $a^9 + 3a^6e^3 + 3a^3e^6 + e^9$
2. The Cubes of the extremes is . . . . . > $a^9 + e^9$
3. The triple ſumm of the Cubes of the means is . . . . } $3a^6e^3 + 3a^3e^6$

Now it is manifeſt that the firſt of thoſe three Quantities is equal to the ſumm of the other two, as the Theorem affirms.

Theor. 23.

If four numbers be continual Proportionals, the difference of the Cubes of the extremes is equal to the triple of the difference of the Cubes of the means, together with the Cube of the difference of the extremes:

As in theſe four, ... $\left\{ \begin{matrix} aaa, & aae, & aee, & eee \\ 8, & 4, & 2, & 1 \end{matrix} \right.$ ÷÷

1. The difference of the Cubes of the extremes is > $a^9 - e^9$
2. The triple of the difference of the Cubes of the means is . . . . } $3a^6e^3 - 3a^3e^6$
3. The Cube of $a^3 - e^3$ (the difference of the extremes) is . . . . } $a^9 - 3a^6e^3 + 3a^3e^6 - e^9$

Now it is manifeſt that the firſt of thoſe three Quantities is equal to the ſumm of the other two. Which was to be proved.

Theor. 24.

Theor. 24.

If four numbers be continual Proportionals; the Cube of the ſumm of the firſt and ſecond is equal to the Product made by the multiplication of the Square of the firſt by the Aggregate of the ſumm of the extremes and the triple ſumm of the means:

As in theſe four, ...	aaa, aae, aee, eee ∺ 8, 4, 2, 1 ∺	
1. The Cube of the ſumm of the firſt and ſecond, to wit, of $a^3 + aae$ is	$a^9 + 3a^8e + 3a^7e^2 + a^6e^3$	
2. The Square of the firſt is	a^6	
3. The Aggregate of the extremes and the triple of the ſumm of the means is	$a^3 + e^3 + 3a^2e + 3ae^2$	

Now it is evident that the firſt of thoſe three Quantities is equal to the Product made by the multiplication of the third by the ſecond. Which was to be proved.

Theor. 25.

If four numbers be continual Proportionals, the Cube of the ſumm of the means is equal to the Product made by the multiplication of the Product of the extremes or means into the Aggregate of the extremes and the triple ſumm of the means:

As in theſe four, ...	aaa, aae, aee, eee ∺ 8, 4, 2, 1 ∺	
1. The Cube of the ſumm of the means, to wit, of $a^2e + ae^2$, is	$a^6e^3 + 3a^5e^4 + 3a^4e^5 + a^3e^6$	
2. The Product of the extremes or means is	a^3e^3	
3. The Aggregate of the extremes and the triple ſumm of the means is	$a^3 + e^3 + 3a^2e + 3ae^2$	

Now it is evident that the firſt of thoſe three Quantities is equal to the Product of the two latter. Which was to be proved.

Theor. 26.

If four numbers be continual Proportionals, the Product made by the multiplication of the ſumm of the extremes by the ſumm of the Squares of the extremes, is equal to the Cubes of the four Proportionals:

As in theſe four, ...	aaa, aae, aee, eee ∺ 8, 4, 2, 1 ∺	
1. The ſumm of the extremes is	$a^3 + e^3$	
2. The ſumm of the Squares of the extremes is	$a^6 + e^6$	
3. The Product of thoſe two ſumms is . . .	$a^9 + a^6e^3 + a^3e^6 + e^9$	
4. The ſumm of the Cubes of the four Proportionals is	$a^9 + a^6e^3 + a^3e^6 + e^9$	

But the Product in the third ſtep is manifeſtly equal to the ſumm in the fourth; as the Theorem affirms.

Theor. 27.

If five numbers be continual Proportionals, the Product of the mean (or third Proportional) into the ſumm of the extremes, is equal to the Squares of the ſecond and fourth:

As in theſe five, . . .	$aaaa$, $aaae$, $aaee$, $aeee$, $eeee$ 16, 8, 4, 2, 1	
1. The Product of the mean into the ſumm of the extremes is	$a^6e^2 + a^2e^6$	
2. And the ſumm of the Squares of the ſecond and fourth is alſo	$a^6e^2 + a^2e^6$	

Therefore the Theorem is manifeſt.

Theor. 28.

Theor. 28.

If five numbers be continual Proportionals, the ſumm of the firſt, third and fifth, ſhall be to the third; as the ſumm of the Squares of the ſecond, third and fourth is to the Square of the third:

As in theſe five, . . . $\begin{cases} aaaa, & aaae, & aaee, & aeee, & eeee \\ 16, & 8, & 4, & 2, & 1 \end{cases}$

1. The ſumm of the firſt, third and fifth is . . . $a^4 + a^2e^2 + e^4$.
2. The third is . . . . a^2e^2. ::
3. The ſumm of the Squares of the ſecond, third and fourth is . . . . $a^6e^2 + a^4e^4 + a^2e^6$.
4. The Square of the third is . . . . a^4e^4.

I ſay thoſe four quantities are Proportionals, in ſuch order as they are above ſeated; for it will appear by multiplication, that the Product of the extremes is equal to the Product of the means; each Product being $a^8e^4 + a^6e^6 + a^4e^8$: therefore the Theorem is manifeſt.

Theor. 29.

If five numbers be continual Proportionals, the ſumm of the extremes more by the double of the mean, the ſumm of the ſecond and fourth, and the mean, are alſo continual Proportionals:

As in theſe five, . . . $\begin{cases} aaaa, & aaae, & aaee, & aeee, & eeee \\ 16, & 8, & 4, & 2, & 1 \end{cases}$

1. The ſumm of the extremes more by the double of the mean is . . . . $a^4 + e^4 + 2a^2e^2$
2. The ſumm of the ſecond and fourth is . . $a^3e + ae^3$
3. The mean is . . . . a^2e^2

I ſay thoſe three quantities are Proportionals; for it will be evident by multiplication that the Product of the firſt and third is equal to the Square of the ſecond: therefore the Theorem is manifeſt.

Theor. 30.

If five numbers be continual Proportionals, the ſumm of the extremes is to the mean, as the difference of the Squares of the extremes, to the difference of the Squares of the ſecond and fourth:

As in theſe five, . . . $\begin{cases} aaaa, & aaae, & aaee, & aeee, & eeee \\ 16, & 8, & 4, & 2, & 1 \end{cases}$

1. The ſumm of the extremes is . . . . $a^4 + e^4$.
2. The mean is . . . . a^2e^2 ::
3. The difference of the Squares of the extremes is $a^8 - e^8$.
4. The difference of the Squares of the ſecond and fourth is . . . . $a^6e^2 - a^2e^6$.

I ſay thoſe four quantities are Proportionals in ſuch order as they are above placed; for it will be evident by multiplication, that the Product of the extremes is equal to the Product of the means, each Product being $a^{10}e^2 - a^2e^{10}$: therefore the Theorem is manifeſt.

Theor. 31.

If five numbers be continual Proportionals, the ſumm of the Squares of the ſecond and fourth, ſhall be to the Square of the mean; as the difference of the Squares of the extremes, to the difference of the Squares of the ſecond and fourth:

As in theſe five, . . . $\begin{cases} aaaa, & aaae, & aaee, & aeee, & eeee \\ 16, & 8, & 4, & 2, & 1 \end{cases}$

1. The ſumm of the Squares of the ſecond and fourth is . . . . $a^6e^2 + a^2e^6$.
2. The Square of the mean is . . . . a^4e^4 ::
3. The difference of the Squares of the extremes is $a^8 - e^8$.
4. The difference of the Squares of the ſecond and fourth is . . . . $a^6e^2 - a^2e^6$.

I say those four quantities are Proportionals in such order as they are above seated; for it will be evident by multiplication, that the Product of the extremes is equal to the Product of the means: therefore the Theorem is manifest.

Theor. 32.

If five numbers be continual Proportionals, the summ of the extremes shall be to the mean; as the summ of the Squares of the second and fourth is to the Square of the mean.

This is evident from the two last preceding Theorems, by exchange of equal Reasons.

Theor. 33.

If five numbers be continual Proportionals, the summ of the Squares of the second and fourth shall be equal to the Product made by the multiplication of the third into the summ of the first and fifth:

As in these five, . . . $\begin{cases} aaaa\ ,\ aaae\ ,\ aaee\ ,\ aeee\ ,\ eeee \\ 16\ ,\ 8\ ,\ 4\ ,\ 2\ ,\ 1 \end{cases}$

1. The summ of the Squares of the second and fourth is . . . . $a^6e^2 + a^2e^6$
2. The mean or third is . . . . a^2e^2
3. The summ of the first and fifth is . . . $a^4 + e^4$

But the Product of the second and third of those three Quantities above-written is equal to the first: therefore the Theorem is manifest.

Chap. VII.

Questions about Quantities in Continual proportion, resolved by Literal Algebra.

QUEST. 1.

THE summ (b) of three proportional Quantities being given, as also (c) the summ of their Squares; to find the Proportionals.

RESOLUTION.

1. For the mean Proportional sought put . . . . a
2. Then subtracting the said mean from (b) the given summ of all the three Proportionals, there will remain the summ of the extremes, to wit, ... $b - a$
3. Therefore the Square of the summ of the extremes is . . . $bb - 2ba + aa$
4. From which Square, if there be subtracted the double of the Square of the mean, to wit, ... $2aa$
5. There will remain (as is manifest by *Theor.* 3. of the preceding *Chap.* 6.) the summ of the Squares of the extremes, to wit, $bb - 2ba - aa$
6. To which summ of the Squares of the extremes if you add (aa) the Square of the mean, the aggregate shall be the summ of the Squares of the three Proportionals sought, to wit, $bb - 2ba$
7. Which summ in the last step must be equal to (c) the given summ of the Squares: Hence this Equation, *viz.* . . . $bb - 2ba = c$
8. Which Equation after due Reduction gives . . . . . . $\dfrac{bb - c}{2b} = a$

And the last Equation in words is this

CANON.

From the Square of the given summ of the three Proportionals sought subtract the given summ of their Squares; then divide the Remainder by the double of the summ of the three Proportionals, and the Quotient is the mean Proportional.

Therefore if 14 be given for the summ of three numbers in continual proportion, and 84 for the summ of their Squares, the mean Proportional will be found 4 by the said Canon. Then the mean being given 4, as also 10 the summ of the extremes; the extremes

extremes will be found 2 and 8, (by the Canon of *Queſt.* 4. *Chap.* 16. of my Firſt Book of *Algebraical Elements:*) and therefore the three Proportionals ſought are 2, 4 and 8.

QVEST. 2.

The ſumm (b) of three proportional Quantities being given, as alſo (c) the ſumm of the Squares of the extremes; to find the Proportionals.

RESOLVTION.

1. For the mean Proportional ſought put . . . . a
2. Then ſubtracting the ſaid mean from (b) the given ſumm of all the three Proportionals, there will remain the ſumm of the extremes, to wit, ... $b - a$
3. Therefore the Square of the ſumm of the extremes is . $bb - 2ba + aa$
4. From which Square if you ſubtract the double of the Square of the mean, to wit, ... $2aa$
5. There will remain (as is manifeſt by the third Theorem of the preceding ſixth Chapter,) the ſumm of the Squares of the extremes, to wit, ... $bb - 2ba - aa$
6. Which ſumm of the Squares of the extremes muſt be equal to the given ſumm (c,) hence this Equation, *viz.* . . $bb - 2ba - aa = c$
7. From which Equation after due Reduction, this will ariſe, $bb - c = aa + 2ba$
8. Therefore by reſolving the laſt Equation, (according to the Canon in *Sect.* 6. *Chap.* 15. of my Firſt Book of *Algebraical Elements*;) the value of (a) the mean Proportional will be made known, *viz.* . . . . . $\sqrt{:2bb - c:} - b = a.$

Which laſt Equation in words is this

CANON.

From the double of the Square of the given ſumm of all the three Proportionals ſought ſubtract the given ſumm of the Squares of the extremes; then from the ſquare Root of the Remainder ſubtract the ſumm of the three Proportionals, ſo ſhall this laſt Remainder be the mean Proportional ſought.

Therefore, if 14 be given for the ſumm of three continual Proportionals, and 68 for the ſumm of the Squares of the extremes, the mean Proportional will be found 4 by the ſaid Canon: Then the mean being given 4, as alſo 10 the ſumm of the extremes; the extremes will be found 2 and 8, (by the Canon of *Queſt.* 4. *Chap.* 15. of my Firſt Book of *Algebraical Elements*;) and therefore the three Proportionals ſought are 2, 4 and 8.

QVEST. 3.

The difference (b) of the extremes of three proportional Quantities being given, as alſo (c) the ſumm of the Squares of the three Proportionals; to find the Proportionals.

RESOLVTION.

1. For the ſumm of the extremes, (to wit, of the firſt and third Proportionals ſought,) put . . . . a
2. Then, foraſmuch as the difference of the extremes is given (b,) and their ſumm is aſſumed to be (a,) therefore (by the Theorem in *Queſt.* 1. *Chap.* 14. of my Firſt Book of *Algebraical Elements*,) the greater extreme ſhall be . $\frac{1}{2}a + \frac{1}{2}b$
3. And by the ſame Theorem the leſſer extreme is . . . $\frac{1}{2}a - \frac{1}{2}b$
4. Then the Product made by the multiplication of the extremes in the ſecond and third ſteps will give the Square of the mean, to wit, ... $\frac{1}{4}aa - \frac{1}{4}bb$
5. And from the ſecond ſtep the Square of the greater extreme is . . . . $\frac{1}{4}aa + \frac{1}{2}ab + \frac{1}{4}bb$
6. And from the third ſtep the Square of the leſſer extreme is $\frac{1}{4}aa - \frac{1}{2}ab + \frac{1}{4}bb$
7. Therefore from the fourth, fifth and ſixth ſteps, the ſumm of the Squares of all the three Proportionals is . . $\frac{3}{4}aa + \frac{1}{4}bb$

8. Which ſumm in the laſt ſtep muſt be equal to (c) the ſumm of the Squares given in the Queſtion; hence this Equation ariſeth, to wit, ... $\frac{3}{4}aa + \frac{1}{4}bb = c$

9. Which Equation after due Reduction will give . . . $aa = \frac{4c - bb}{3}$

10. Therefore by extracting the ſquare Root out of each part of the laſt Equation the ſumm of the extreme Proportionals is diſcovered, to wit, ... $a = \sqrt{\frac{4c - bb}{3}}$.

Which laſt Equation gives this

CANON.

From four times the given ſumm of the Squares of the three Proportionals ſought, ſubtract the Square of the given difference of the extremes; then the ſquare Root of one third part of that Remainder ſhall be the ſumm of the extreme Proportionals.

Then half the ſumm of the extremes increaſed with half their difference gives the greater extreme, and half the ſaid ſumm leſſened by half the ſaid difference leaves the leſſer extreme.

Laſtly, the ſquare Root of the Product made by the mutual multiplication of the extremes is the mean Proportional.

Therefore if 16 be given for the difference of the extremes of three Proportionals, and 364 for the ſumm of the Squares of all the three Proportionals, the Proportionals are alſo given ſeverally, to wit, 2, 6, 18 $\div\div$.

QVEST. 4.

One extreme (b) of three proportional Quantities being given, as alſo (c) the ſumm of the Squares of the other extreme and the mean; to find out that other extreme and mean.

RESOLVTION.

1. For the extreme Proportional ſought put . . . . . a
2. Which multiplied by the given extreme (b) produceth the Square of the mean, to wit, ... ba
3. But from the firſt ſtep the Square of the extreme Proportional ſought is . . . . aa
4. Therefore from the ſecond and third ſteps the ſumm of the Squares of the two Proportionals ſought is . . . $aa + ba$
5. Which ſumm in the laſt ſtep muſt be equal to (c) the ſumm given in the Queſtion; hence this Equation ariſeth, *viz.* $aa + ba = c$
6. Which Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. of my firſt Book of *Algebraick Elements*, will diſcover the extreme Proportional ſought, to wit, . . . $a = \sqrt{:c + \frac{1}{4}bb:} - \frac{1}{2}b$.

The laſt Equation in words is this

CANON.

To the given ſumm add the Square of half the extreme Proportional given, and out of this ſumm extract the ſquare Root; then this ſquare Root leſſened by half the given extreme will give the other extreme.

Therefore if 18 be given for one of the extremes of three Proportionals, and 40 for the ſumm of the Squares of the other two Proportionals, the Canon will diſcover 2 for the extreme ſought. Laſtly, the ſquare Root of the Product of the extremes, to wit, 6 is the mean ſought. Therefore the three Proportionals are 18, 6 and 2.

QVEST. 5.

The difference (b) between the extremes of three proportional Quantities being given, as alſo the Proportion which the difference of the Squares of the extremes hath to the ſumm of the Squares of all the three Proportionals, ſuppoſe the difference be to the ſumm as (r) to (s;) to find the Proportionals. But (r) muſt be leſs than (s.)

RESOLVTION.

1. For the ſumm of the extremes put . . . . a
2. Then for as much as their difference is given . . . . b
3. Therefore the difference of the Squares of the extremes ſhall be ba; (for the Product of the multiplication of the ſumm of any two numbers into their difference is equal to the difference of their Squares;) . . . . ba

4. Then

4. Then from the first and second steps, (by the *Theor.* of *Quest.* 1. *Chap.* 14. of my First Book of *Algebraical Elements*,) the greater extreme shall be . . . . . . $\frac{1}{2}a + \frac{1}{2}b$
5. And (by the same *Theor.*) the lesser extreme shall be . . $\frac{1}{2}a - \frac{1}{2}b$
6. Therefore from the fourth step the Square of the greater extreme is . . . . $\frac{1}{4}aa + \frac{1}{4}bb + \frac{1}{2}ba$
7. And from the fifth step the Square of the lesser extreme is $\frac{1}{4}aa + \frac{1}{4}bb - \frac{1}{2}ba$
8. And because the Product made by the mutual multiplication of the extremes is equal to the Square of the mean, therefore the extremes in the fourth and fifth steps being multiplied one by the other, will give the Square of the mean, to wit, $\frac{1}{4}aa - \frac{1}{4}bb$
9. Therefore by adding together the Squares in the three last steps, the summ of the Squares of the three Proportionals shall be . . . . $\frac{3}{4}aa + \frac{1}{4}bb$
10. Then according to the Question, As r is to s, so must the difference in the third step be to the summ in the ninth step; hence this Analogy ariseth, *viz.*

$$r \;.\; s \;::\; ba \;.\; \tfrac{3}{4}aa + \tfrac{1}{4}bb,$$

11. Whence, by comparing the Product made by the mutual multiplication of the extremes to the Product of the means, this Equation comes forth, *viz.*

$$sba = \tfrac{3}{4}raa + \tfrac{1}{4}rbb.$$

12. From which Equation, after due Reduction, there will arise

$$\frac{4sb}{3r}a - aa = \frac{bb}{3}.$$

13. Therefore (*per* Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) the two Roots or values of a in the last Equation are these, to wit,

$$a = \frac{2sb + \sqrt{:4ssbb - 3rrbb:}}{3r} \text{ the greater.} \quad a = \frac{2sb - \sqrt{:4ssbb - 3rrbb:}}{3r} \text{ the lesser.}$$

14. But the greater of those two values of (a) is the desired summ of the extreme Proportionals sought; for if we should suppose the lesser value to be the summ of the extremes, it ought to exceed (b) the difference of the extremes; but from that supposition it will follow, that (r) is greater than (s,) and consequently that the difference of the Squares of the extremes is greater than the summ of the Squares of all the three Proportionals; which is impossible. Now to prove the said consequence;

15. Suppose . . . . $\frac{2sb - \sqrt{:4ssbb - 3rrbb:}}{3r} \sqsubset b.$
16. Then by multiplying each part by $3r$, it follows, that . . . . $2sb - \sqrt{:4ssbb - 3rrbb:} \sqsubset 3rb.$
17. And by adding $\sqrt{:4ssbb - 3rrbb:}$ to each part in the sixteenth step, . . $2sb \sqsubset 3rb + \sqrt{:4ssbb - 3rrbb:}$
18. And by subtracting $3rb$ from each part in the seventeenth step, ... $2sb - 3rb \sqsubset \sqrt{:4ssbb - 3rrbb:}$
19. And by squaring each part in the eighteenth step, ... $4ssb - 12srbb + 9rrbb \sqsubset 4ssbb - 3rrbb.$
20. And by adding $3rrbb$ to each part in the nineteenth step, ... $4ssbb - 12srbb + 12rrbb \sqsubset 4ssbb.$
21. And by adding $12srbb$ to each part in the twentieth step, ... $4ssbb + 12rrbb \sqsubset 4ssbb + 12srbb.$
22. And by subtracting $4ssbb$ from each part in the twenty-first step, ... $12rrbb \sqsubset 12srbb.$
23. Wherefore by dividing each part in the twenty-second step by $12rbb$, ... $r \sqsubset s.$
24. Thus, from a supposition that the lesser value of (a) in the thirteenth step is greater than (b) the given difference of the extremes, it follows by just consequence that (r) is greater than (s,) which is impossible; for in regard the difference of the Squares of the extremes is less than the summ of the Squares of all three Proportionals, and that according to the Question the said difference is to the said summ as (r) to (s,) therefore (r) is less than (s;) and because the series of Inferences drawn from the said Supposition ends in an Impossibility, therefore that which was supposed cannot be true; *viz.* The lesser value

value of (a) is not greaterthan (b) the given difference of the extremes, and consequently it cannot be equal to the summ of the extremes. Which was to be proved.

But by the like argumentation it may be proved that the greater value of (a) in the thirteenth step exceeds (b) the given difference of the extremes; and if it be exprest by Words, it will give the following Canon to find out the summ of the extreme Proportionals sought; whence by the help of the given difference of the extremes, the extremes are severally given.

CANON.

From four times the Square of the latter or greater term (s) of the given Reason subtract thrice the Square of the first term (r,) and multiply the Remainder by the Square of the given difference of the extreme Proportionals sought; then add the square Root of that Product to the double of the Product made by the multiplication of the latter term (s) into the difference of the extremes, and divide the summ of that addition by the triple of the first term (r;) so shall the Quotient be the summ of the extreme Proportionals: lastly, half the summ of the extremes increased with half their difference gives the greater extreme, but the said half summ lessened by the said half difference leaves the lesser extreme.

As, for example, if 6 be given for the difference of the extremes of three continual Proportionals, and the difference of the Squares of the extremes hath such proportion to the summ of the Squares of all the three Proportionals as 5 to 7, then by the Canon, the three Proportionals will be found 2, 4 and 8.

Again, if $2\frac{1}{4}$ be given for the difference of the extremes, and the difference of the Squares of the extremes be to the summ of the Squares of all the three Proportionals as 123 to 427, the Proportionals will be found 4, 5, and $6\frac{1}{4}$.

QUEST. 6.

The summ (b) of the extremes, and the summ (c) of the means of four Quantities in continual proportion being given; to find out the Proportionals: but (b) must exceed (c.)

RESOLUTION.

1. For one of the means put . . . . a
2. Then by subtracting that mean from (c) the given summ of the means, the Remainder is the other mean, to wit, . . $c - a$
3. And by dividing the Square of the latter mean by the former, the Quotient gives one of the extremes, to wit, . . $\dfrac{cc - 2ca + aa}{a}$
4. In like manner the Square of the first mean (a) being divided by the other mean ($c-a$,) gives the other extreme, to wit, $\dfrac{aa}{c-a}$
5. Therefore from the third and fourth steps the summ of the two extremes is . . . . $\dfrac{ccc - 3cca + 3caa}{ca - aa}$
6. Which summ must be equal to (b) the given summ of the extremes; hence this Equation ariseth; to wit, . . . $\dfrac{ccc-3cca+3caa}{ca-aa} = b$
7. From which Equation after due Reduction this ariseth, to wit, ... $\dfrac{ccc}{3c+b} = ca - aa$
8. Wherefore by resolving the last Equation by the Canon in *Sect.* 10. *Chap.* 15. *Book* I. the two values of (a,) to wit, the mean Proportionals sought will be made known, *viz.*

$$a = \tfrac{1}{2}c + \sqrt{:\frac{cc}{4} - \frac{ccc}{3c+b}}: \quad \text{the greater mean;}$$

$$a = \tfrac{1}{2}c - \sqrt{:\frac{cc}{4} - \frac{ccc}{3c+b}}: \quad \text{the lesser mean.}$$

Which values of (a) give this

CANON.

Divide the Cube of the summ of the means by the aggregate of the triple summ of the means and the summ of the extremes; subtract the Quotient from the Square of half the summ of the means, and extract the square Root of the Remainder; then the said square Root being added to and subtracted from half the summ of the means, the Summ and Remainder shall be the means sought.

Then

Then the Square of the leſſer mean being divided by the greater will give the leſſer extreme; and the Square of the greater mean divided by the leſſer gives the greater extreme.

Therefore if 18 be given for the ſumm of the extremes, and 12 for the ſumm of the means of four continual Proportionals, the Proportionals are given ſeverally by the ſaid Canon, to wit, 2, 4, 8 and 16.

QVEST. 7.

The difference (b) of the extremes, and the difference (c) of the means of four Quantities continually proportional being given; to find out the four Proportionals.

RESOLVTION.

1. For the leſſer mean Proportional put	a
2. Which added to (c) the given difference of the means gives the greater mean, to wit,	$c+a$
3. Then the Square of the ſaid greater mean being divided by the leſſer, gives for the greater extreme	$\frac{cc+2ca+aa}{a}$
4. Likewiſe by dividing (aa) the Square of the leſſer mean by the greater, there ariſeth for the leſſer extreme	$\frac{aa}{c+a}$
5. Therefore the difference of the two extremes in the third and fourth ſteps is	$\frac{ccc+3cca+3caa}{ca+aa}$
6. Which difference muſt be equal to (b) the given difference of the extremes, hence this Equation ariſeth, *viz.*	$\frac{ccc+3cca+3caa}{ca+aa}=b$
7. From which Equation, after due Reduction, this ariſeth, to wit,	$\frac{ccc}{b-3c}=ca+aa$

8. Wherefore by reſolving the laſt Equation by the Canon in *Sect. 6. Chap. 15. Book 1.* the value of (a,) to wit, the leſſer mean Proportional ſought will be made known, *viz.*

$$a=\sqrt{:\frac{cc}{4}+\frac{ccc}{b-3c}:}-\tfrac{1}{2}c.$$

Which Equation in words is this

CANON.

Divide the Cube of the given difference of the means by the exceſs of the given difference of the extremes above the triple of the difference of the means; add the Quotient to the Square of half the difference of the means: then from the ſquare Root of that ſumm ſubtract half the difference of the means, ſo ſhall this Remainder be the leſſer mean.

Then to the leſſer mean add the difference of the means, and the ſumm is the greater. Laſtly, the Square of the greater mean divided by the leſſer gives the greater extreme, and the Square of the leſſer mean divided by the greater gives the leſſer extreme.

Therefore if 52 be given for the difference of the extremes of four continual Proportionals, and 12 for the difference of the means, the Proportionals will be found 2, 6, 18, 54.

QVEST. 8.

The ſumm (b) of four Quantities in continual proportion being given, as alſo (c) the ſumm of their Squares; to find the Proportionals.

RESOLVTION.

1. For the ſumm of the means put	a
2. Which ſubtracted from (b) the given ſumm of all the four Proportionals, leaves the ſumm of the extremes; to wit,	$b-a$
3. The Square of (b) the given ſumm of all the four Proportionals is	bb
4. Now (according to *Theor.* 16. of the preceding *Chap.* 6.) from the ſaid Square (bb) I ſubtract (c) the given ſumm of the Squares of the four Proportionals, and from the half of the Remainder I alſo ſubtract (aa) the Square of the ſumm of the means, ſo this Quantity remains, to wit,	$\tfrac{1}{2}bb-\tfrac{1}{2}c-aa$

5. Which Remainder, to wit, $\tfrac{1}{2}bb-\tfrac{1}{2}c-aa$, (by the ſaid *Theor.* 16.) ſhall be to the given ſumm of the Squares of the four Proportionals, as the ſumm of the means is to the ſumm of all the four Proportionals; hence this Analogy ariſeth, *viz.*

$$\tfrac{1}{2}bb-\tfrac{1}{2}c-aa\ .\ c\ ::\ a\ .\ b$$

6. Which

6. Which Analogy, by comparing the Product made by the mutual multiplication of the extremes to the Product of the means, will be converted into this Equation, *viz.*

$$\tfrac{1}{2}bbb - \tfrac{1}{2}bc - baa = ca$$

7. Whence after due Reduction this Equation ariseth, to wit,

$$\tfrac{1}{2}bb - \tfrac{1}{2}c = aa + \frac{c}{b}a$$

Which Equation being resolved (*per* Canon in *Sect.* 6. *Chap.* 15. *Book* 1.) gives this following

CANON.

From the Square of the given summ of the four Proportionals subtract the given summ of their Squares, and to the half of the Remainder add the Square of half the Quotient that ariseth by dividing the summ of the Squares of the four Proportionals by the summ of the four Proportionals. Then extract the square Root of the summ of that addition, and from the said square Root subtract half the Quotient aforesaid, so shall the Remainder be the summ of the two desired mean Proportionals.

Then the summ of the means of four continual Proportionals being given, as also the summ of the extremes, the Proportionals shall be given severally by the Canon of the preceding *Quest.* 6. of this *Chapt.*

So if 30 be given for the summ of four Proportionals, and 340 for the summ of their Squares; first, by the Canon above exprest, the summ of the means will be found 12; which subtracted from 30 the given summ of the four Proportionals, leaves 18 for the summ of the extremes: then the summ of the means being given 12, and the summ of the extremes 18, the four Proportionals (by the Canon of the preceding sixth Question,) will be found 2, 4, 8, 16.

QUEST. 9.

The summ (b) of four Quantities in continual proportion being given, as also (c) the summ of the Squares of the means; to find the Proportionals.

RESOLUTION.

1. For the summ of the means put . . . . a
2. Then, because (by *Theor*: 12. of the preceding *Chap.*6.) the summ of four Quantities continually proportional is to the summ of the means, as the summ of the Squares of the means is to the Product made by the mutual multiplication of the means or extremes, say, by the Rule of Three,

   If . . . . . b . a :: c . $\frac{ca}{b}$.

   Whence the Product of the means or extremes is found . . . $\frac{ca}{b}$
3. And because if from the Square of the summ of the means there be subtracted the summ of the Squares of the means, there will remain the double Product of the means or extremes; therefore if from (aa) you subtract (c,) the half of the Remainder shall be the Product of the means or extremes, to wit, ... $\tfrac{1}{2}aa - \tfrac{1}{2}c$
4. Which Product, to wit, $\tfrac{1}{2}aa - \tfrac{1}{2}c$ must be equal to $\frac{ca}{b}$ the Product in the second step; hence this Equation ariseth, to wit, $\tfrac{1}{2}aa - \tfrac{1}{2}c = \frac{ca}{b}$
5. From which Equation after due Reduction there ariseth . . . $aa - \frac{2c}{b}a = c$

Which last Equation being resolved (by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1.) gives this following

CANON.

To the given summ of the Squares of the means add the Square of the Quotient that ariseth by dividing the said summ by the given summ of the four Proportionals, and out of the summ made by that addition extract the square Root; then this square Root added to the aforesaid Quotient gives the summ of the mean Proportionals sought.

Then the summ of the means being given, as also the summ of the extremes, (for the summ of the means found out being subtracted from the given summ of all the four Proportionals leaves the summ of the extremes,) the four Proportionals will be discovered by the Canon of the sixth Question of this Chapter.

Therefore,

Therefore, If 30 be given for the ſumm of four continual Proportionals, and 80 for the ſumm of the Squares of the means, the four Proportionals are alſo ſeverally given; to wit, 2, 4, 8, 16; by the Canon above-expreſt.

QVEST. 10.

The ſumm (b) of four Quantities continually proportional being given, as alſo (c) the ſumm of the Squares of the extremes; to find out the Proportionals.

RESOLVTION.

1. For the ſumm of the means put . . . . a
2. Which ſubtracted from (b) the given ſumm of the four Proportionals leaves the ſumm of the extremes, to wit, ... $b - a$
3. Therefore the Square of the ſumm of the extremes is . . . $bb - 2ba + aa$
4. From which Square, if (c) the given ſumm of the Squares of the extremes be ſubtracted, there will remain the double Product made by the mutual multiplication of the extremes or means; therefore the Product of the means is . . . . $\frac{bb - 2ba + aa - c}{2}$
5. And, becauſe if from aa the Square of the ſumm of the means there be ſubtracted $bb - 2ba + aa - c$ the double Product of the means, there will remain the ſumm of the Squares of the means; therefore the ſumm of the Squares of the means is . . . . $2ba - bb + c$
6. And becauſe by *Theor.* 12. in the preceding *Chap.* 6. the ſumm of the Squares of the means is to the Product of the means, as the ſumm of all the four Proportionals is to the ſumm of the means; therefore from the premiſes this following Analogy ariſeth, *viz.*

$$2ba - bb + c \;.\; \frac{bb - 2ba + aa - c}{2} :: b \;.\; a$$

7. From which Analogy, by comparing the Product of the extremes to the Product of the means, this Equation ariſeth, *viz.*

$$2baa - bba + ca = \frac{bbb - 2bba + baa - bc}{2}$$

8. Which Equation, after due Reduction, gives this following Equation, *viz.*

$$aa + \frac{2c}{3b}a = \frac{bb - c}{3}.$$

Whence (*per* Canon in *Sect.* 6. *Chap.* 15. *Book* 1.) there ariſeth this following

CANON.

Divide the given ſumm of the Squares of the extremes by the triple of the given ſumm of all the four Proportionals, and to the Square of the Quotient add one third part of the exceſs of the Square of the ſumm of the four Proportionals above the ſumm of the Squares of the extremes; then from the ſquare Root of the ſumm made by that Addition ſubtract the Quotient firſt found out: ſo ſhall the Remainder be the deſired ſumm of the mean Proportionals.

Then the ſumm of the means being given, as alſo the ſumm of the extremes, (for the ſumm of the means being ſubtracted from the given ſumm of the four Proportionals leaves the ſumm of the extremes,) the four Proportionals will be diſcovered by the Canon of the ſixth Queſtion of this Chapter.

Therefore, If 80 be given for the ſumm of four continual Proportionals, and 2920 for the ſumm of the Squares of the extremes, the four Proportionals will be found 2, 6, 18, 54.

QVEST. 11.

The ſumm (b) of the Squares of the extremes of four Quantities in continual proportion being given, as alſo (c) the ſumm of the Squares of the means; to find out the Proportionals.

RESOLVTION.

1. Add the two given ſumms into one, that you may have the ſumm of the Squares of the four Proportionals ſought, for which laſt mentioned ſumm put . . . . d
2. Then for the ſumm of the Squares of the firſt and ſecond Proportionals put . . . . a
3. Therefore the ſumm of the Squares of the third and fourth Proportionals is . . . . $d - a$

4. Then, because (by *Theor.*13. of the preceding *Chap.*6.) the summ of the Squares of the two means is a mean Proportional between the summ of the Squares of the first and second, and the summ of the Squares of the third and fourth, this Analogy is manifest, *viz.* $a . c :: c . d-a$
5. Therefore by comparing the Product made by the multiplication of the extremes of that Analogy to the Product of the means, this Equation ariseth, *viz.* . . . . $da - aa = cc$
6. Which Equation being resolved by the Canon in *Sect.* 10. *Chap.* 16. *Book* 1. gives this following

CANON.

Add the given summ of the Squares of the extremes to the given summ of the Squares of the means, and reserve half of the summ: from the Square of this half summ subtract the Square of the summ of the Squares of the means and extract the square Root of the Remainder: add this square Root to the half summ before reserved, and also subtract it from the same half sum; so the Summ shall be the summ of the Squares of the first and second Proportionals, and the Remainder shall be the summ of the Squares of the third and fourth.

Then (according to *Theor.* 3. of the preceding *Chap.* 6.) add severally the summ of the Squares of the first and second Proportionals, and the summ of the Squares of the third and fourth, to the summ of the Squares of the means, and out of each summ extract the square Root; so shall one of these Roots be the summ of the first and third Proportionals, and the other shall be the summ of the second and fourth: which two last mentioned summs being added together give the summ of the four Proportionals sought.

Lastly, the summ of four Proportionals being given, as also the summ of the Squares of the means, the Proportionals shall be given severally by the ninth Question of this *Chapt.*

Therefore if 260 be given for the summ of the Squares of the extremes of four continual Proportionals, and 80 for the summ of the Squares of the means, the Proportionals will be found 16, 8, 4, 2.

QUEST. 12.

The summ (b) of the extremes of four Quantities in continual proportion being given, as also (c) the summ of the Cubes of the means; to find out the Proportionals.

RESOLUTION.

1. For one of the extreme Proportionals put . . . . . . a
2. Then the other extreme, by subtracting (a) from (b) the given summ of the extremes, shall be . . . . $b-a$
3. Therefore the Product made by the mutual multiplication of the extremes is . . . . $ba-aa$
4. And because (*per* Theor. 21. of the preceding *Chap.* 6.) the Product made by the multiplication of the means or extremes into the summ of the extremes, is equal to the summ of the Cubes of the means; therefore if you multiply $ba-aa$ by b, this Product shall be equal to (c) the given summ of the Cubes of the means; hence ariseth this Equation, *viz.* . . . . $bba - baa = c$
5. And by dividing every term of that Equation by (b,) there ariseth . . . . $ba - aa = \frac{c}{b}$

Which last Equation being resolved (by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) gives this following

CANON.

6. From the Square of half the given summ of the extremes subtract the Quotient that ariseth by dividing the given summ of the Cubes of the means by the summ of the extremes, and extract the square Root of the Remainder, then half the summ of the extremes being increased & also lessened by the said square Root, gives the extremes severally.

Then you may find out the means by a new work, thus,

7. Let the greater extreme found out as above be . . . . . f
8. And the lesser extreme . . . . g
9. Then for the greater mean put . . . . a
10. Therefore by dividing (aa) the square of the greater mean by the greater extreme (f,) the Quotient shall be the lesser mean, to wit, $\frac{aa}{f}$

11. But

11. But the Square of the leſſer mean is equal to the Product of the leſſer extreme multiplied by the greater mean; therefore from the three laſt preceding ſteps this Equation ariſeth, *viz.* . . . . $\frac{aaaa}{ff} = ga$

12. Which Equation, after due Reduction, gives . . . . $aaa = ffg$

13. Therefore by extracting the cubick Root out of each part of the laſt Equation the greater mean is made known, *viz.* . . $a = \sqrt{(3)}ffg$

Which laſt Equation, together with that in the tenth ſtep, will give this

CANON.

14. Multiply the Square of the greater extreme by the leſſer, then the cubick Root of the Product ſhall be the greater mean. Laſtly, the Square of the greater mean divided by the greater extreme gives the leſſer mean.

Therefore if 18 be given for the ſumm of the extremes of four numbers in contiuual proportion, and 576 for the ſumm of the Cubes of the means, then by the firſt Canon of this Queſtion the extremes will be found 16 and 2; and laſtly, by the latter Canon, the means will be found 8 and 4: wherefore the four continual Proportionals ſought are 16, 8, 4, 2.

QUEST. 13.

The ſumm (b) of the Cubes of the extremes of four Quantities in continual proportion being given, as alſo (c) the ſumm of the Cubes of the means; to find the four Proportionals.

RESOLUTION.

1. For the ſumm of the extremes put . . . . a
2. Therefore the Cube of that ſumm is . . . . aaa
3. Then becauſe by *Theor.* 22. of the preceding *Chapt.* 6. if four Quantities be continually proportional, the ſumm of the Cubes of the extremes more by the triple of the Cubes of the means is equal to the Cube of the ſumm of the extremes; therefore if to b you add $3c$, it gives the Cube of the ſumm of the extremes, which Cube muſt be equal to aaa; hence this Equation, ... $b + 3c = aaa$
4. Therefore by extracting the cubick Root out of each part of that Equation, the ſumm of the extremes is made known, *viz.* . . $\sqrt{(3)}:\overline{b+3c}: = a$

Which laſt Equation in words is this following

CANON.

Add the triple of the given ſumm of the Cubes of the means to the given ſumm of the Cubes of the extremes, and out of the ſumm made by that Addition extract the cubick Root, which ſhall be the ſumm of the extremes ſought.

Then the ſumm of the extremes being given, as alſo the ſumm of the Cubes of the means, the four Proportionals ſhall be given ſeverally by the Canon of the preceding twelfth Queſtion. As, for example, if 157472 be given for the ſumm of the Cubes of the extremes of four numbers in continual proportion, and 6048 for the ſumm of the Cubes of the means; firſt, by the Canon of this Queſtion the ſumm of the extremes will be found 56, and then by the Canon of the preceding twelfth Queſtion, the four Proportionals will be found 2, 6, 18, 54.

QUEST. 14.

The ſumm of the extremes (b) of five Quantities in continual proportion being given, as alſo (c) the ſumm of the three means; to find the five Proportionals.

RESOLUTION.

1. For the third Proportional, that is, the middle term of all the five, put . . . . a
2. Then ſubtract that middle term (a) from (c) the given ſumm of the three means, and there will remain the ſumm of the ſecond and fourth, *viz.* . . . . $c - a$
3. And becauſe (by *Theor.* 29. of the preceding *Chap.* 6.) the ſumm of the extremes of five continual Proportionals together with the double of the mean, the ſumm of the ſecond and fourth, and the mean, are alſo in continual proportion; therefore this Analogy is manifeſt, *viz.* . . . . $b + 2a \,.\, c - a :: c - a \,.\, a$

4. From which Analogy, by comparing the Product made by the multiplication of the extremes to the Product of the means, this Equation is produced, *viz.* $ba + 2aa = cc - 2ca + aa$

5. Which Equation, after due Reduction, gives . . $aa + ba + 2ca = cc$.

Lastly, by resolving the last Equation according to the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. there will arise this following

CANON.

Add the summ of the extremes to the double of the summ of the three means, and take the half of the summ made by such Addition; then to the Square of the said half summ add the Square of the summ of the three means, and out of this summ extract the square Root; from which Root subtract the half summ first taken, and the Remainder shall be the middle (or third) Proportional of the five sought.

Then by subtracting the said third Proportional from the summ of the three means, the Remainder is the summ of the second and fourth; by which Summ and the third Proportional, the second and fourth shall be given severally, (by the Canon of *Quest.* 4. *Chap.* 16. *Book* 1.) Then the Square of the second Proportional being divided by the third gives the first, and the Square of the fourth being divided by the third gives the fifth.

Therefore, if 34 be given for the summ of the first and fifth of five continual Proportionals, and 28 for the summ of the three means, the five Proportionals shall be given severally, *viz.* 2, 4, 8, 16, 32 $\div\!\div$.

QUEST. 15.

The summ (b) of the first, third and fifth of five Quantities in continual proportion being given, as also (c) the summ of the second and fourth; to find the five Proportionals.

RESOLUTION.

1. For the third Proportional, that is, the middle term of the five, put a
2. Then subtract that middle term (a) from the given summ (b) and the Remainder is the summ of the first and fifth, *viz.* . . $b - a$
3. And because (by *Theor.* 27. of the preceding *Chap.* 6.) the Product made by the multiplication of the third or middle term of five continual Proportionals into the summ of the first and fifth is equal to the Squares of the second and fourth; therefore (from the first and second steps) the summ of the Squares of the second and fourth Proportionals is . . . . $ba - aa$
4. The Square of the third Proportional (a) is equal to the Product of the second multiplied into the fourth; therefore the double of that Product is . . . . $2aa$
5. Therefore, from the two last steps, the Aggregate of the Squares and the double Product of the second and fourth Proportionals is $aa + ba$
6. But the Aggregate of the Squares and the double Product of the second and fourth Proportionals is equal to the Square of their summ, therefore the Aggregate in the fifth step must be equal to the Square of the given summ (c,) *viz.* . . . . $aa + ba = cc$.

Which Equation being resolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. will give this following

CANON.

Add the Square of half the given summ of the first third, and fifth Proportionals to the Square of the given summ of the second and fourth; then from the square Root of the summ made by that Addition subtract the said half summ, and the Remainder shall be the third Proportional.

Then by subtracting the said third Proportional from the given summ of the first, third and fifth, the Remainder is the summ of the first and fifth; by which summ and the third (or mean) Proportional, the first and fifth, (to wit, the extremes) shall be given severally by the Canon of *Quest.* 4. *Chap.* 16. *Book* 1. Then the third Proportional being multiplied into the first and fifth severally, and the square Root being extracted out of each Product, these Roots shall be the second and fourth Proportionals.

Therefore, if 42 be given for the summ of the first, third and fifth of five numbers in continual Proportion, and 20 for the summ of the second and fourth, the five Proportionals will be found these, to wit, 2, 4, 8, 16, 32.

QUEST. 16.

The third Proportional (b) of five Quantities in continual proportion being given; as alfo (c) the fumm of the other four; to find out the five Proportionals.

RESOLUTION.

1. For the fumm of the fecond and fourth Proportionals put . . . a
2. Then fubtract that fumm (a) from (c) the given fumm of the firft, fecond, fourth and fifth Proportionals, and there will remain the fumm of the firft and fifth, to wit, ... $c - a$
3. The Square of the third (that is, of the mean) Proportional (b) is equal to the Product of the fecond multiplied into the fourth, therefore the double of that Product is . . . . $2bb$
4. Which double Product ($2bb$) fubtracted from (aa) the Square of the fumm of the fecond and fourth Proportionals, leaves for the fumm of the Squares of the fecond and fourth, ... $aa - 2bb$
5. And becaufe (by *Theor.* 33. of the preceding *Chap.* 6.) the fumm of the Squares of the fecond and fourth of five continual Proportionals is equal to the Product of the third (or mean) multiplied by the fumm of the firft and fifth; therefore, if ($aa - 2bb$) the fumm of the Squares of the fecond and fourth be divided by the mean (b) the Quotient fhall be the fumm of the firft and fifth, *viz.* $\frac{aa - 2bb}{b}$
6. Which fumm found out in the laft ftep, muft be equal to the fumm of the firft and fifth Proportionals found out in the fecond ftep; hence this Equation arifeth, *viz.* . . . . $\frac{aa - 2bb}{b} = c - a$
7. Which Equation, after due Reduction, gives . . . . . . $aa + ba = 2bb + bc.$

Wherefore by refolving the laft Equation (according to the Canon in *Sect.* 6. *Chap.* 15. *Book* 1.) there will come forth this following

CANON.

To the Square of the half of the given third (or mean) Proportional add the double of the Square of the faid mean, as alfo the Product of the faid mean multiplied into the given fumm of the other four Proportionals, and out of the fumm of that Addition extract the fquare Root; this Root leffened by half the given mean, gives the fumm of the fecond and fourth Proportionals.

Then from the given fumm of the firft, fecond, fourth and fifth Proportionals fubtract the fumm of the fecond and fourth (found out as above,) and the Remainder is the fumm of the firft and fifth; by which fumm and the third (or mean) Proportional, the faid firft and fifth fhall be given feverally by the Canon of *Queft.* 4. *Chap.* 16. *Book* 1.

Laftly, the fquare Roots of the Product of the firft multiplied into the third, and of the Product of the third into the fifth, fhall be the fecond and fourth Proportionals.

Therefore, if 8 be given for the third of five numbers in continual proportion, and 54 for the fumm of the other four; the five Proportionals will be found thefe, to wit, 2, 4, 8, 16, 32.

QUEST. 17.

The fumm (b) of the extremes of five Quantities in continual proportion being given, as alfo (c) the fumm of the Squares of the three means; to find the five Proportionals.

RESOLUTION.

1. For the mean (or third) Proportional put . . . . . . a
2. Then (by *Theor.* 33. of the preceding *Chap.* 6.) the mean (a) multiplied by (b) the given fumm of the extremes, produceth the fumm of the Squares of the fecond and fourth Proportionals, *viz.* ba
3. Therefore if to (aa) the Square of the mean, you add (ba) the fumm of the Squares of the fecond and fourth, there will come forth the fumm of the Squares of the fecond, third and fourth Proportionals, *viz.* . . . . $aa + ba$
4. Which fumm found out in the laft ftep muft be equal to the given fumm (c;) hence this Equation arifeth, *viz.* . . . . . . $aa + ba = c.$

Wherefore,

Wherefore by resolving that Equation (according to the Canon in *Sect. 6. Chap. 15. Book 1.*) there will arise this following

CANON.

Add the Square of half the given summ of the extremes to the given summ of the Squares of the three means, and out of the summ of that Addition extract the square Root; this Root lessened by half the summ of the extremes, will give the mean (or third) Proportional.

Then the mean (or third) Proportional being given, and the summ of the extrmes, (*viz.* of the first and fifth,) the said extremes shall be given severally by the Canon of *Quest. 4. Chap. 16. Book 1.*

Lastly, the square Roots of the Products of the first into the third, and of the third into the fifth shall be the second and fourth Proportionals.

Therefore, if 34 be given for the summ of the extremes of five numbers in continual Proportion, and 336 for the summ of the Squares of the three means, the five Proportionals shall be also given, to wit, 2, 4, 8, 16, 32.

QUEST. 18.

The summ (b) of the extremes of five Quantities in continual proportion being given; as also (c) the summ of the Squares of the second and fourth; to find the five Proportionals.

RESOLUTION.

1. For the mean Proportional put . . . . a
2. Then (by *Theor.* 33. of the preceding *Chap. 6.*) the mean (a) multiplied by (b) the summ of the extremes, produceth the summ of the Squares of the second and fourth, *viz.* . . . . ba
3. Which summ must be equal to the given summ (c,) therefore $ba = c$
4. Wherefore, by dividing each part of that Equation by (b,) the mean Proportional will be made known, *viz.* . . . . $a = \frac{c}{b}$.

Which last Equation, in words, is this following

CANON.

Divide the given summ of the Squares of the second and fourth Proportionals by the given summ of the first and fifth, so shall the Quotient be the mean or third Proportional.

Then the mean (or third) Proportional being given, as also the summ of the first and fifth, these shall be given severally by the Canon of *Quest. 4. Chap. 16. Book 1.*

Lastly, the square Roots of the Products of the first into the third, and of the third into the fifth shall be the second and fourth Proportionals.

Therefore, if 34 be given for the summ of the extremes of five numbers in continual proportion, and 272 for the summ of the Squares of the second and fourth, the Proportionals will be discovered severally, *viz.* 2, 4, 8, 16, 32.

QUEST. 19.

A Vintner having a vessel full of Wine containing 16 (or b) Gallons, draws out 4 (or c) Gallons, and then pours into the vessel as much Water as he drew out Wine; then out of that mix'd quantity of Wine and Water he draws out the same number of Gallons as before, and pours in the same quantity of Water; again he makes a third draught of the same quantity as at first: The Question is, to find how much pure Wine remained in the vessel after the third draught.

RESOLUTION.

1. The number of Gallons of Wine in the vessel at first was b
2. Out of which quantity, (c) Gallons being drawn, there remained of pure Wine in the vessel . . . . . . $b - c$
3. To which remaining quantity of pure Wine, (c) Gallons of Water being added, the vessel is again full, and contains (b) Gallons of Wine and Water together; out of which drawing again (c) Gallons, we must seek how much pure Wine was in this second draught; saying by the Rule of Three,

$$\begin{array}{lcccc} & \textit{mixt,} & \textit{Wine,} & \textit{mixt,} & \textit{Wine.} \\ \text{If} & b \;. & b - c \;:: & c \;. & \left(\frac{bc - cc}{b}\right. \end{array}$$

Whence it is found, that the quantity of pure Wine in the second draught was . . . . $\frac{bc - cc}{b}$

4. Which

4. Which quantity $\frac{bc-cc}{b}$ being subtracted from $b-c$ the quantity of pure Wine in the vessel before the second draught was made, there remains for the quantity of pure Wine in the vessel after the second draught, ... $\frac{bb-2bc+cc}{b}$

5. To which remaining quantity of pure Wine add (c) Gallons of Water, so the vessel is again full, and contains (b) Gallons of Wine and Water together; out of which drawing again (c) Gallons, we must seek how much pure Wine was in this third draught, saying,

As $\overset{mixt,}{b} \; . \; \overset{Wine,}{\frac{bb-2bc+cc}{b}} \; :: \; \overset{mixt.}{c}$. to a fourth Proportional or quantity of pure Wine in the third draught, which will be found . . . . $\frac{bbc-2bcc+ccc}{bb}$

6. Then by subtracting the said fourth Proportional or quantity of pure Wine in the third draught, from $\frac{bb-2bc+cc}{b}$ the quantity of pure Wine in the vessel when the third draught was made, there remains for the desired quantity of pure Wine in the vessel after the third draught . . . . . $\frac{bbb-3bbc+3bcc-ccc}{bb}$

Which Quantity last found out is the Answer of the Question; and if it be resolved into numbers it gives $6\frac{3}{4}$ for the number of Gallons of pure Wine that remained in the vessel after the third draught. Moreover, if the first, second, fourth and sixth steps of the Resolution be well examined and compared with *Sect.* 2, 5, and 6. *Chap.* 5. of this Second Book, it will be manifest that the quantity of pure Wine in the vessel at first, and the several quantities of Wine remaining in the vessel after each draught are in Continual Proportion;

$$Viz. \begin{cases} b \; . \; b-c \; . \; \frac{bb-2bc+cc}{b} \; . \; \frac{bbb-3bbc+3bcc-ccc}{bb} \; \because \\ 16 \; . \; 12 \; . \; 9 \; . \; 6\frac{3}{4} \; \because \end{cases}$$

Of which continual Proportionals the first is the given quantity of Wine in the vessel at first; the second is the excess of the same quantity above the given quantity drawn out at each draught; and then the fourth continual Proportional is the quantity of pure Wine remaining in the vessel when three draughts have been made, according to the import of the Question; but the fifth continual Proportional when four draughts; the sixth when five draughts; the seventh when six draughts shall be the remaining quantity of pure Wine sought by the Question. Lastly, the first and the second Terms of a Rank of numbers in continual proportion being given, any of the following Terms shall be given by the Rule in *Sect.* 5, and 6. *Chap.* 5. of this Second Book.

QUEST. 20.

A Vintner having a vessel full of Wine containing 16 (or b) Gallons, draws out a certain quantity, and then pours into the vessel as much Water as he drew out Wine: again, out of that mixt quantity of Wine and Water he draws out the same quantity as before, and pours in the same quantity of Water: then he makes a third draught of the same quantity as at first, and after this third draught there remained $6\frac{3}{4}$ (or d) Gallons of pure Wine. The Question is, to find what quantity of pure Wine was drawn out at the first draught, or what quantity of Wine and Water together at the second or third draught, (for the three draughts were Equal quantities.)

RESOLUTION.

1. The number of Gallons of Wine in the vessel at first was . b
2. For the number of Gallons of Wine drawn out at the first draught put . . . . a
3. Then the quantity of Wine remaining in the vessel after the first draught was, ... $b-a$
4. By prosecuting the search as in the preceding nineteenth Question, saving that (a) is to be

be used here instead of (c) there, you will find this quantity, *viz.* $\frac{bbb-3bba+3baa-aaa}{bb}$ to be the number of Gallons of pure Wine remaining in the vessel after the third draught, and therefore it must be equal to the given quantity $6\frac{3}{4}$, (or d;) hence ariseth this Equation, *viz.*

$$\frac{bbb - 3bba + 3baa - aaa}{bb} = d,$$

5. Therefore by multiplying each part of that Equation by the Denominator bb, there will come forth this Equation in Integers, *viz.*

$$bbb - 3bba + 3baa - aaa = bbd,$$

6. And by extracting the Cubick Root out of each part of the last Equation, there ariseth

$$b - a = \sqrt{(3)}bbd,$$

7. Wherefore from the last Equation after due Transposition, the value of (a) will be made known, *viz.* $a = b - \sqrt{(3)}bbd = 4.$

Whence it is manifest that four Gallons were drawn out at every one of the three draughts. But if the Resolution had been wrought out at large, as in the preceding nineteenth Question, then it would appear, that if between (b) and (d,) *viz.* the quantity of Wine first given and the quantity of Wine remaining after the last draught, there be found the greater of two mean Proportionals when three draughts are proposed, or the greatest of three means when four draughts, and so forwards; then the mean so found out being subtracted from the greater extreme (b) leaves the Quantity drawn out at each draught. The manner of finding out mean Proportional numbers between any two numbers given for Extremes hath already been shewn in *Sect.* 14. *Chap.* 5. of this Second Book.

If the Reader desires more variety of Questions about Quantities in continual Proportion, he may consult the *Algebra* of *Jac. de Billy*, entituled *Nova Geometriæ Clavis* and the First Part of our Learned Dr. *Wallis* his Mathematical Works.

CHAP. VIII.

The manner of finding out all the Aliquot parts *both of Numbers and Algebraical Quantities, as also the smallest numbers that shall have given multitudes of* Aliquot parts.

I. IN the Resolution of knotty Questions about Quantity, there is oftentimes great use of finding out all the *Aliquot parts*, or *just Divisors*, as well of Numbers, as of Quantities represented by Letters; and therefore in this Chapter I shall shew how that work may be done; as also, how to find out the least number that shall have a given multitude of *Aliquot parts*, according to the method of *Fran. van Schooten* in *Sect.* 2, 3, and 4. of his *Miscellanies*, and in his *Principia Mathes. universal.*

II. A *Prime* or *Incompofit number* is that which can only be measured or divided by it self, or by Unity, and leave no Remainder: as, 2, 3, 5, 7, 11, 13, &c. are *Prime* numbers.

III. A *Composit number* is that which may be divided by some number less than the *Composit* it self, but greater than Unity: as, 4, 6, 8, 9, 10, &c. are *Composits.*

IV. *Just Divisors* are such numbers or quantities as will divide a given number or quantity and leave no Remainder; every one of which Divisors, except that which is equal to the given Quantity, is called an *Aliquot part*, because if it be taken (*Aliquoties*, that is,) certain times, it will precisely constitute the given Quantity: As, if 6 be a number proposed, its just Divisors are 1, 2, 3, and 6; but the Aliquot parts of 6 are only 1, 2, and 3: for 6 cannot be a part of 6, but it may be a Divisor to it self, that is, 6 may be divided by 6, and the Quotient is Unity. Hence it is manifest, that the *just Divisors* of a number are more in multitude by one than the number of its *Aliquot parts.*

V. The Aliquot parts of a whole number may be found out in this manner, *viz.* First, if the number proposed be even, divide it by 2, and reserve the Divisor; again, if the Quotient be even divide it by 2 and reserve the Divisor; and continue the Division of every

of every following Quotient by 2 until the Quotient be an odd number: But if either the number firſt propoſed, or the Quotient reſulting from ſuch Diviſion by 2, be odd, divide it by 3, if it will give an Integer Quotient, and continue the Diviſion by 3 in like manner as before by 2, ſo long as the Quotient is an Integer without any Fraction; likewiſe, when the Diviſion by 3 ceaſeth, divide by 5, 7, 11, 13, 17, 19, *&c.* that is, by every Prime number, until you find a Quotient leſs than the Diviſor; and if no ſuch Diviſor will give an Integer Quotient before the Quotient is leſs than the Diviſor; you may conclude the number firſt propoſed to be Incompoſit, (*viz.* ſuch as hath no Diviſor but it ſelf or Unity,) and that laſt Diviſor to be greater than the ſquare Root of the propoſed number: then by the help of thoſe Prime Diviſors to the given number, all the reſt may be found out by the Operation directed in the following Examples.

Example 1.

Suppoſe it be deſired to find out all the Aliquot parts and Diviſors of 360: Firſt, I divide 360 by 2, and the Quotient is 180, this divided by 2 gives 90, which divided by 2 gives 45, this being an odd number, the Diviſion by 2 ceaſeth: then I divide the ſaid 45 by 3 and the Quotient is 15, this divided by 3 gives the Quotient 5, and ſo the Diviſion by 3 ceaſeth; then I divide 5 by it ſelf, and the Quotient is Unity.

360	180	90	45	15	5	1
2	2	2	3	3	5	

Now by the help of thoſe Diviſors or Prime numbers, which (as may eaſily be proved,) are ſuch, that if they be continually multiplied will produce the given number 360, all the reſt of the juſt Diviſors of the ſaid 360 may be found out thus:

Firſt, I ſet every one of the ſaid Prime Diviſors, 2, 2, 2, 3, 3 and 5 at the Head of a Columel, as you ſee in this Table; then I multiply the firſt Diviſor 2 by the ſecond Diviſor 2, and ſet the Product 4 under 2 in the ſecond Columel; again, I multiply the ſaid 4 by 2, (which ſtands at the Head of the third Columel,) and ſet the Product 8 under 2 in the third Columel. Then I multiply every one of the numbers in the firſt, ſecond and third Columels, by 3, which ſtands at the Head of the fourth Columel, and write the Products under 3 in the ſaid fourth Columel; except ſuch Products which happen to be the ſame with any of thoſe before written, (for one and the ſame Product muſt not be written twice;) ſo multiplying 2, 4 and 8 by 3, I ſet the Products 6, 12 and 24 under 3 in the fourth Columel. Again, I multiply every one of the numbers in the firſt, ſecond, third and fourth Columels by 3, (which ſtands at the top of the fifth Columel,) and ſet the Products under the ſaid 3; except (as before) ſuch Products which happen to be the ſame with any of thoſe before written in any of the precedent Columels: ſo the Products written under 3 in the fifth Columel are 9, 18, 36 and 72. Laſtly, I multiply every one of the numbers in the firſt, ſecond, third, fourth and fifth Columels by 5, (which ſtands at the Head of the laſt Columel,) and write the ſeveral Products, (except as is before excepted,) under the ſaid 5: So at length all the juſt Diviſors to the given number 360 are found theſe, to wit, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180 and 360; every one of which Diviſors except the greateſt, (which is always equal to the number firſt propoſed,) is an *Aliquot part* of 360, which (as you ſee) hath 23 *Aliquot parts*, and 24 *Diviſors*.

2	2	2	3	3	5
	4	8	6	9	10
			12	18	20
			24	36	40
				72	15
					30
					60
					120
					45
					90
					180
					360

Example 2.

Again, if it be required to find out all the Aliquot parts and Divisors of 2310, the Operation will be like that in Example 1. For, first the Prime Divisors will be found these, to wit, 2, 3, 5, 7, 11; then after the said Prime Divisors are set at the heads of so many Columels, as you see in the Table in the Margin, the rest of the Divisors will be found out by Multiplication according to the foregoing directions; which in summ amount to this, *viz.* Each Prime Divisor standing at the head of every Columel following the first, is to be multiplied by every one of the numbers in the foregoing Columels, (except such which make the same Products as were before produced,) and the Products are to be set under each Prime Divisor respectively by which they were produced: So all the Divisors to the given number 2310 are discovered to be these, to wit, 1, 2, 3, 5, 6, 7, 10, 11, 14, 15, 21, 22, *&c.* as you see in this Table; every one of which Divisors except the greatest, to wit, 2310, (which is the same with the number proposed,) is an Aliquot part of the said 2310, which hath 31 Aliquot parts, but 32 Divisors.

2310	1155	385	77	11	1
2	3	5	7	11	

2	3	5	7	11
	6	10	14	22
		15	21	33
		30	42	66
			35	55
			70	110
			105	165
			210	330
				77
				154
				231
				462
				385
				770
				1155
				2310

Upon the same Foundation the Divisors of Quantities exprest by Letters may be found out; as will appear by the following Examples. But this work requires that the Analyst be well exercis'd in the Rules of Algebraical Multiplication, Division, and the Extraction of Roots; for the finding out of the Primitive or Incomposit Divisors, when the given Quantity is compos'd of many large Members connected by different Signs, is oftentimes both difficult and laborious.

Example 3.

Let it be required to find out all the Divisors and Aliquot Parts of this Quantity *aaabbc*. First, I divide the said *aaabbc* by *a*, and the Quotient is *aabbc*, which divided by *a* gives *abbc*, this divided by *a* gives *bbc*; and so the Division by *a* ceaseth. Then I divide *bbc* by *b*, and the Quotient is *bc*, this divided by *b* gives *c*, which being a Primitive or Incomposit quantity I divide by it self, and the Quotient is 1: So all the Primitive Divisors of the proposed Quantity *aaabbc* are found *a*, *a*, *a*, *b*, *b* and *c*; which are manifestly such as being multiplied continually will produce the given quantity *aaabbc*.

aaabbc	*aabbc*	*abbc*	*bbc*	*bc*	*c*	1
a	*a*	*a*	*b*	*b*	*c*	

Now out of those Divisors, after they are set at the heads of so many Columels as you see in this Table, I search out the rest of the Divisors by Algebraical multiplication, in like manner as in Example 1.) So all the different Divisors to the given quantity *aaabbc* are found these, to wit, 1, *a*, *aa*, *aaa*, *b*, *ab*, *aab*, *aaab*, *bb*, *abb*, *aabb*, *aaabb*, *c*, *ac*, *aac*, *aaac*, *bc*, *abc*, *aabc*, *aaabc*, *bbc*, *abbc*, *aabbc*, *aaabbc*; every one of which Divisors except the last and greatest is an Aliquot part of the given Quantity *aaabbc*, which hath 23 Parts, and 24 Divisors.

a	*a*	*a*	*b*	*b*	*c*
	aa	*aaa*	*ab*	*bb*	*ac*
			aab	*abb*	*aac*
			aaab	*aabb*	*aaac*
				aaabb	*bc*
					abc
					aabc
					aaabc
					bbc
					abbc
					aabbc
					aaabbc

Note, That this third Example differs not from Example 1. saving that Algebraical Division and Multiplication is used here, in stead of vulgar Division and Multiplication in numbers there.

Exam-

Example 4.

After the ſame manner, 31 Aliquot parts and 32 Diviſors will be found to this quantity $abcde$, viz. 1, a, b, ab, c, ac, bc, abc, d, ad, bd, &c. as you ſee them expreſt in the following Table.

	$abcde$	$bcde$	cde	de	e	1
Primitive Diviſors,	a	b	c	d	e	

a	b	c	d	e
	ab	ac	ad	ae
		bc	bd	be
		abc	abd	abe
			cd	ce
			acd	ace
			bcd	bce
			$abcd$	$abce$
				de
				ade
				bde
				$abde$
				cde
				$acde$
				$bcde$
				$abcde$

Compare this Example with the precedent Example 2.

Example 5.

Again, to find all the Diviſors of this Compound quantity $aaabc - abbbc$; Firſt, I ſearch out all its Prime Diviſors thus, *viz.* I divide the ſaid Compound quantity by a, and the Quotient is $aabc - bbbc$; this divided by b gives $aac - bbc$, which divided by c gives the Quotient $aa - bb$: This divided by $a - b$ gives the Quotient $a + b$, which being a Primitive quantity I divide it by it ſelf and the Quotient is 1. So the prime Diviſors are found a, b, c, $a - b$ and $a + b$, which are to be reſerved.

$aaabc - abbbc$	$aabc - bbbc$	$aac - bbc$	$aa - bb$	$a + b$	1
a	b	c	$a - b$	$a + b$	

Then, (as in the foregoing Examples;) I ſet the ſaid Primitive Diviſors at the heads of ſo many Columels, and from thoſe Diviſors, (according to the directions in Example 1.) I find out all the reſt by Multiplication: ſo at length it appears that $aaabc - abbbc$ the Compound quantity propoſed hath 31 Aliquot parts and 32 Diviſors; to wit, 1, a, b, ab, c, ac, bc, abc, $a - b$, $aa - ab$, $ab - bb$, &c. as you ſee them expreſt in the following Table.

a	b	c	$a - b$	$a + b$
	ab	ac	$aa - ab$	$aa + ab$
		bc	$ab - bb$	$ab + bb$
		abc	$aab - abb$	$aab + abb$
			$ac - bc$	$ac + bc$
			$aac - abc$	$aac + abc$
			$abc - bbc$	$abc + bbc$
			$aabc - abbc$	$aabc + abbc$
				$aa - bb$
				$aaa - abb$
				$aab - bbb$
				$aaab - abbb$
				$aac - bbc$
				$aaac - abbc$
				$aabc - bbbc$
				$aaabc - abbbc$

Example 6.

Again; to find out all the Divifors of this Quantity $aaabbc - 2aabbbc + abbbbc$; Firft, (as before,) I fearch out the Primitive Divifors, *viz.* I divide the Quantity propofed by a, and the Quotient is $aabbc - 2abbbc + bbbbc$; which divided by b gives the Quotient $aabc - 2abbc + bbbc$; this divided again by b gives $aac - 2abc + bbc$, which divided by c gives $aa - 2ab + bb$: this laft Quotient being a Square whofe fide is either $a - b$ or $b - a$, according as a is greater or lefs than b, I fhall fuppofe a to be greater than b, and then dividing the faid Square $aa - 2ab + bb$ by its fide $a - b$ the Quotient is alfo $a - b$; and laftly, by dividing $a - b$ by it felf, (becaufe 'tis a Primitive quantity,) the Quotient is 1. Thus the Primitive Divifors of the quantity propofed are found a, b, b, c, $a - b$ and $a - b$. Then every one of them being fet at the head of a Columel, and multiplication made according to the Operation in the precedent Examples, the reft of the defired Divifors to the quantity $aaabbc - 2aabbbc + abbbbc$ will be found out; and at length all the Divifors to the faid quantity are difcovered to be thefe, *viz.* 1, a, b, ab, bb, abb, c, ac, bc, abc, bbc, $abbc$, $a - b$, $aa - ab$, $ab - bb$, &c. as you fee them expreft in the following Table.

a	b	b	c	$a - b$	$a - b$
	ab	bb	ac	$aa - ab$	$aa - 2ab + bb$
		abb	bc	$ab - bb$	$aaa - 2aab + abb$
			abc	$aab - abb$	$aab - 2abb + bbb$
			bbc	$abb - bbb$	$aaab - 2aabb + abbb$
			$abbc$	$aabb - abbb$	$aabb - 2abbb + bbbb$
				$ac - bc$	$aaabb - 2aabbb + abbbb$
				$aac - abc$	$aac - 2abc + bbc$
				$abc - bbc$	$aaac - 2aabc + abbc$
				$aabc - abbc$	$aabc - 2abbc + bbbc$
				$abbc - bbbc$	$aaabc - 2aabbc + abbbc$
				$aabbc - abbbc$	$aabbc - 2abbbc + bbbbc$
					$aaabbc - 2aabbbc + abbbbc$

Example 7.

In like manner, if it be defired to find out all the Divifors of this Quantity $aaaaaa + 2aaaacc + aacccc$, that is, $a^6 + 2a^4cc + aac^4$; I divide it firft by a, and the Quotient is $a^5 + 2a^3cc + ac^4$, this divided again by a gives $a^4 + 2aacc + c^4$. Now 'tis evident that this laft Quotient cannot be divided by a or by c, or the like quantity; but becaufe (by *Sect.* 4. *Chap.* 8. *Book* 1.) the faid $a^4 + 2aacc + c^4$ is a Square, whofe Root is $aa + cc$, I divide the faid Square by its Root $aa + cc$, and the Quotient is alfo the fame Root $aa + cc$; which being a Primitive quantity, I divide it by it felf, and the Quotient is 1. So the Divifors to be referved are a, a, $aa + cc$ and $aa + cc$.

$a^6 + 2a^4cc + aac^4$	$a^5 + 2a^3cc + ac^4$	$a^4 + 2aacc + c^4$	$aa + cc$	1
a	a	$aa + cc$	$aa + cc$	

Then after thofe Divifors are fet at the heads of fo many Columels, (as you fee in the following Table,) I proceed to find out the reft of the Divifors by Multiplication according to the directions in Example 1. *viz.* I multiply each primitive Divifor ftanding at the head of every Columel following the firft by every one of the Quantities in the preceding Columels, and fet the Products under the refpective primitive Divifor, with this Caution, that one and the fame Product be not written down twice: So at length I find all the different Divifors to be thefe, *viz.* 1; a; aa; $aa + cc$; $a^3 + acc$; $a^4 + aacc$; $a^4 + 2aacc + c^4$; $a^5 + 2a^3cc + ac^4$; and $a^6 + 2a^4cc + aac^4$: all which Divifors except the laft are Aliquot parts of the propofed Quantity $a^6 + 2a^4cc + aac^4$.

a	a	$aa + cc$	$aa + cc$
	aa	$a^3 + acc$	$a^4 + 2aacc + c^4$
		$a^4 + aacc$	$a^5 + 2a^3cc + ac^4$
			$a^6 + 2a^4cc + aac^4$

VI. By

VI. By this skill of finding out all the Divisors of Quantities, we may reduce two or more given Quantities, when they are not Prime between themselves, to others in the same Reason (or Proportion) with those given, and in the smallest Terms: As, to reduce these three quantities, $aaa - abb$; $aab - bbb$; and $aaa + aab - abb - bbb$ to the smallest quantities in the same Proportion with those proposed; First, I seek (by the Method before delivered) all the different Divisors to every one of those three given Quantities, so I find the Divisors of the first quantity $aaa - abb$ to be these, *viz.* 1; a; $a + b$; $a - b$; $aa + ab$; $aa - ab$; $aa - bb$; $aaa - abb$: and the Divisors of the second quantity $aab - bbb$ to be these, *viz.* 1; b; $a - b$; $ab - bb$; $a + b$; $ab + bb$; $aa + bb$; and $aab - bbb$: also the Divisors of the third quantity $aaa + aab - abb - bbb$ to be these, to wit, 1; $a - b$; $a + b$; $aa - bb$; $aa + 2ab + bb$ and $aaa + aab - abb - bbb$. Now because among those three Companies of Divisors, these three $a - b$, $a + b$ and $aa - bb$ are found in each Company, we may by the help of any one of those three Divisors reduce the given Quantities, to others more simple and in the same Proportion with those given: But to find out the smallest Terms, I divide the proposed Quantities $aaa - abb$; $aab - bbb$ and $aaa + aab - abb - bbb$ severally by $aa - bb$, (to wit,) such of the said three Divisors which hath most Dimensions,) and there arise a, b and $a + b$; which three Quantities are the smallest Terms that can be found in the same Proportion with the three Quantities first proposed.

Note. The Quantities propos'd to be reduced are said to be Prime the one to the other when they have no common Divisor besides 1, (to wit, Unity,) in which case the Quantities proposed are already in their smallest Terms.

VII. The finding out of Divisors may very fitly be applied to the reducing of Fractions to their smallest Terms: As, to abbreviate this Fraction,

$$\frac{aaa + aab - abb - bbb}{aaa - abb}$$

First, the Divisors of the Numerator (by the precedent Method) are found 1; $a - b$; $a + b$; $aa - bb$; $aa + 2ab + bb$; and $aaa + aab - abb - bbb$: likewise, the Divisors of the Denominator are 1; a; $a + b$; $a - b$; $aa + ab$; $aa - ab$; $aa - bb$; and $aaa - abb$. Then because among those Divisors, these three, to wit, $a + b$; $a - b$ and $aa - bb$ are common both to the Numerator and Denominator, I divide the Numerator and Denominator severally by $aa - bb$, (to wit, that common Divisor which hath most Dimensions;) so there ariseth $a + b$ for a new Numerator, and a for a new Denominator, which gives this Fraction $\frac{a + b}{a}$, (or $1 + \frac{b}{a}$) equal to that proposed, and in the smallest Terms; as was desired.

In like manner to abbreviate $\frac{aaa - abb}{aa + 2ab + bb}$, because the greatest Divisor common to the Numerator and Denominator is $a + b$, I divide the Numerator and Denominator severally by $a + b$, and there ariseth $\frac{aa - ab}{a + b}$; which is equal to the Fraction proposed, and in the smallest Terms.

VIII. *Observations upon the Examples in the foregoing* Sect. V.

First, When two, three, or more of the formost Letters (towards the left hand) of a Simple quantity are equal to one another, (*viz.* exprest by one and the same Letter,) then mark well how many equal Letters stand formost together, for so many Aliquot parts they will give: As, in Example 3. in *Sect.* 5. where the Quantity proposed is $aaabbc$, the three first letters a, a, a, (that is, aaa) give three Aliquot parts, to wit, 1, a, aa; but four Divisors, 1, a, aa, aaa. In like manner, if four equal Letters stand formost together, as a, a, a, a, or $aaaa$, they will afford these four Parts, 1, a, aa, aaa; but five Divisors, to wit, 1, a, aa, aaa, $aaaa$. The like property ensues, when five or more equal Letters stand formost together.

Hence it is evident that every Power hath so many Aliquot Parts as there be Dimensions in the Power; As, the Square aa whose Index (or number of Dimensions) is 2, hath two Parts, to wit, 1 and a; likewise the Cube aaa, or a^3, hath three Parts; the fourth Power $aaaa$ (or a^4) hath four Parts; and so forwards.

Secondly,

Secondly, It is evident from all the precedent Examples in *Sect.* 5. that when among the Primitive Diviſors, (which are ſet at the tops of the Columels,) a following Diviſor differs from the next precedent Primitive Diviſor, then the multitude of Diviſors in the Columel of the ſaid following Diviſor is more by one than the multitude of all the different Diviſors in the precedent Columels: As; in Example 3. in *Sect.* 5. where the Quantity propoſed is *aaabbc*, the letter (or Primitive Diviſor) *b* which follows and is different from the next foregoing Primitive Diviſor *a*; gives four Diviſors, to wit, *b*, *ab*, *aab*, and *aaab*; which are more in multitude by one than all the foregoing different Diviſors *a*, *aa*, and *aaa*.

Again, in Example 4. in *Sect.* 5. where the Quantity propoſed is *abcde*, the Diviſors *b* and *ab* in the ſecond Columel are more in number by one than *a* in the firſt; likewiſe the Diviſors *c*, *ac*, *bc*, and *abc* in the third Columel are more in multitude by one than *a*, *b* and *ab*, to wit, all the Diviſors in the firſt and ſecond Columels: alſo *d*, *ad*, *bd*, *abd*, *cd*, *acd*, *bcd* and *abcd* in the fourth Columel, are more in multitude by one than all the Diviſors in the firſt, ſecond and third Columels, and ſo forward. The Reaſon is manifeſt; for every Primitive Diviſor which ſtands at the top of a following Columel is multiplied into all the different Diviſors ſeverally in all the foregoing Columels; and therefore if that multiplying Primitive Diviſor be added to the number of thoſe Products, the total multitude muſt neceſſarily be more by one than the multitude of different Diviſors in all the foregoing Columels.

Thirdly, It is alſo evident, that when the ſaid Primitive Diviſors are all different, then the numbers which expreſs the multitude of Diviſors in every Columel are in continual Proportion increaſing from Unity in a Duple Reaſon: As, in the fourth Example in *Sect.* 5. where the Primitive Diviſors *a*, *b*, *c*, *d*, *e* are all different, there is one Diviſor in the firſt Columel; two in the ſecond; four in the third; eight in the fourth; and ſixteen in the fifth; which numbers of multitude, to wit, 1, 2, 4, 8 and 16 are manifeſtly in Duple Proportion. Therefore when all the Primitive Diviſors of a Quantity propoſed are different, or unlike, then if ſo many of the formoſt Terms of the ſaid continual Proportionals 1, 2, 4, 8, 16, &c. be added together, as there be Primitive Diviſors, (to wit, thoſe Incompoſit quantities, which being continually multiplied will produce the Quantity propoſed,) the the ſumm ſhall be the number of Aliquot Parts contained in that Quantity; and the number of Diviſors ſhall be more by one than that ſumm.

As, for Example, if the number of Aliquot Parts in the quantity *ab* be deſired, I add 1 and 2 together, (to wit, the two firſt Terms of the ſaid Geometrical Progreſſion 1, 2, 4, 8, 16, &c.) and the ſumm 3 ſhews that *ab* contains three Aliquot Parts, and 4 (that is, 3 + 1) Diviſors. Likewiſe if there be propoſed the Quantity *abc*, (which conſiſts of three different letters,) the ſumm of 1, 2, 4, (to wit, of the three firſt Terms of the ſaid Geometrical Progreſſion,) is 7; which ſhews that *abc* contains ſeven Parts, but eight (or 7 + 1) Diviſors. Again, if *abcd* (which conſiſts of four different letters,) be propoſed, the ſumm of 1, 2, 4, 8, (the four formoſt Terms of the ſaid Progreſſion,) is 15; which ſhews that the quantity *abcd* contains fifteen Aliquot Parts, and ſixteen (or 15 + 1) Diviſors, and ſo forward. But becauſe the ſaid Proportionals proceed in a Duple Reaſon from Unity, the ſumm of any number of Terms may be found out by this brief Rule, *viz.* The third Term (or Proportional) leſſened by Unity, (the firſt Term) gives the ſumm of the firſt and ſecond Terms; likewiſe the fourth Term leſſened by 1, gives the ſumm of the firſt, ſecond and third Terms; and the fifth Term leſſened by 1, gives the ſumm of the firſt, ſecond, third and fourth Terms; and ſo forward, infinitely. All which may be further illuſtrated by the ten Quantities, and their reſpective multitudes of Aliquot Parts, expreſt in the following Table.

Quantities given.	Multitude of Parts.	Summs of Terms in continual Proportion, proceeding from 1 in Duple Reaſon.
a	hath 1 =	1
ab	. . 3 =	1+2
abc	. . 7 =	1+2+4
abcd	. . 15 =	1+2+4+8
abcde	. . 31 =	1+2+4+8+16
abcdef	. . 63 =	1+2+4+8+16+32
abcdefg	. 127 =	1+2+4+8+16+32+64
abcdefgh	. 255 =	1+2+4+8+16+32+64+128
abcdefghi	. 511 =	1+2+4+8+16+32+64+128+256
abcdefghik	. 1023 =	1+2+4+8+16+32+64+128+256+512

Fourthly,

Fourthly, When two, three or more equal letters in a Simple quantity ſtand together, and follow ſome different foregoing letter or letters; then as many Aliquot Parts as the firſt of thoſe following equal letters produceth, (according to *Obſervat.* 2.) ſo many Parts every one of the reſt of the ſaid following letters will produce. As, in Example 3. in *Sect.* 5. where this quantity *aaabbc* is propoſed, the three firſt letters, *a*, *a*, *a*, (or *aaa*) give three Parts; (by *Obſervat.* 1.) and the firſt following letter *b*, in regard it differs from the next preceding letter *a*, gives four Parts, (by *Obſervat.* 2.) now I ſay the ſecond *b* ſhall alſo give four Parts, and if there had been a third *b*, or a fourth *b*, &c. every one of them would give four parts, to wit, as many as the firſt *b* produced.

In like manner, if this quantity *abbbbb* or ab^5 be propoſed, the firſt letter *a* gives one Part; then (by *Obſervat.* 2.) the next following letter *b* (in regard it differs from *a*) gives two Parts: now I ſay every *b* following the firſt *b* will alſo give two Parts, and ſo *bbbbb* will give ten (to wit, five times two) Parts, which added to one Part noted for *a* makes 11 Parts; whence I conclude that the quantity *abbbbb* contains 11 Aliquot Parts, and 12 Diviſors. All which may be produced particularly by the Rule in the foregoing *Sect.* 5.

Again, if this quantity *abcddd* be propoſed, firſt, (by *Obſervat.* 3.) *abc* will give ſeven Parts, and (by *Obſervat.* 2.) the next following letter *d* gives eight Parts; therefore (by this fourth *Obſervat.*) every *d* following the firſt *d* gives alſo eight Parts, and conſequently *ddd* gives 24 Parts, which added to the ſeven Parts before noted for *abc*, makes 31 Parts. So that the quantity *abcddd* hath 31 Aliquot Parts, and 32 Diviſors; and the ſame number of Parts and Diviſors will be found in the number produced by the continual Multiplication of theſe five Prime numbers, 2, 3, 5, 7, 7, 7.

Fifthly, From what hath been ſaid in the precedent Obſervations 'tis eaſie to diſcover how many Aliquot Parts are contained in any Simple quantity deſign'd by letters, without producing the particular Parts: As, if *aaabbc* be propoſed, firſt, three Parts are to be noted for *aaa*, (according to *Obſervat.* 1.) and eight Parts more for *bb*, (by *Obſervat.* 4.) which eight Parts added to the three Parts before noted make eleven Parts; then for *c*, twelve Parts are to be noted, (to wit, 11 + 1, according to *Obſervat.* 2.) which added to the ſaid eleven Parts makes 23 Parts: whence I conclude that the quantity *aaabbc* hath 23 Aliquot Parts, and 24 Diviſors; which are particularly expreſt in Example 3. *Sect.* 5.

In like manner, we may diſcover that this quantity *aaaaabbbbcccdd* or $a^5b^4c^3d^2$ hath 359 Aliquot Parts, and 360 Diviſors; for firſt, I note 5 Parts for a^5, (according to *Obſervat.* 1.) then (by *Obſervat.* 4.) *bbbb* or b^4 gives 24 Parts, which added to the five Parts before noted makes 29 Parts; and becauſe one ſingle *c* gives 30 Parts, to wit 29 + 1, (by *Obſervat.* 2.) *ccc* or c^3 will give 90, to wit, 3 times 30 Parts, (by *Obſervat.* 4.) which added to 29 Parts before noted makes 119 Parts; laſtly, becauſe the letter *d* is written twice, and one ſingle *d* gives 120, to wit, 119 + 1 Parts, (by *Obſervat.* 2.) *dd* will give 240 Parts; (by *Obſervat.* 4.) which added to 119 Parts before noted, makes 359 Parts; which is the multitude of Aliquot Parts the propoſed Quantity hath, but its number of Diviſors is 360.

And with the like facility we may diſcover the multitude of Parts and Diviſors of a given number, after its Primitive Diviſors are found out; As, for example, to find how many Parts and Diviſors 15876000 hath, I ſearch out by Diviſion (in like manner as in the Examples in *Sect.* 5.) all the Primitive Diviſors which being continually multiplied will produce the ſaid given number, and find them to be theſe, to wit, 2, 2, 2, 2, 2, 3, 3, 3, 3, 5, 5, 5, 7, 7, which may be noted by $a^5b^4c^3dd$; but this Quantity (as before hath been ſhewn) hath 359 Aliquot Parts and 360 Diviſors, and therefore the ſaid 15876000 hath the ſame number of Parts and Diviſors; which may be particularly found out by the method in the precedent Examples in *Sect* 5.

Sixthly, If a Quantity be compoſed of different letters or Powers, and Unity be added ſeverally to the Indices of thoſe Powers, that is, to the numbers expreſſing how oft each letter is found in that Quantity, then the numbers reſulting by thoſe Additions being multiplied one into the other continually, will produce a number greater by Unity than the number of Aliquot Parts that Quantity hath: As, for example, if *aaaabbb* or a^4b^3 be propoſed; I add 1 to 4 and 3 ſeverally, (becauſe the Indices of *aaaa* and *bbb* are 4 and 3,) and it makes 5 and 4; theſe multiplied one into the other make 20, which is greater by 1 than 19 the number of Aliquot Parts that the propoſed quantity a^4b^3 hath. The reaſon of this property is not difficult to be conceived; for ſince (by *Obſervat.* 1.)

aaaa

aaaa hath four Parts, that is, five Parts wanting one Part; and *bbb* following *aaaa* hath thrice five Parts, (by *Obſervat.* 4.) therefore the whole quantity *aaaabbb* (or a^4b^3) hath 4×5 Parts wanting one Part, *viz.* 19 Parts; which numbers 4 and 5 exceed 3 and 4 the Indices of *bbb* and *aaaa*, ſeverally by Unity.

Again, if *aaaabbbcc* be propoſed, the Indices of *aaaa*, *bbb* and *cc* are 4, 3 and 2, which increaſed ſeverally by 1, make 5, 4 and 3; theſe multiplied continually produce 60, which is greater by Unity than 59, the number of Aliquot Parts which the propoſed quantity *aaaabbbcc* hath. For ſince (for the reaſon in the laſt preceding Example) *aaaabbb* hath 4×5 Parts wanting one Part, and *cc* following *aaaabbb* hath (by *Obſervat.* 4.) $2 \times 4 \times 5$ Parts, the propoſed quantity *aaaabbbcc* hath conſequently $3 \times 4 \times 5$ Parts wanting one Part, that is, 59 Parts; which numbers 3, 4 and 5 do ſeverally exceed the Indices of *cc*, *bbb* and *aaaa* by Unity.

Seventhly, From the preceding *Obſervat* 6. it follows, That if a Compoſit number be reſolved into any two or more of ſuch of its Factors, the leaſt of which exceeds Unity, and if from every one of thoſe Factors Unity be ſubtracted, the Remainders ſhall be Indices of ſo many ſeveral Powers expreſſible by different letters that being joyned together, (that is, multiplied one into the other,) will give a Quantity having a number of Aliquot Parts leſs by Unity than the Compoſit number propoſed: As, for example, if 20 be propoſed; for as much as 5 and 4 multiplied one by the other produce 20, I ſubtract 1 from 5 and 4 ſeverally; ſo the Remainders 4 and 3 do ſhew, that if the fourth Power of ſome quantity *a*, as *aaaa*, be multiplied into the third Power of ſome other quantity *b*, as into *bbb*, the Quantity produced, to wit, *aaaabbb* hath 19 Aliquot Parts, which 19 is leſs by Unity than 20 the number propoſed. Again, becauſe the Product of 10 into 2 doth alſo make 20, I ſubtract 1 from 10 and 2 ſeverally; ſo the Remainders 9 and 1 do ſhew, that if the ninth Power of ſome quantity *a*, as a^9, be multiplied by ſome other different quantity *b*, the Quantity produced, to wit, a^9b hath alſo 19 Aliquot Parts. Hence it is manifeſt, that often times many Quantities may be found out, every one of which ſhall have a given multitude of Aliquot Parts; as will appear in the next following Section.

IX. *The manner of finding out all ſuch Quantities as ſhall have a given multitude of Aliquot Parts.*

If the multitude of Aliquot Parts deſired be any of the numbers of the ſecond Columel of the Table in *Obſervat.* 3. *Sect.* 8. the Quantity there ſtanding on the left hand of that number, and on the ſame line with it, hath the number of Parts deſired. As, if it be deſired to find a Quantity that hath 63 Aliquot Parts, that Table ſhews that *abcdef* hath 63 Parts; and therefore if ſix Prime numbers, ſuppoſe 2, 3, 5, 7, 11, 13 be taken for the values of thoſe ſix letters, *a*, *b*, *c*, *d*, *e*, *f*, the Product made by the continual multiplication of the ſaid Prime numbers, to wit, 30030, ſhall have 63 Aliquot Parts, and 64 Diviſors.

But without reſpect to that Table, by the help of the Obſervations in the foregoing *Sect.* 8. many Quantities for the moſt part, and alwayes one Quantity may eaſily be found out that ſhall have a given multitude of Aliquot Parts; as will be made manifeſt by the following Examples.

Example 1.

Let it be required to find out all ſuch ſimple Quantities expreſſible by letters, that may every one of them have 15 Aliquot Parts, and 16 Diviſors.

1. To the ſaid 15 I add 1 and it makes 16, this I divide by 2 and the Quotient is 8, which divided by it ſelf gives 1; then from each of the Diviſors 2 and 8, (the Product of whoſe multiplication makes the firſt Dividend 16,) I ſubtract 1; ſo the Remainders 1 and 7 do ſhew that if ſome letter, as *a*, be written once, and next after it another different letter *b* ſeven times, the Quantity ſo compoſed, to wit, *abbbbbbb* or ab^7 ſhall have 15 Aliquot Parts, and 16 Diviſors; as was deſired.

16	8	1
2	8	

2. Again, I divide the ſaid 16 (to wit, 15 + 1,) by 2, and the Quotient is 8; this divided again by 2 gives 4, which divided again by 2 gives 2, which divided by it ſelf gives 1; then from every one of the Diviſors 2, 2, 2, 2 I ſubtract 1; ſo the Remainders 1, 1, 1, 1 do ſhew that if 4 different ſingle letters be ſet together, as *abcd*, this quantity ſhall have 15 Parts, and 16 Diviſors; as before.

16	8	4	2	1
2	2	2	2	

3. Again,

3. Again, I divide 16 by 2, and the Quotient is 8; this divided by 2 gives 4, which divided by it ſelf gives 1; then from every one of the Diviſors 2, 2, 4 I ſubtract 1, and the Remainders 1, 1, and 3 do ſhew, that if two different letters *a* and *b* be joyned together, and next after them a third different from each of them, as *c* be written thrice, the Quantity ſo compoſed, to wit *abccc*, ſhall have 15 Aliquot Parts, and 16 Diviſors; as before.

16	8	4	1
2	2	4	

4. Again, I divide 16 by 4, and the Quotient is 4, this divided by it ſelf gives 1; then from each of the Diviſors 4 and 4, I ſubtract 1, and the Remainders 3 and 3 do ſhew, that if ſome letter *a* be written thrice, as *aaa*, and next after the ſame another letter different from *a*, as *b*, be likewiſe written thrice, the Quantity ſo compoſed, to wit, *aaabbb*, or a^3b^3 ſhall have 15 Aliquot Parts, and 16 Diviſors; as before.

16	4	1
4	4	

5. Laſtly, I divide 16 by it ſelf and the Quotient is 1, then from 16 I ſubtract 1, and the Remainder 15 ſhews that if ſome letter *a* be written 15 times, as *aaaaaaaaaaaaaaa*, or a^{15}, this Quantity ſhall have 15 Parts, and 16 Diviſors; as before.

16	1
16	

Hence, becauſe 16 cannot be divided by any other ways than thoſe five before expreſt, we may conclude that the five Quantities found out, and thoſe only, to wit, ab^7, *abcd*, abc^3, a^3b^3 and a^{15}, have each of them 15 Aliquot Parts, and 16 Diviſors. All which Operations do clearly reſult from *Obſervat.* 6, and 7. in the precedent *Sect.* 8.

Example 2.

Let it be required to find out all ſuch Quantities expreſſible by letters, which may every one of them have 23 Aliquot Parts, and 24 Diviſors.

Firſt, (as before) I addd 1 to 23, and it makes 24; this may be divided by its Factors in a ſeven-fold manner before the Quotient be Unity, as here you ſee.

24	8	4	2	1
3	2	2	2	

24	6	2	1
4	3	2	

24	4	2	1
6	2	2	

24	4	1
6	4	

24	3	1
8	3	

24	2	1
12	2	

24	1
24	

Whence I conclude that ſeven different Quantities may be produced, every one of which ſhall have 23 Aliquot Parts, and 24 Diviſors; now to find out the ſaid Quantities, I ſubtract 1, (to wit Unity,) from every one of the Diviſors of the foregoing ſeven-fold Diviſion, ſo the Diviſors, 3, 2, 2, 2 of the firſt Diviſion being ſeverally leſſened by Unity give 2, 1, 1, 1; whence, according to the precedent directions in Example 1. of this *Sect.* 9. this Quantity may be compoſed, to wit, *aabcd*; and by proceeding in like manner with the reſt of the Diviſors, ſeven different Quantities, every one of which hath 23 Aliquot Parts and 24 Diviſors, are diſcovered; and may be expreſt either

Thus,		Or thus,
	aabcd;	a^2bcd.
	aaabbc;	a^3b^2c.
	aaaaabc;	a^5bc.
	aaaaabbb;	a^5b^3.
	aaaaaaabb;	a^7b^2.
	aaaaaaaaaaab;	$a^{11}b$.
	aaaaaaaaaaaaaaaaaaaaaaa;	a^{23}.

Example 3.

Let it be required to find out a Quantity which hath 42 Aliquot Parts.

Firſt, (as before) I add 1 to 42 and it makes 43, which being a Prime number, (that is, ſuch as cannot be divided by any number but by it ſelf or Unity,) doth ſhew, that there is only one Quantity can be found that hath 42 Aliquot Parts; *viz.* ſome letter, as *a* being written 42 times one after another, or a ſingle *a* with its Index 42, as a^{42}, doth expreſs a Quantity (to wit, the forty-ſecond Power of *a*) which hath 42 Aliquot Parts, and 43 Diviſors. The like is to be underſtood of other Quantities, when the multitude of Aliquot Parts deſired being increaſed with Unity makes a Prime number.

 For

For further illustration of the premises, the Learner may view the following Table, which shews all the various Quantities exprest by Letters, that have a given multitude of Aliquot Parts not exceeding 50; and upon the grounds before explained, the Table may be continued as far as you please.

Quantities.		Aliquot Parts.
a	hath	1
aa	hath	2
ab, a^3	have each	3
a^4	&c.	4
aab, a^5		5
a^6		6
a^3b, abc, a^7		7
$aabb$, a^8		8
a^4b, a^9		9
a^{10}		10
a^2bc, a^3b^2, a^5b, a^{11}		11
a^{12}		12
a^6b, a^{13}		13
a^4bb, a^{14}		14
a^3bc, $abcd$, a^3b^3, a^7b, a^{15}		15
a^{16}		16
a^2b^2c, a^5b^2, a^8b, a^{17}		17
a^{18}		18
a^4bc, a^4b^3, a^9b, a^{19}		19
a^6b^2, a^{20}		20
$a^{10}b$, a^{21}		21
a^{22}		22
a^3b^2c, a^2bcd, a^5bc, a^5b^3, a^7b^2, $a^{11}b$, a^{23}		23
a^4b^4, a^{24}		24
$a^{12}b$, a^{25}		25
$a^2b^2c^2$, a^8b^2, a^{26}		26
a^6bc, a^6b^3, $a^{13}b$, a^{27}		27
a^{28}		28
a^4b^2c, a^5b^4, a^9b^2, $a^{14}b$, a^{29}		29
a^{30}		30
a^3bcd, a^3b^3c, a^7bc, $abcde$, a^7b^3, $a^{15}b$, a^{31}		31
$a^{10}b^2$, a^{32}		32
$a^{16}b$, a^{33}		33
a^6b^4, a^{34}		34
a^2b^2cd, a^5b^2c, $a^3b^2c^2$, a^8bc, a^8b^3, a^5b^5, $a^{11}b^2$, $a^{17}b$, a^{35}		35
a^{36}		36
$a^{18}b$, a^{37}		37
$a^{12}b^2$, a^{38}		38
a^4bcd, a^4b^3c, a^9bc, a^7b^4, a^9b^3, $a^{19}b$, a^{39}		39
a^{40}		40
a^6b^2c, a^6b^5, $a^{13}b^2$, $a^{20}b$, a^{41}		41
a^{42}		42
$a^{10}bc$, $a^{10}b^3$, $a^{21}b$, a^{43}		43
$a^4b^2c^2$, a^8b^4, $a^{14}b^2$, a^{44}		44
$a^{22}b$, a^{45}		45
a^{46}		46
a^3b^2cd, a^5bcd, a^5b^3c, a^2bcde, $a^3b^3c^2$, a^7b^2c, $a^{11}bc$, a^7b^5, $a^{11}b^3$, $a^{15}b^2$, $a^{23}b$, a^{47}		47
a^6b^6, a^{48}		48
a^4b^4c, a^9b^4, $a^{24}b$, a^{49}		49
$a^{16}b^2$, a^{50}		50

X. How

X. *How to find out the smallest number that shall have a given multitude of Aliquot Parts.*

First; by the foregoing *Sect.* 9. search out all the Quantities expressible by letters, every one of which may have the number of Aliquot Parts desired; then to the different letters by which every one of those Quantities is exprest, assign the smallest Prime numbers, and find out by continual Multiplication the Products of those Prime numbers correspondent to the said Quantities. Again, let the values of those letters be exprest by the same Prime numbers, varied as many wayes as is possible, and find out their respective Products, as before. Lastly, all those Products being compared one to another, the least of them shall be the smallest number that hath the prescribed multitude of Aliquot Parts.

Example 1.

Let it be required to find the smallest number that hath 15 Aliquot Parts.

First, all the different Quantities that can be found to have severally 15 Aliquot Parts, (as appears by the precedent *Sect.* 9.) are these, to wit, $abcd$, a^3bc, a^3b^3, a^7b, a^{15}; then by assigning to a, b, c, d, the smallest Prime numbers, 2, 3, 5, 7, for $abcd$ there will be found 210, (by multiplying 2, 3, 5, 7 one into the other continually;) for a^3bc, 120; for a^3b^3, 216; for a^7b, 384; and for a^{15}, 32768; the least of which Products is 120. But before we can determine whether 120 be the least number or not that hath 15 Aliquot Parts, enquiry must be made by exchanging the values of those letters with the said Prime numbers all manner of ways, *viz.* we may suppose $a = 3$; $b = 2$; $c = 5$; and $d = 7$: or, $a = 5$; $b = 2$; $c = 3$; and $d = 7$: or again, $a = 7$; $b = 2$; $c = 3$; $d = 5$; and many other wayes the values of a, b, c, d may be exprest by the said Prime numbers 2, 3, 5, 7: and consequently from those variations, the quantities $abcd$, a^3bc, ab^3b^3, a^7b, a^{15} will be expounded by various numbers, which must be compared together, and then the least among them all is the number sought. So after all variations are made, it will appear that a^3bc is that Quantity by which 120, the smallest number having 15 Aliquot Parts and 16 Divisors will be found out.

Example 2.

Again, if the least number that hath 23 Aliquot Parts, or 24 Divisors, be desired.

First, by *Sect.* 9. all the Quantities which have severally 23 Parts will be found these, to wit, a^2bcd; a^3bbc; a^5bc; a^5b^3; a^7b^2; $a^{11}b$; and a^{23}: Then by assuming for the values of a, b, c, d the least Prime numbers 2, 3, 5, 7; for a^2bcd there will be found 420; for a^3b^2c, 360; for a^5bc, 480; for a^5b^3, 864; for a^7b^2, 1152; for $a^{11}b$, 6144; and for a^{23}, 8388608: and after all other possible variations made with the said letters and Prime numbers, by taking sometimes one, sometimes another of the said numbers for the value of a, b, &c. it will at length appear that a^3b^2c finds out 360, the least number that hath the desired multitude of 23 Aliquot Parts, and 24 Divisors.

If there be not occasion to find the least, but any number that hath a given multitude of Aliquot Parts, suppose 15, then you may indifferently use any one of these five quantities, $abcd$, a^3bc, a^3b^3, a^7b, a^{15}; by assigning to a, b, c, d Prime numbers at pleasure, and taking sometimes one, sometimes another of those numbers; or alwayes new Prime numbers for the values of a, b, c, d; whence innumerable numbers may be found out, every one of which shall have 15 Aliquot Parts. As, if we suppose $a = 2$; $b = 3$; and $c = 5$, there will be found for a^3bc, 120: but by putting $a = 3$; $b = 2$; and $c = 5$, there will be found for a^3bc, 270. Or also by assuming $a = 7$; $b = 11$; and $c = 13$, there will be produced for a^3bc, 49049: or if we put $a = 17$; $b = 19$; and $c = 23$, then $a^3bc = 2146981$. And in like manner you may use every one of the other four quantities $abcd$, a^3b^3, a^7b, and a^{15}. The like also is to be understood of every one of these, a^2bcd; a^3b^2c; a^5bc; a^5b^3; a^7b^2; $a^{11}b$; and a^{23}, for the finding out innumerable numbers which have severally 23 Aliquot Parts, and 24 Divisors.

Lastly, to find the least number that hath 42 Parts, and 43 Divisors; for as much as a Quantity having this multitude of Parts and Divisors can be designed only in one manner, *viz.* by writing a^{42}; let the least Prime number 2 be taken for the value of a, and then seek the forty second Power of the Root 2, by writing down 2 forty-two times seperately, and multiplying those numbers one into another, according to the Rule of Continual Multiplication, so the last Product will be 4398046511104; which is the least number that hath the desired multitude of 42 Aliquot Parts. And so of others.

For further illuſtration, the Learner may view the following Table, which ſhews the leaſt number that hath any given multitude of Aliquot Parts under 51. *Note*, That the number of Diviſors to any number is alwayes more by one than its number of Aliquot Parts; for albeit a number cannot properly be called a Part of it ſelf, yet 'tis contained in it ſelf once, and therefore may be ſaid to be a Diviſor to it ſelf.

Each number in the firſt of theſe Columels is the ſmalleſt that can be found to have ſuch a multitude of Aliquot Parts as is expreſt in the latter Columel.

2	hath	1	Aliquot Part.
4	hath	2	Aliquot Parts.
6		3	
16		4	
12		5	
64		6	
24		7	
36		8	
48		9	
1024		10	
60		11	
4096		12	
192		13	
144		14	
120		15	
65536		16	
180		17	
262144		18	
240		19	
576		20	
3072		21	
4194304		22	
360		23	
1296		24	
12288		25	
900		26	
960		27	
268435456		28	
720		29	
1073741824		30	
840		31	
9216		32	
196608		33	
5184		34	
1260		35	
68719476736		36	
786432		37	
36864		38	
1680		39	
1099511627776		40	
2880		41	
4398046511104		42	
15360		43	
3600		44	
12582912		45	
70368744177664		46	
2520		47	
46656		48	
6480		49	
589824		50	

Chap. IX.

The Arithmetick both of Surd Numbers, and Surd Quantities exprest by Letters. The Constitution and Invention of six Binomials in numbers, agreeable to those expounded in Prop. 49, 50, 51, 52, 53, 54. Elem. 10. Euclid. *with Rules to extract the Square Root out of every one of them; as also, what Root you please out of any Binomial in numbers, having such a Binomial Root as is desired.*

Sect. I. *Definitions concerning Surd Roots, and their Fundamental Operations.*

Every Absolute (or Ordinary) number, whether it be a whole number, or a Fraction, or a whole number with a Fraction annext to it, is called *Rational*: As, 1, 2, 3, 4, *&c.* also, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{8}$, $\frac{11}{21}$, *&c.* and $2\frac{1}{2}$, (or $\frac{5}{4}$,) $5\frac{1}{3}$, (or $\frac{16}{3}$,) $20\frac{11}{12}$, *&c.* are called Rational numbers; so also a, ab, $\frac{bc}{a}$, $a+\frac{bc}{a}$, &c. represent Rational Quantities.

But when the Square Root, Cubick Root, or any other Root cannot be perfectly extracted out of a Rational number, that Root is called *Irrational* or *Surd*; and because it cannot be exactly exprest by any Rational number, it is usual to set some Character (which is called the Radical Sign) before the Rational number out of which the Root ought to be extracted, to design or signifie the same Root: As, $\surd$ or $\surd(2)$ prefixt before any Rational number, signifies the Square Root of that number; $\surd(3)$ the Cubick Root; $\surd(4)$ the the Biquadratick Root; $\surd(5)$ the Root of the fifth Power, *&c.*

Hence $\surd 12$, or $\surd(2)12$ denotes or represents the square Root of 12, which Root is called Irrational or Surd, because it cannot be perfectly exprest by any Rational number; for 3 multiplied by it self produceth 9, which is less than 12; and 4 multiplied by it self produceth 16, which is greater than 12: and although there be innumerable mixt numbers consisting of 3 and some Fraction, which fall between 3 and 4, yet none of them multiplied into it self quadratickly can produce the whole number 12.

In like manner, $\surd(3)5$, which represents the cubick Root of 5, is called an Irrational or Surd number, because no number can be found, which being multiplied into it self cubically will produce 5 exactly: so also $\surd a$, $\surd bc$, $\surd(3)bb$, &c. represent Surd quantities.

There are two sorts of Irrational or Surd numbers, Simple and Compound: a Simple Surd number is exprest by one single term; such are $\surd 5$, $\surd 10$, $\surd(3)16$, $\surd(4)8$, *&c.* but a Compound Surd number consists of many simple or single terms, and is formed by the Addition or Subtraction of Simple terms; such are $\surd 5+\surd 2$; $\surd 5-\surd 2$; $\surd 8+\surd 6-\surd 2$; $\surd(3):7+\surd 2$: which last is called an Universal Root, and signifies the cubick Root of the summ of 7, and the square Root of 2. (See *Sect.* 28. *Chap.* 1. *Book* 1. concerning the designing of Surd numbers.)

The Arithmetick of Surd Numbers, and Surd Quantities design'd by Letters, depends chiefly upon these six Primary or Fundamental Operations in simple Surds, *viz.*

1. The Reduction of Rational numbers and Rational quantities exprest by letters, to the form of surd Roots, which shall have a given radical Sign.
2. The Reduction of simple surd Roots having different radical Signs, to other Surds which shall have one common radical Sign, and be equal in value to the given Surds.
3. Multiplication in simple Surds.
4. Division in simple Surds.
5. The Reduction of a given Surd number or quantity to another more simple, when it may be done.
6. How to discover whether two simple Surd numbers or quantities be *Commensurable* or not, *viz.* whether their *Reason* or *Proportion* can be exprest by Rational numbers or quantities, or not. These six Operations I shall handle in order.

Sect. II.

Sect. II. *How to Reduce Rational numbers and quantities designed by Letters, to the form of Surd Roots, which shall have the same Radical Sign with any Surd Root prescribed.*

Multiply the given Rational number or quantity into it self, so often as is requisite to produce a Power of the same Degree with that Power which is denoted by the radical Sign of the prescribed Surd, and then set the said radical Sign before the Power produced by the said multiplication.

As, to reduce 6 to the form of a surd Root which shall have the same radical Sign with $\surd 12$ (or $\surd(2)12$,) I multiply 6 into it self quadratickly, and it makes 36; then $\surd 36$ (that is 6,) and $\surd 12$ have the same radical Sign, to wit, $\surd$ or $\surd(2)$.

Again, to reduce 5 to the same radical Sign with $\surd(3)12$, I multiply 5 into it self cubically, (*viz.* 5 into 5, and the Product into 5,) and it produceth 125; then $\surd(3)125$ (that is, 5,) and $\surd(3)12$ have the same radical Sign, to wit, $\surd(3)$.

Likewise, to reduce 3 to the same radical Sign with $\surd(4)12$, I seek the fourth Power of 3, which (by multiplying the Square of 3 into it self) will be found 81; then $\surd(4)81$ and $\surd(4)12$ are of the same kind. And so of others.

By the help of this Rule, when the radical Sign of a simple Surd Fraction hath reference only to one of its Terms, we may reduce the Fraction to another whose radical Sign shall refer both to the Numerator and Denominator: As if $\frac{\surd 2}{5}$ be proposed, which signifies that $\surd 2$ is divided or to be divided by 5, we may take $\surd 25$ instead of 5, and then that Fraction will be reduced to this $\surd\frac{2}{25}$, whose radical Sign refers as well to the Denominator as the Numerator; *viz.* $\surd\frac{2}{25}$ signifies that $\surd 2$ is divided by $\surd 25$.

Likewise $\frac{5}{\surd(3)4}$ may be reduced to $\surd(3)\frac{125}{4}$, by setting 125 the Cube of 5 for a Numerator instead of 5, and the radical Sign $\surd(3)$ against the middle of the Fraction; so that $\surd(3)\frac{125}{4}$ (which signifies that $\surd(3)125$ is divided by $\surd(3)4$) imports as much as $\frac{5}{\surd(3)4}$, (that is, 5 divided by $\surd(3)4$.

Nor will the Operation be otherwise in reducing Rational quantities designed by letters to the form of Surd quantities; (respect being had to the Rules of Algebraical Multiplication before delivered.) As to reduce the quantity a, so as it may have the same radical Sign with $\surd b$, I multiply a into it self quadratickly, and it makes aa; then $\surd aa$ (that is, a) and $\surd b$ have the same radical Sign.

Again, to reduce $a+b$ to the same radical Sign with $\surd bc$, I square $a+b$ and it makes $aa+2ab+bb$; then $\surd{:}\overline{aa+2ab+bb}{:}$ (that is, $a+b$) and $\surd bc$ have the same radical Sign.

Likewise, to reduce b to the same radical Sign with $\surd(3)ab$, I multiply b into it self cubically, and it makes bbb; then $\surd(3)bbb$ (that is, b) and $\surd(3)ab$ have the same radical Sign, to wit, $\surd(3)$.

Hence also $\frac{a}{\surd b}$ may be reduced to $\surd\frac{aa}{b}$; and $\frac{\surd(3)ab}{3c}$ to $\surd(3)\frac{ab}{27ccc}$.

Sect. III. *How to reduce two simple Surd numbers or quantities having different radical Signs, to two others that may have a common radical Sign.*

This Reduction is like that of reducing Vulgar Fractions to a common Denominator; but how 'tis wrought, I shall shew by Examples, first in Surd numbers, and then in Surd quantities exprest by letters.

Example 1.

Let it be required to reduce $\surd(4)10$ and $\surd(6)7$ into two other Roots that may have a common radical Sign, and be equal in value to those given.

First divide the given Indices (4) and (6) by their greatest common Divisor (2), and set the Quotients (2) and (3) under their respective Dividends as here you see; then multiply cross-wise, *viz.* the first Dividend or Index (4), by the second Quotient (3); (or the second Dividend (6) by the

$$\begin{array}{cccc} (2)\,) & \surd(4)10 & \times & \surd(6)7 \\ & (2) & & (3) \\ & \surd(12)1000 & & \surd(12)49 \end{array}$$

the firſt Quotient (2),) and the Product is (12), before which ſetting √ it gives √(12), which is to be reſerved for the common radical Sign ſought. Then multiply the Powers of the given Roots according to the altern Quotients, *viz.* multiply the firſt Power 10 cubically, becauſe the ſecond Quotient is (3); and latter Power 7 quadratickly, becauſe the firſt Quotient is (2): ſo the Products will be 1000 and 49, before each of which prefixing √(12) the common radical Sign before found, there ariſe √(12)1000 and √(12)49 the two ſurd Roots ſought, which are equal in value to the given Surds reſpectively; *viz.* √(12)1000 is equal to √(4)10, and √(12)49 is equal to √(6)7; and the Surds found out have a common Radical Sign, as was required.

Example 2.

In like manner, √(2)5 and √(3)6 will be reduced to √(6)125 and √(6)36; and the work will ſtand as here you ſee underneath.

$$(1)\Big)\quad \begin{array}{c} \surd(2)5 \\ (2) \\ \surd(6)125 \end{array} \quad \times \quad \begin{array}{c} \surd(3)6 \\ (3) \\ \surd(6)36. \end{array}$$

Example 3.

Again, if $\frac{\surd 7}{3}$ and $\frac{5}{\surd(3)4}$ be propoſed to be reduced to a common Radical Sign, firſt by the Rule in the preceding *Sect.* 2. I reduce them to $\surd\frac{7}{9}$ (or $\surd(2)\frac{7}{9}$) and $\surd(3)\frac{125}{4}$, which according to the Rule in the firſt Example of this Section will be reduced to theſe, to wit, $\surd(6)\frac{343}{729}$ and $\surd(6)\frac{15625}{16}$; and the work will ſtand as here you ſee.

$$(1)\Big)\quad \begin{array}{c} \surd(2)\frac{7}{9} \\ (2) \\ \surd(6)\frac{343}{729} \end{array} \quad \times \quad \begin{array}{c} \surd(3)\frac{125}{4} \\ (3) \\ \surd(5)\frac{15625}{16}. \end{array}$$

The like work is to be done in reducing two Surd quantities expreſt by letters, which have different radical Signs, to two others which ſhall have a common radical Sign, as will appear in the following Examples.

Example 4.

Suppoſe it be deſired to reduce √(2)*a* and √(6)*aa* to a common radical Sign.

Firſt, I divide the given Indices (2) and (6) ſeverally by their greateſt common Diviſor (2) and ſet the Quotients (1) and (3) under their reſpective Dividends as here you ſee; then I multiply croſs-wiſe, *viz.* the firſt Dividend (2) by the ſecond Quotient (3), (or the latter Dividend (6) by the firſt Quotient (1), and the Product is (6); before which

$$(2)\Big)\quad \begin{array}{c} \surd(2)a \\ (1) \\ \surd(6)aaa \end{array} \quad \times \quad \begin{array}{c} \surd(6)aa \\ (3) \\ \surd(6)aa \end{array}$$

ſetting √ it gives √(6) for the Common Radical ſign ſought. Then I multiply the Powers of the given Roots according to the alternate Quotients, *viz.* the firſt Power *a* cubically, becauſe the latter Quotient is (3), but the ſecond Power *aa*, becauſe the firſt Quotient (1) is a lateral Index, is not to be multiplied into it ſelf at all. So the Products are *aaa* and *aa*, before each of which prefixing √(6), (the common radical Sign before found,) there ariſe √(6)*aaa* and √(6)*aa* the two ſurd Roots ſought; which are equal in value to the given Surds reſpectively, *viz.* √(6)*aaa* is equal to √(2)*a*, and √(6)*aa* is equal to √(6)*aa*; and the ſurd Roots found out have a common radical Sign, to wit, √(6). Therefore that is done which was required.

Example 5.

After the ſame manner √(4)3*b* and √(10)5*ac* will be reduced to √(20)243*bbbbb* and √(20)25*aacc*, and the work will ſtand as here you ſee.

$$(2)\Big)\quad \begin{array}{c} \surd(4)3b \\ (2) \\ \surd(20)243bbbbb \end{array} \quad \times \quad \begin{array}{c} \surd(10)5ac \\ (5) \\ \surd(20)25aacc. \end{array}$$

Sect. IV.

Sect. IV: *Multiplication in simple Surd Quantities.*

Before Addition and Subtraction can be perform'd in Surd Quantities, the manner of their Multiplication and Division must first be learnt; I shall therefore begin with Multiplication, which requires that the surd Roots proposed to be multiplied be of the same kind; and therefore if they be of different kinds, they must first of all be reduced to the same Radical sign, (by the Rule in the foregoing *Sect.* 3.) Then,

1. Multiply the numbers or quantities standing next after their common Radical sign one into another, without any regard had to the said Sign; and to the Product of that multiplication prefix the common Radical sign: so this new Root shall be the Product sought.

As, for example, to multiply $\surd 5$ by $\surd 3$, I multiply 5 by 3 and it makes 15; to which I prefix $\surd$, (the Radical sign of each of the Surds given to be multiplied,) and there ariseth $\surd 15$ for the Product sought.

Likewise if $\surd 6$ be multiplied by $\surd 5$, it produceth $\surd 30$.

Also, $\surd\frac{3}{4}$ multiplied by $\surd\frac{1}{2}$, makes $\surd\frac{3}{8}$.

And $\surd 2\frac{1}{2}$ (or $\surd\frac{5}{2}$) into $\surd 2\frac{1}{3}$ (or $\surd\frac{7}{3}$,) gives $\surd\frac{35}{6}$.

Again, $\surd(3)4$ multiplied by $\surd(3)5$, produceth $\surd(3)20$.

Likewise $\surd(4)\frac{5}{2}$ into $\surd(4)2$, produceth $\surd(4)5$.

And if $\surd(2)5$ be to be multiplied into $\surd(3)6$, the Product will be $\surd(6)4500$; for, first, the given Roots being of different kinds are reduced to these, to wit, $\surd(6)125$ and $\surd(6)36$, which multiplied one into another make $\surd(6)4500$.

After the same manner, Multiplication in simple Surd quantities exprest by Letters is performed: as, if $\surd a$ be to be multiplied by $\surd b$, the Product will be $\surd ab$. For (according to the Rule of Algebraical Multiplication) the quantity a multiplied by the quantity b, produceth ab; to which I prefix the given Radical sign $\surd$, and it gives $\surd ab$ the Product sought.

Likewise $\surd ab$ into $\surd cd$, produceth $\surd abcd$.

And $\surd\frac{2ab}{3c}$ multiplied by $\surd\frac{9ad}{2b}$, maketh $\surd\frac{3aad}{c}$.

Again, to multiply $\surd(2)d$ by $\surd(3)ab$, first (by the Rule in the foregoing *Sect.* 3.) I reduce them to $\surd(6)ddd$ and $\surd(6)aabb$, which multiplied one into another, give $\surd(6)dddaabb$ for the Product required.

2. When any Surd Root is to be multiplied into it self according to the Index of its own Power, *viz.* if a surd Square Root be to be squared, or a surd Cubick Root be to be cubed; cast away the Radical sign, and take the number or quantity remaining for the Product sought, which in this case is alwayes Rational: as, to multiply $\surd 5$ into it self, I cast away the Radical sign $\surd$, and take 5 for the Product, or Square of $\surd 5$; (for $\surd 5$ into $\surd 5$ makes $\surd 25$, that is, 5.) Likewise, the Square of $\surd 8$ is 8, and the Square of $\surd 4$ is 4.

In like manner, to multiply $\surd(3)5$ into it self cubically, I take 5 for the Product, to wit, the Cube of $\surd(3)5$: (for $\surd(3)5$ into $\surd(3)5$ makes $\surd(3)25$, and this again into $\surd(3)5$ produceth $\surd(3)125$, that is, 5.)

Again, $\surd(4)12$ multiplied into it self biquadratickly, produceth 12; for $\surd(4)12$ into $\surd(4)12$ maketh $\surd(4)144$, (which is the Square of $\surd(4)12$;) then $\surd(4)144$ again into $\surd(4)12$ makes $\surd(4)1728$, (which is the Cube of $\surd(4)12$;) lastly, $\surd(4)1728$ again into $\surd(4)12$ produceth $\surd(4)20736$, that is 12; which is the fourth Power of $\surd(4)12$, the Root proposed.

The like is to be done in Surd quantities exprest by Letters; as, if $\surd ab$ be to be multiplied into it self, or squared, I cast away the Radical sign, and write ab for the Product or Square of $\surd ab$. Likewise, if $\surd(3)bcd$ be to be multiplied into it self cubically, the Product or Cube thereof will be bcd.

3. When a Surd quantity is given to be multiplied by a Rational quantity, reduce the Rational into the form of a Surd of the same kind with the given Surd, (by the foregoing Rule in *Sect.* 2.) and then multiply according to the first Rule of this fourth Section; as, to multiply $\surd 8$ by 2, I first reduce 2 to $\surd 4$, then $\surd 8$ into $\surd 4$ gives $\surd 32$, the Product desired: likewise $\surd 7$ multiplied by 5, that is, by $\surd 25$, gives the Product $\surd 175$.

Again, if $\surd(3)6$ be to be multiplied by 2, I reduce 2 to $\surd(3)8$, (by multiplying 2 into it self cubically;) then $\surd(3)6$ multiplied by $\surd(3)8$, gives $\surd(3)48$ for the Product desired.

Like-

Likewiſe $\sqrt{(4)}8$ multiplied by 5, that is, by $\sqrt{(4)}625$, gives $\sqrt{(4)}5000$ for the Product ſought.

After the ſame manner, to multiply the Surd quantity $\sqrt{a}$ by the Rational quantity b, I firſt reduce b to $\sqrt{bb}$, then $\sqrt{a}$ into $\sqrt{bb}$ makes $\sqrt{abb}$ the Product ſought; likewiſe, $\sqrt{(3)}a$ into b makes $\sqrt{(3)}abbb$, (b being firſt reduced to $\sqrt{(3)}bbb$.)

Again, $\sqrt{3}$ into $4a$ gives the Product $\sqrt{48aa}$.

4. But when a Surd quantity is given to be multiplied by a Rational quantity, it will oftentimes be very convenient to omit their multiplication, and only to connect them ſo as that the Rational quantity may ſtand on the left hand of the given Surd, to ſignifie the Product of their multiplication; as, to multiply $\sqrt{8}$ by 2, I write $2\sqrt{8}$ for the Product, which ſignifies twice the ſquare Root of 8. Likewiſe $20\sqrt{3}$ repreſents the Product of the multiplication of $\sqrt{3}$ by 20, *viz.* it imports $\sqrt{3}$ to be taken 20 times; which amounts to as much as $\sqrt{1200}$, found out by the preceding third Rule of this Section.

Again, $\frac{8}{3}\sqrt{7}$ ſignifies the Product of $\sqrt{7}$ multiplied by $\frac{8}{3}$, (or $\frac{8}{3}$ by $\sqrt{7}$;) and $\frac{2}{3}\sqrt{\frac{7}{5}}$ denotes the Product of $\frac{2}{3}$ multiplied into $\sqrt{\frac{7}{5}}$, (or $\sqrt{\frac{7}{5}}$ into $\frac{2}{3}$;) alſo, 4 into $20\sqrt{3}$ makes $80\sqrt{3}$, that is, $20\sqrt{3}$ taken four times. Likewiſe $2\sqrt{(3)}6$ ſignifies twice the Cubick Root of 6, and is of equal value with $\sqrt{(3)}48$; likewiſe $\frac{5}{3}\sqrt{(3)}80$ denotes the Product of the Cubick Root of 80 multiplied by $\frac{5}{3}$, or $\frac{5}{3}$ of $\sqrt{(3)}80$, which is equivalent to $\sqrt{(3)}\frac{10000}{27}$; and $3\sqrt{(3)}5$ multiplied by 6 makes $18\sqrt{(3)}5$, that is, $\sqrt{(3)}29160$.

The like may be done in Surd quantities expreſt by Letters. As, if $\sqrt{a}$ be to be multiplied by b, I write $b\sqrt{a}$ to ſignifie the Product; alſo, 5 into $b\sqrt{a}$ makes $5b\sqrt{a}$; and c into $b\sqrt{a}$, gives the Product $cb\sqrt{a}$; likewiſe, $4a$ into $\sqrt{3}$ makes $4a\sqrt{3}$.

Again, if $\sqrt{ab}$ be to be multiplied by $b-d$, the Product may be expreſt thus, $\overline{b-d}\times\sqrt{ab}$; or thus, $\overline{b-d}\sqrt{ab}$.

Alſo, if $\sqrt{(3)}\frac{2ab}{c}$ be to be multiplied by d, the Product may be expreſt thus, $d\sqrt{(3)}\frac{2ab}{c}$; and $\sqrt{(3)}a$ into b, makes $b\sqrt{(3)}a$, which is equivalent to $\sqrt{(3)}abbb$.

5. When two Rational quantities, whether they be equal or unequal, are multiplied ſeverally into one common ſurd ſquare Root, according to the method in the preceding fourth Rule, and it is deſired to multiply thoſe Products one into the other, (which Products are called Commenſurable quantities, for the reaſon hereafter given in *Sect.* 7.) multiply the Rational by the Rational, and that which is produced multiply by the ſaid common Surd, omitting its Radical ſign; ſo the laſt Product is that which is ſought, and will be entirely Rational.

As, for example, to multiply $3\sqrt{5}$ by $2\sqrt{5}$, I multiply 3 by 2, and the Product 6 by 5, ſo it makes 30; which is the Product of $3\sqrt{5}$ multiplied by $2\sqrt{5}$, (or of $\sqrt{45}$ into $\sqrt{20}$.)

Likewiſe, $2\sqrt{3}$ multiplied by $2\sqrt{3}$, (*viz.* the Square of $2\sqrt{3}$,) makes 12; and $20\sqrt{3}$ into $8\sqrt{3}$ makes 480, (by multiplying 20, 8 and 3 one into another continually;) again, $\frac{8}{3}\sqrt{12}$ into $5\sqrt{12}$, produceth 160.

After the ſame manner, to multiply $a\sqrt{c}$ by $b\sqrt{c}$, I multiply a by b, and the Product ab by c; ſo there ariſeth abc for the Product ſought. The Reaſon of this Rule is evident, for $\sqrt{aac}$, (that is, $a\sqrt{c}$) multiplied into $\sqrt{bbc}$, (that is, $b\sqrt{c}$) makes $\sqrt{aabbcc}$, that is abc; as before.

In like manner, $5\sqrt{b}$ into $5\sqrt{b}$ produceth $25b$, to wit, the Square of $5\sqrt{b}$; and $2a\sqrt{b}$ into $5a\sqrt{b}$ gives the Product $10aab$: alſo, $5a\sqrt{12d}$ multiplied by $\frac{8}{3}a\sqrt{12d}$, produceth $160aad$.

But here is to be noted, that this fifth Rule of multiplication takes place only when the common ſurd Root into which Rational numbers are multiplied is a ſurd ſquare Root; ſo that if $4\sqrt{(3)}5$ be to be multiplied by $2\sqrt{(3)}5$, the ſaid fifth Rule will be ineffective, and the Product is to be found out by the following ſixth Rule.

6. When two Rational quantities, whether they be equal or unequal, are multiplied into two unequal ſurd Roots of the ſame kind, or into one common Surd above the quadratick kind, according to the method in the foregoing fourth Rule of this Section, and it is deſired to multiply thoſe Products one into another; multiply the Rational by the Rational, and the Surd by the Surd, and joyn theſe Products together, ſo as the Rational Product may ſtand on the left hand; then thoſe two Products ſo connected ſhall be the Product ſought.

As, for example, to multiply $5\sqrt{8}$ by $2\sqrt{3}$, I multiply 5 by 2, and the Product is 10; alſo, $\sqrt{8}$ into $\sqrt{3}$ make $\sqrt{24}$: then thoſe two Products connected make $10\sqrt{24}$, (that is, $\sqrt{2400}$,)

$\surd 2400$,) the Product sought. In like manner, $2\surd 8$ into $2\surd 3$ makes $4\surd 24$, that is, $\surd 384$.

Again, $20\surd 5$ multiplied by $18\surd 3$ produceth $360\surd 15$; and $8\surd 27$ into $2\surd 3$ makes $16\surd 81$, that is, 144; also, $5\surd(3)4$ into $3\surd(3)5$ produceth $15\surd(3)20$, that is, $\surd(3)3375$; likewise, $4\surd(3)5$ into $2\surd(3)5$ maketh $8\surd(3)25$; and $3\surd(4)5$ into $2\surd(4)6$, makes $6\surd(4)30$.

After the same manner, to multiply $a\surd bc$ into $g\surd ad$; first, I multiply a by g, and it makes ag; then, $\surd bc$ into $\surd ad$ produceth $\surd bcad$; lastly, ag into $\surd bcad$ gives $ag\surd bcad$, the Product sought.

Likewise, $2\surd ab$ multiplied by $3c\surd bc$, produceth $6c\surd abbc$; and $2\surd a$ into $2\surd b$ makes $4\surd ab$.

Also, $\frac{2bc}{a}\surd ddd$ multiplied by $\frac{aa}{2c}\surd ac$, gives the Product $ab\surd acddd$; and $b\surd(3)dd$ into $c\surd(3)f$ makes $bc\surd(3)ddf$; again, $a\surd(3)c$ into $b\surd(3)c$ makes $ab\surd(3)cc$.

7. When a simple Surd quantity whose Radical sign hath for its Index some even number greater than 2 is to be squared, prefix a Radical sign whose Index is half the given Index, before the Power of the given Surd; so shall this new Surd be the Square of that given. As, if $\surd(4)5$ be to be squared or multiplied into it self, take $\surd(2)5$, or $\surd 5$, for the Square or Product sought: likewise, the Square of $\surd(6)10$ is $\surd(3)10$: and $\surd(8)10$ into $\surd(8)10$ makes $\surd(4)10$.

After the same manner, to multiply $\surd(4)bc$ into it self quadratickly, I write $\surd(2)bc$, or $\surd bc$, for the Product, or Square of $\surd(4)bc$: likewise, the Square of $\surd(8)10bc$ is $\surd(4)10bc$: and $\surd(10)a$ into $\surd(10)a$, makes $\surd(5)a$: moreover, $2ab\surd(4)d$ into $3\surd(4)d$ makes $6ab\surd d$; for $2ab$ into 3 makes $6ab$, and $\surd(4)d$ being squared makes $\surd(2)d$ or $\surd d$.

But when a simple Surd quantity whose Radical sign hath for its Index some ternary number greater than 3, as 6, 9, &c. is to be multiplied into it self cubically, prefix a Radical sign with an Index that may be a third part of the given Index before the Power of the given surd Root, so shall this new Surd be the Cube of that given: As, if $\surd(6)64$ be to be multiplied into it self cubically, then $\surd(2)64$ or $\surd 64$ shall be the Cube sought; likewise, the Cube of $\surd(9)512$ is $\surd(3)512$.

More Examples to exercise the precedent Rules of Multiplication in simple Surd Numbers.

Multiply	$\surd 5$	$\surd(3)4$	$\surd(4)8$
by	$\surd 8$	$\surd(3)7$	$\surd(4)2$
Product	$\surd 40$	$\surd(3)28$	$\surd(4)16$, that is, 2.

Multiply	$\surd 32$	Multiply these three continually, $\surd(3)50$
by	$\surd 32$	$\surd(3)50$
		$\surd(3)50$
Product	32	50

Multiply	$\surd 27$	12
by	6	$\surd(3)5$
Product	$6\surd 27$, or, $\surd 972$	$12\surd(3)5$, or, $\surd(3)8640$

Multiply	$18\surd 5$	$24\surd 6\frac{3}{8}$	$6\surd 7$
by	$4\surd 5$	$5\surd 6\frac{3}{8}$	$5\surd 3$
Product	360	765	$30\surd 21$

Multiply	$\surd 8$	that is, $\surd(6)512$	$4\surd 5$
by	$\surd(3)4$	$\surd(6)16$	$4\surd 5$
Product	. . . . . .	$\surd(6)8192$	80

Multiply	$5\surd 8$	$12\surd(3)4$	$\surd(4)12$
by	4	$2\frac{1}{2}$	$\surd(4)12$
Product	$20\surd 8$	$30\surd(3)4$	$\surd 12$

More

More Examples to exercise the precedent Rules of Multiplication in simple Surd Quantities exprest by Letters.

Multiply	$\sqrt{12a}$	$\sqrt{\frac{8}{3}ab}$
by	$\sqrt{3a}$	$\sqrt{\frac{3}{2}ac}$
Product	$\sqrt{36aa}$, or, $6a$	$\sqrt{4aabc}$, or, $2a\sqrt{bc}$.

Multiply	$\sqrt{a}$	that is,	$\sqrt{(6)aaa}$
by	$\sqrt{(3)aa}$		$\sqrt{(6)aaaa}$
Product	. . . .		$\sqrt{(6)a^7}$.

Multiply	$\sqrt{27aa}$	Multiply these three continually,	$\sqrt{(3)aa}$
by	$\sqrt{27aa}$		$\sqrt{(3)aa}$
			$\sqrt{(3)aa}$
Product	$27aa$		aa

Multiply	$\sqrt{3bc}$	$5b$
by	2	$\sqrt{(3)2a}$
Product	$2\sqrt{3bc}$, or, $\sqrt{12bc}$	$5b\sqrt{(3)2a}$, or, $\sqrt{(3)250abbb}$.

Multiply	$3a\sqrt{5}$	$7\sqrt{bc}$	$\frac{8}{3}a\sqrt{bc}$
by	$2b\sqrt{5}$	$4\sqrt{bc}$	$\frac{3}{4}b\sqrt{bc}$
Product	$30ab$	$28bc$	$2abbc$.

Multiply	$5\sqrt{ab}$	$3a\sqrt{5}$	$\frac{2bc}{a}\sqrt{d}$
by	$3\sqrt{ac}$	$2b\sqrt{6}$	$\frac{aa}{2c}\sqrt{d}$
Product	$15\sqrt{aabc}$	$6ab\sqrt{30}$	abd.

The certainty of the first Rule of this fourth Section, (upon which all the rest depend) for the multiplication of two simple Surd numbers of the same kind, may be Demonstrated in manner following. First, let there be two square Roots given to be multiplied, suppose $\sqrt{5}$ and $\sqrt{3}$, then (by the said Rule) the Product of their Multiplication is $\sqrt{15}$; now we must prove that $\sqrt{15}$ is the true Product of $\sqrt{5}$ multiplied by $\sqrt{3}$.

Demonstration.

By the Definition of Multiplication, these are Proportionals, *viz.*	$1 \,.\, \sqrt{5} :: \sqrt{3} \,.$	Product,
Therefore their Squares shall be also Proportionals, (*per* 22. *prop.* 6. *Elem.* Euclid.) *viz.* . . .	$1 \,.\, 5 :: 3 \,.$	Square of the Product.
But these are Proportionals, (*per* 19. *prop.* 7. *Elem.* Euclid.) . .	$1 \,.\, 5 :: 3 \,.$	15.

Therefore, from the two last Analogies, 15 is equal to the Square of the Product; and consequently $\sqrt{15}$ is the Product of $\sqrt{5}$ into $\sqrt{3}$: which was to be proved.

Likewise in Cubick Roots, if $\sqrt{(3)5}$ be to be multiplied by $\sqrt{(3)4}$, the Product (by the same Rule) is $\sqrt{(3)20}$. For,

By the Definition of Multiplication, these are Proportionals, *viz.*	$1 \,.\, \sqrt{(3)5} :: \sqrt{(3)4} \,.$	Product,
Therefore their Cubes are also Proportionals, (*per prop.* 37. *Elem.* 11. Euclid.) *viz.*	$1 \,.\, 5 :: 4 \,.$	Cube of the Product.
But, as	$1 \,.\, 5 :: 4 \,.$	20.

There-

Therefore 20 is equal to the Cube of the Product; and consequently the cubick Root of 20, to wit, $\surd(3)20$ is the Product of $\surd(3)5$ multiplied by $\surd(3)4$: which was to be proved.

Moreover, because (by *Sect.* 11. *Chap.* 5.) if four numbers be Proportionals, their fourth Powers, fifth Powers, *&c.* are also Proportionals, this Demonstration may be extended to prove the certainty of the said Rule for multiplying any two simple Surd numbers of the same kind.

Sect. V. *Division in simple Surd Quantities.*

As before in Multiplication, so here in Division, if the given Surd Roots, to wit, the Dividend and Divisor be not of the same kind, they must be reduced to a common Radical sign by the preceding *Sect.* 3. Then,

1. Divide the Number or Quantity following the Radical sign of the Dividend, by the Number or Quantity following the same Radical sign of the Divisor, without any regard to the Sign, and to the Quotient prefix the said common Radical sign; so this new Root shall be the Quotient sought.

As, for example, to divide $\surd 15$ by $\surd 3$; I divide 15 by 3, and there ariseth 5, before which I prefix $\surd$, (the Radical sign common to the given Surds,) so $\surd 5$ is the Quotient sought.

Likewise, if $\surd 30$ be divided by $\surd 5$, the Quotient is $\surd 6$.

Also, $\surd\frac{3}{8}$ divided by $\surd\frac{3}{4}$ gives the Quotient $\surd\frac{1}{2}$.

And $\surd 5\frac{5}{6}$, or $\surd\frac{35}{6}$, divided by $2\frac{1}{3}$, or $\frac{7}{3}$, gives the Quotient $2\frac{1}{2}$.

Again, $\surd(3)20$ divided by $\surd(3)5$, gives the Quotient $\surd(3)4$; for 20 divided by 5 gives 4, before which setting $\surd(3)$ the Radical sign belonging to each of the given Surds, there ariseth $\surd(3)4$ for the Quotient sought.

Likewise $\surd(4)5$ divided by $\surd(4)\frac{5}{2}$, gives the Quotient $\surd(4)2$.

Moreover, if $\surd(6)4500$ be given to be divided by $\surd(2)5$, the Quotient will be $\surd(3)6$; for first, the given Roots being of different kinds are reduced to these, to wit, $\surd(6)4500$ and $\surd(6)125$; then by dividing $\surd(6)4500$ by $\surd(6)125$ there ariseth $\surd(6)36$, whose square Root being extracted, (because 36 is a square number, and the Index (6) an even number,) it gives $\surd(3)6$ for the Quotient sought.

After the same manner, Division is perform'd in simple Surd Quantities exprest by Letters. As, to divide $\surd ab$ by $\surd a$; I divide ab by a and there ariseth b, then setting $\surd$ before b, it gives $\surd b$ for the Quotient sought; to wit, the Quotient that ariseth by dividing $\surd ab$ by $\surd a$.

Also, $\surd b$ divided by $\surd a$, gives the Quotient $\surd\frac{b}{a}$.

Likewise, $\surd abcd$ divided by $\surd ab$ gives the Quotient $\surd cd$.

Also, $\surd\frac{3aad}{c}$ divided by $\frac{2ab}{3c}$ gives the Quotient $\surd\frac{9ad}{2b}$.

Again, to divide $\surd(6)dddaabb$ by $\surd(3)ab$, I first reduce them to $\surd(6)dddaabb$ and $\surd(6)aabb$; then I divide $\surd(6)dddaabb$ by $\surd(6)aabb$, and there ariseth $\surd(6)ddd$, that is, $\surd(2)d$, for the Quotient sought.

2. When a Rational number or quantity is to be divided by its square Root, that Root is the Quotient; as, if 5 be divided by its square Root, to wit by $\surd 5$, the Quotient will be $\surd 5$: also, 8 divided by $\surd 8$ gives $\surd 8$ for the Quotient.

In like manner if the quantity bc be divided by its square Root, to wit, by $\surd bc$, the Quotient will be $\surd bc$; and $5a$ divided by $\surd 5a$, gives the Quotient $\surd 5a$.

3. When a Surd number or quantity is to be divided by a Rational number or quantity, or a Rational number or quantity by a Surd; reduce the Rational into the form of a Surd, (by *Sect.* 2. of this *Chapt.*) and then divide according to the first Rule of this *Sect.* 5.

As, to divide $\surd 32$ by 2, I first reduce 2 to $\surd 4$; then by dividing $\surd 32$ by $\surd 4$, there ariseth $\surd 8$ for the Quotient.

Likewise $\surd 175$ divided by 5, that is, $\surd 25$, gives the Quotient $\surd 7$.

Also 12, that is, $\surd 144$, divided by $\surd 3$ gives the Quotient $\surd 48$.

Again, if $\surd(3)48$ be to be divided by 2, I first reduce 2 to $\surd(3)8$; then by dividing $\surd(3)48$ by $\surd(3)8$, there ariseth $\surd(3)6$ for the Quotient sought: also, $\surd(4)5000$ divided by 5, (that is, by $\surd(4)625$) gives the Quotient $\surd(4)8$.

After

After the same manner, to divide the quantity $\sqrt{abb}$ by b, I first reduce b to $\sqrt{bb}$; and then by dividing $\sqrt{abb}$ by $\sqrt{bb}$, there ariseth $\sqrt{a}$ the Quotient sought. Again, $\sqrt{48aa}$ divided by $4a$, that is by $\sqrt{16aa}$, gives the Quotient $\sqrt{3}$. Also $\sqrt{(3)}abbb$ divided by b, that is by $\sqrt{(3)}bbb$, gives the Quotient $\sqrt{(3)}a$.

Likewise, to divide the Rational quantity $\frac{bc}{a}$ by $\sqrt{(3)}bbcc$; I first reduce $\frac{bc}{a}$ to $\sqrt{(3)}\frac{bbbccc}{aaa}$, then I divide $\sqrt{(3)}\frac{bbbccc}{aaa}$ by $\sqrt{(3)}bbcc$, and there ariseth $\sqrt{(3)}\frac{bc}{aaa}$, or $\frac{\sqrt{(3)}bc}{a}$, the Quotient sought.

4. When the Product of a Rational number or quantity multiplied into a Surd number or quantity is to be divided by the same Surd, the Quotient will be the said multiplying Rational number or quantity. As, $5\sqrt{3}$ divided by $\sqrt{3}$ gives the Quotient 5: also, $20\sqrt{(3)}4$ divided by $\sqrt{(3)}4$ gives the Quotient 20.

In like manner, $5a\sqrt{b}$ divided by $\sqrt{b}$ gives the Quotient $5a$; and $4b\sqrt{(3)}12$ divided by $\sqrt{(3)}12$ gives the Quotient $4b$.

5. When the Dividend and Divisor are the Products of two Rational numbers or quantities multiplied severally into one common Surd, according to the fourth Rule of Multiplication in *Sect.* 4. (which Products are called Commensurable Surd Roots, as hereafter will appear in *Sect.* 7. of this *Chapt.*) divide the Rational part of the Dividend by the Rational part of the Divisor, and that which ariseth shall be the Quotient sought. As, for example, to divide $6\sqrt{3}$ by $2\sqrt{3}$, I divide 6 by 2, and there ariseth 3 the Quotient sought; (for $2\sqrt{3}$ multiplied by 3, produceth $6\sqrt{3}$.)

Again, $5\sqrt{6}$ divided by $2\sqrt{6}$ gives the Quotient $\frac{5}{2}$, or $2\frac{1}{2}$.

Also, $2\sqrt{6}$ divided by $5\sqrt{6}$ gives the Quotient $\frac{2}{5}$; and $2\sqrt{5}$ divided by $2\sqrt{5}$, gives the Quotient 1.

So also $8\sqrt{(3)}7$ divided by $4\sqrt{(3)}7$, gives the Quotient 2; and $3\sqrt{(4)}5$ divided by $4\sqrt{(4)}5$, gives $\frac{3}{4}$ for the Quotient.

In like manner, to divide $4a\sqrt{7}$ by $2a\sqrt{7}$; I divide $4a$ by $2a$, and there ariseth 2, the Quotient sought; (for $2a\sqrt{7}$ into 2 produceth $4a\sqrt{7}$:) also, $3\sqrt{b}$ divided by $5\sqrt{b}$ gives the Quotient $\frac{3}{5}$; and $2\sqrt{b}$ divided by $2\sqrt{b}$, gives the Quotient 1.

Again, $5a\sqrt{3b}$ divided by $3a\sqrt{3b}$ gives the Quotient $\frac{5}{3}$.

And $7ab\sqrt{(3)}dd$ divided by $3b\sqrt{(3)}dd$, gives the Quotient $\frac{7}{3}a$.

6. When the Dividend and Divisor are the Products of two Rational numbers or quantities multiplied into two unequal Surd numbers or quantities, according to the fourth Rule of Multiplication in the preceding *Sect.* 4. (which Products are called Incommensurable Surd Roots, as hereafter will appear;) divide the Rational part of the Dividend by the Rational part of the Divisor, and the Surd part by the Surd part, then connect the Quotients so as the Rational quotient may stand on the left hand, and this new quantity shall be the Quotient sought.

As, for example, if $4\sqrt{15}$ be to be divided by $2\sqrt{5}$, first I divide 4 by 2, and there ariseth 2; also I divide $\sqrt{15}$ by $\sqrt{5}$, and there ariseth $\sqrt{3}$: then those two Quotients joyned together make $2\sqrt{3}$ (or $\sqrt{12}$,) the Quotient sought.

In like manner $4\sqrt{12}$ divided by $3\sqrt{2}$ gives the Quotient $\frac{4}{3}\sqrt{6}$; for 4 divided by 3, (to wit, the Rational by the Rational,) gives $\frac{4}{3}$; and $\sqrt{12}$ divided by $\sqrt{2}$, (to wit, the Surd by the Surd,) gives $\sqrt{6}$: then by joyning together those two Quotients there ariseth $\frac{4}{3}\sqrt{6}$, or $1\frac{1}{3}\sqrt{6}$, (or $\sqrt{2\frac{2}{3}}$) for the Quotient sought.

Again, $2\sqrt{7}$ divided by $3\sqrt{5}$ gives the Quotient $\frac{2}{3}\sqrt{\frac{7}{5}}$; and $2\sqrt{3}$ divided by $2\sqrt{5}$ gives the Quotient $1\sqrt{\frac{3}{5}}$, or $\sqrt{\frac{3}{5}}$.

Likewise to divide $4\sqrt{(3)}64$ by $2\sqrt{(3)}8$, I divide 4 by 2, and it gives 2; also, $\sqrt{(3)}64$ divided by $\sqrt{(3)}8$ gives $\sqrt{(3)}8$; then those two Quotients joyned together make $2\sqrt{(3)}8$, that is 4, the Quotient sought. Moreover, $5\sqrt{(3)}20$ divided by $3\sqrt{(3)}4$ gives the Quotient $\frac{5}{3}\sqrt{(3)}5$.

After the same manner, $4a\sqrt{fb}$ divided by $2a\sqrt{f}$ gives the Quotient $2\sqrt{b}$; for $4a$ divided by $2a$ gives 2; and $\sqrt{fb}$ divided by $\sqrt{f}$ gives $\sqrt{b}$; then connecting those two Quotients there ariseth $2\sqrt{b}$ for the Quotient sought.

So also, $6ab\sqrt{cd}$ divided by $6a\sqrt{df}$ gives the Quotient $b\sqrt{\frac{c}{f}}$.

And

And $a\sqrt{(3)}cc$ divided by $b\sqrt{(3)}dd$, gives the Quotient $\frac{a}{b}\sqrt{(3)}\frac{cc}{dd}$.

The Demonſtration of the aforeſaid firſt Rule of Diviſion (which is the Riſe of all the reſt) may be formed like that of Multiplication in the preceding *Sect.* 4. if there be laid, as a ground-work , this Analogy ; *viz.* As the Diviſor is to 1 (or Unity ,) ſo is the Dividend to the Quotient. But waving the Demonſtration, I ſhall give more Examples of Diviſion in ſimple Surds, both in Numbers and quantities expreſt by Letters.

More Examples to exerciſe Diviſion in ſimple Surd Numbers.

Dividend	$\sqrt{117}$	$\sqrt{(3)}16\frac{1}{3}$, or, $\sqrt{(3)}\frac{49}{3}$	$\sqrt{(4)}256$
Diviſor	$\sqrt{6\frac{1}{2}}$	$\sqrt{(3)}\ 3\frac{1}{2}$, or, $\sqrt{(3)}\ \frac{7}{2}$	$\sqrt{(4)}16$
Quotient	$\sqrt{18}$	$\sqrt{(3)}\ 4\frac{2}{3}$, or, $\sqrt{(3)}\frac{14}{3}$	2

Dividend	$\sqrt{(12)}5125$ } that is,	$\sqrt{(12)}5125$	$\sqrt{(6)}8192$
Diviſor	$\sqrt{(4)}5$ }	$\sqrt{(12)}125$	$\sqrt{(2)}8$
Quotient	. . . .	$\sqrt{(12)}49$, or, $\sqrt{(6)}7$	$\sqrt{(3)}4$

Dividend	12	$5\sqrt{8}$	$16\sqrt{(3)}25$
Diviſor	$\sqrt{12}$	$\sqrt{8}$	$\sqrt{(3)}25$
Quotient	$\sqrt{12}$	5	16

Dividend	$\sqrt{245}$	$\sqrt{(3)}686$	$\sqrt{(5)}23328$
Diviſor	$3\frac{1}{2}$	$3\frac{1}{2}$	6
Quotient	$\sqrt{20}$	$\sqrt{(3)}16$	$\sqrt{(5)}3$

Dividend	$20\sqrt{14}$	$\frac{2}{3}\sqrt{20}$	$5\sqrt{(3)}3$
Diviſor	$2\sqrt{14}$	$\frac{2}{15}\sqrt{20}$	$2\sqrt{(3)}3$
Quotient	10	5	$\frac{5}{2}$, or, $2\frac{1}{2}$

Dividend	$15\sqrt{18}$	$3\sqrt{8}$	$6\sqrt{(3)}24$
Diviſor	$3\sqrt{6}$	$3\sqrt{3}$	$9\sqrt{(3)}4$
Quotient	$5\sqrt{3}$	$\sqrt{\frac{8}{3}}$	$\frac{2}{3}\sqrt{(3)}6$

More Examples to exerciſe Diviſion in ſimple Surd quantities expreſt by Letters.

Dividend	$\sqrt{15bc}$	$\sqrt{(3)}4bbddd$	$\sqrt{(4)}32aa$
Diviſor	$\sqrt{3a}$	$\sqrt{(3)}4bb$	$\sqrt{(4)}2aa$
Quotient	$5\frac{bc}{a}$	$\sqrt{(3)}ddd$, or, d	$\sqrt{(4)}16$, or, 2

Dividend	$\sqrt{(6)}675aaaaabbbbb$ } that is,	$\sqrt{(6)}675a^5b^5$
Diviſor	$\sqrt{(2)}3ab$ }	$\sqrt{(6)}27a^3b^3$
Quotient	. . . .	$\sqrt{(6)}25aabb$, or, $\sqrt{(3)}5ab$

Dividend	$\sqrt{80aaabbb}$	$9bcdd$ (or , $\sqrt{81bbccd^4}$)
Diviſor	$4ab$, (or, $\sqrt{16aabb}$)	$\sqrt{27bcd}$
Quotient	$\sqrt{5ab}$	$\sqrt{3bcddd}$

Dividend	bc	$b\sqrt{df}$	$2d\sqrt{(3)}bb$
Diviſor	$\sqrt{bc}$	$\sqrt{df}$	$\sqrt{(3)}bb$
Quotient	$\sqrt{bc}$	b	$2d$

Dividend

Dividend	$12\sqrt{dc}$	$\frac{2bc}{a}\sqrt{d}$	$ab\sqrt{(3)}f$
Divisor	$3\sqrt{dc}$	$\frac{2c}{b}\sqrt{d}$	$b\sqrt{(3)}f$
Quotient	4	$\frac{bb}{a}$	a
Dividend	$2bc\sqrt{d}$	$b\sqrt{af}$	$6aa\sqrt{(3)}bbbd$
Divisor	$c\sqrt{a}$	$c\sqrt{f}$	$2a\sqrt{(3)}d$
Quotient	$2b\sqrt{\frac{d}{a}}$	$\frac{b}{c}\sqrt{a}$	$3ab.$

Note. By the help of Division, Surd quantities may oftentimes be reduced into others more simple, which being a very useful work, I shall explain it in the next Section.

Sect. VI. *How to reduce a Surd quantity to another more simple, when it may be done.*

When the Power of a Surd quantity, the Radical sign being omitted, can be divided just without any Remainder, by a Power which hath a Rational Root of the same kind with that which is denoted by the said Radical sign; then divide the Surd quantity proposed by that Rational Root, and prefix this Root before the Quotient: so you have a new Surd quantity equal to that proposed, and in more simple Terms.

As, if $\sqrt{63}$ be proposed, because 63 may be divided by the square number 9 without any Remainder, I divide $\sqrt{63}$ by $\sqrt{9}$, (that is, by 3,) and it gives the Quotient $\sqrt{7}$, before which I set the Rational Divisor 3, and it makes $3\sqrt{7}$, (that is 3 into the square Root of 7, or thrice the square Root of 7,) which is equal to $\sqrt{63}$ first proposed; (for the Quotient $\sqrt{7}$ multiplied by the Divisor 3 makes the Dividend $\sqrt{63}$:) so that instead of $\sqrt{63}$ I write $3\sqrt{7}$.

Likewise, instead of $\sqrt{50}$ we may write $5\sqrt{2}$, (which signifies five times the square Root of 2;) for in regard 50 divided by the Square 25 gives 2, I divide $\sqrt{50}$ by $\sqrt{25}$, that is, by 5, and the Quotient is $\sqrt{2}$; and because every Quotient multiplied by the Divisor produceth the Dividend: Therefore $5\sqrt{2}$ shall be equal to the Dividend $\sqrt{50}$.

After the same manner, instead of $\frac{\sqrt{75}}{2}$, or $\sqrt{2\frac{3}{4}}$, we may write $\frac{1}{2}\sqrt{3}$; for $2\frac{3}{4}$ divided by the square number $\frac{2}{4}$ gives the Quotient 3; and consequently, $\sqrt{2\frac{3}{4}}$ divided by $\sqrt{\frac{2}{4}}$, that is by $\frac{1}{2}$, gives the Quotient $\sqrt{3}$: Therefore $\frac{1}{2}\sqrt{3}$ shall be equal to $\frac{\sqrt{75}}{2}$ or $\sqrt{2\frac{3}{4}}$.

Again, instead of $\sqrt{(3)}40$, we may write $2\sqrt{(3)}5$, (which signifies twice the cubick Root of 5;) for 40 divided by the Cube 8 gives the Quotient 5; and consequently, $\sqrt{(3)}40$ divided by $\sqrt{(3)}8$, that is by 2, gives $\sqrt{(3)}5$: Therefore $2\sqrt{(3)}5$ shall be equal to $\sqrt{(3)}40$.

Likewise for $\sqrt{(3)}2\frac{4}{8}$, (or $\frac{\sqrt{(3)}54}{2}$,) we may write $\frac{1}{2}\sqrt{(3)}2$; for $2\frac{4}{8}$ divided by the Cube $\frac{2}{8}$ gives 2; and consequently $\sqrt{(3)}2\frac{4}{8}$ divided by $\sqrt{(3)}\frac{2}{8}$, that is by $\frac{1}{2}$, will give $\sqrt{(3)}2$: Wherefore $\frac{1}{2}\sqrt{(3)}2$ shall be equal to $\sqrt{(3)}2\frac{4}{8}$.

The like Operation is to be done, in reducing Surd quantities exprest by Letters to others more Simple: as, if $\sqrt{75aa}$ be proposed; For as much as $75aa$ divided by the Square $25aa$ gives the Quotient 3, and consequently $\sqrt{75aa}$ divided by $\sqrt{25aa}$, that is, by $5a$ will give $\sqrt{3}$; Therefore the Divisor $5a$ multiplied into the Quotient $\sqrt{3}$, produceth $5a\sqrt{3}$, equal to the Dividend $\sqrt{75aa}$; and therefore instead of $\sqrt{75aa}$, we may write $5a\sqrt{3}$.

After the same manner $\sqrt{10aabb}$ may be reduced to $ab\sqrt{10}$; also $\sqrt{5aa}$ to $a\sqrt{5}$; and $\sqrt{(3)}4ddd$ to $d\sqrt{(3)}4$.

Again, for as much as $\overline{aaab + aabb}$ may be divided by the Square aa and there ariseth $ab + bb$, and consequently $\sqrt{:\overline{aaab + aabb}:}$ divided by $\sqrt{aa}$, that is by a, gives the Quotient $\sqrt{:\overline{ab + bb}:}$ therefore a into $\sqrt{:\overline{ab + bb}:}$ shall be equal to $\sqrt{:\overline{aaab + aabb}:}$ So that instead of $\sqrt{:\overline{aaab + aabb}:}$ we may write a into $\sqrt{:\overline{ab + bb}:}$ or $a\sqrt{:\overline{ab + bb}:}$

Likewise,

Likewiſe, for $\sqrt{:aabbc + 2afbbc + ffbbc:}$ we may write $\overline{a+f}$ into $\sqrt{bbc}$, or $\overline{a+f}\sqrt{bbc}$; for $aabbc+2afbbc+ffbbc$ divided by the Square $aa+2af+ff$ gives bbc, and conſequently $\sqrt{:aabbc+2afbbc+ffbbc:}$ divided by $\sqrt{:aa+2af+ff:}$ that is by $a+f$, gives the Quotient $\sqrt{bbc}$: Therefore $\overline{a+f}\sqrt{bbc}$ imports as much as $\sqrt{:aabbc+2afbbc+ffbbc:}$

After the ſame manner, inſtead of $\sqrt{(3)}\frac{27aaaabbb}{8b-8a}$ we may write $\frac{3ab}{2}$ into $\sqrt{(3)}\frac{a}{b-a}$, or $\frac{3ab}{2}\sqrt{(3)}\frac{a}{b-a}$; for ſince the Power of the Surd propoſed is produced by the multiplication of $\frac{a}{b-a}$ into the Cube $\frac{27aaabbb}{8}$ whoſe cubick Root is $\frac{3ab}{2}$, and conſequently $\sqrt{(3)}\frac{27aaaabbb}{8b-8a}$ divided by $\sqrt{(3)}\frac{27aaabbb}{8}$, that is by $\frac{3ab}{2}$ gives the Quotient $\sqrt{(3)}\frac{a}{b-a}$: Therefore $\frac{3ab}{2}\sqrt{(3)}\frac{a}{b-a}$ ſhall be equal to $\sqrt{(3)}\frac{27aaaabbb}{8b-8a}$.

So alſo, for $\sqrt{:}\frac{aaoomm+4aammmp}{ppzz}:$ we may write $\frac{am}{pz}\sqrt{:oo+4mp:}$ for, if the Power of the Surd propoſed be divided by the Square $\frac{aamm}{ppzz}$ the Quotient will be $oo+4mp$; and conſequently, if the Surd propoſed be divided by $\sqrt{:}\frac{aamm}{ppzz}:$ that is, by $\frac{am}{pz}$, the Quotient will be $\sqrt{:oo+4mp:}$ Therefore the Diviſor $\frac{am}{pz}$ multiplied into the Quotient $\sqrt{:oo+4mp:}$ (viz. $\frac{am}{pz}\sqrt{:oo+4mp:}$) denotes as much as $\sqrt{:}\frac{aaoomm+4aammmp}{ppzz}:$ the Surd propoſed.

Likewiſe, for $\sqrt{:}\frac{oozz+4mpzz}{aa}:$ we may write $\frac{z}{a}\sqrt{:oo+4mp:}$

But when a Square, or Cube, &c. by which the Diviſion neceſſary to ſuch Contraction is to be performed, cannot be readily diſcerned, firſt, (by the Rules of the preceding eighth Chapter) ſearch out all the Diviſors of the Power of the Surd quantity propoſed, and then ſee whether any of them be a Square or Cube, &c. to wit, ſuch a Power as the Radical ſign denotes, which if you find, you may uſe in the aforeſaid manner to free the Surd quantity, in part, from the Radical ſign.

As, if $\sqrt{288}$ be propoſed, becauſe among the Diviſors of 288 there are found the Square numbers 4, 9, 16, 36 and 144, which dividing 288 will give the Quotients 72, 32, 18, 8 and 2; inſtead of $\sqrt{288}$ we may write $2\sqrt{72}$, or $3\sqrt{32}$, or $4\sqrt{18}$, or $6\sqrt{8}$, or laſtly, $12\sqrt{2}$.

In like manner, if $\sqrt{:aaab+aabb:}$ be propoſed, becauſe among the Diviſors of the quantity $aaab+aabb$, there is found the Square aa, the ſaid $\sqrt{:aaab+aabb:}$ may be reduced to $a\sqrt{:aa+bb:}$ as before.

Again, for as much as $a^3b - aabb + 2aabc + abcc - ab^3 + bbcc - 2b^3c + b^4$ is produced by the multiplication of $ab+bb$ into the Square $aa+2ac+cc-2ab-2bc+bb$, whoſe Root is $a+c-b$; we may inſtead of $\sqrt{:a^3b-aabb+2aabc+abcc-ab^3+bbcc-2b^3c+b^4:}$ write $\overline{a+c-b}$ into $\sqrt{:ab+bb:}$ or $\overline{a+c-b}\sqrt{:ab+bb:}$

Likewiſe, becauſe among the Diviſors of $1200aabb$ there are found the Squares $4aabb$, $16aabb$, $25aabb$, $100aabb$ and $400aabb$, which dividing the ſaid $1200aabb$, will give the Quotients 300, 75, 48, 12 and 3; we may for $\sqrt{1200aabb}$ write $2ab\sqrt{300}$, or $4ab\sqrt{75}$, or $5ab\sqrt{48}$, or $10ab\sqrt{12}$, or laſtly, $20ab\sqrt{3}$.

Sect. VII. *Two Surd Roots being given, to find whether they be Commenſurable or Incommenſurable.*

Commenſurable Surd Roots are ſuch whoſe Reaſon or Proportion to one another may be expreſt by Rational Numbers, or Quantities; and thoſe Surd Roots whoſe Proportion cannot be expreſt by Rational Numbers or Quantities are called Incommenſurable.

The

The Rule to try whether two Surd Roots of the same kind, (that is, such as have a common Radical sign,) be Commensurable or not, is this that follows, *viz.*

Divide the given Roots severally by their greatest Common Divisor; then if the Quotients be Rational Numbers or Quantities, the Roots proposed are Commensurable; but if the Quotients be Irrational or Surd, the given Roots are Incommensurable.

As, for example, to try whether $\surd 12$ and $\surd 3$ be Commensurable or not, I divide them severally by their greatest common Divisor $\surd 3$, and find the Quotients $\surd 4$ and $\surd 1$, that is, 2 and 1 to be Rational numbers, whence I conclude that $\surd 12$, that is $2\surd 3$, hath such Proportion to $\surd 3$, that is $1\surd 3$, as 2 to 1, *viz.* as a Rational number to a Rational number; and consequently $\surd 12$ and $\surd 3$ (according to the Definition above given) are Commensurable. But that $\surd 12$ is to $\surd 3$ as 2 to 1, may be demonstrated thus, *viz.* It is evident (by reason of the common Factor $\surd 3$,) that $2\surd 3 \,.\, 1\surd 3 :: 2 \,.\, 1$, and (by Division as above,) $\surd 12 = 2\surd 3$, and $\surd 3 = 1\surd 3$; therefore $\surd 12 \,.\, \surd 3 :: 2 \,.\, 1$. Otherwise thus,

For as much as 12 and 3 divided severally by their common Divisor 3 give the Quotients 4 and 1, therefore, As $\left.\right\}\ 12 \,.\, 3 :: 4 \,.\, 1.$

Wherefore the square Roots of those Proportionals shall be Proportionals also, (*per* 22. *Prop.* 6. *Elem.* Euclid) *viz.* $\left.\right\}\ \surd 12 \,.\, \surd 3 :: 2 \,.\, 1.$

Which was to be demonstrated.

After the same manner, $\surd 18$ and $\surd 8$ will be found Commensurable, for the former is to the latter as 3 to 2, to wit, as a Rational number to a Rational number; for if $\surd 18$ and $\surd 8$ be severally divided by their greatest common Divisor $\surd 2$, the Quotients will be $\surd 9$ and $\surd 4$, that is, 3 and 2. Therefore $\surd 18$ is to $\surd 8$, as 3 to 2; and instead of $\surd 18$ and $\surd 8$ we may write $3\surd 2$ and $2\surd 2$, to wit, the Products of the Rational Quotients 3 and 2 multiplied into the common Divisor $\surd 3$.

Again, $\surd 48$ and $\surd 75$ (that is, $4\surd 3$ and $5\surd 3$) are Commensurable, for the former is to the latter as 4 to 5, (to wit, as a Rational number to a Rational number;) for $\surd 48$ and $\surd 75$ being severally divided by their greatest common Divisor $\surd 3$, give the Quotients $\surd 16$ and $\surd 25$, to wit, 4 and 5. Therefore $\surd 48 \,.\, \surd 75 :: 4 \,.\, 5 :: 4\surd 3 \,.\, 5\surd 3$.

Moreover, $\surd(3)320$ and $\surd(3)135$ (that is, $4\surd(3)5$ and $3\surd(3)5$,) having such proportion one to the other as 4 to 3 are Commensurable; for $\surd(3)320$ and $\surd(3)135$ being severally divided by their greatest common Divisor $\surd(3)5$, will give the Quotients $\surd(3)64$ and $\surd(3)27$, to wit, 4 and 3. Therefore, $\surd(3)320 \,.\, \surd(3)135 :: 4 \,.\, 3 :: 4\surd(3)5 \,.\, 3\surd(3)5$.

So also $\surd(4)3888$ and $\surd(4)243$ (that is, $2\surd(4)243$ and $1\surd(4)243$ are Commensurable, the former having such proportion to the latter as 2 to 1; for if they be severally divided by their greatest common Divisor $\surd(4)243$, the Quotients will be $\surd(4)16$ and $\surd(4)1$, to wit, 2 and 1. Therefore, $\surd(4)3888 \,.\, \surd(4)243 :: 2 \,.\, 1 :: 2\surd(4)243 \,.\, 1\surd(4)243$.

If two Surd Fractions, or mixt numbers standing fraction-wise, be proposed, and have not a common Denominator, reduce them to their smallest common Denominator, and then try (in like manner as before) whether the new Surd Numerators be Commensurable or not, for if these be Commensurable, the Surd Fractions first proposed shall be also Commensurable. As, if $\surd\frac{2}{3}$ and $\surd\frac{24}{25}$ be proposed; I reduce them to $\surd\frac{50}{75}$ and $\surd\frac{72}{75}$, then I divide the new Numerators only, to wit, $\surd 50$ and $\surd 72$ by their greatest common Divisor $\surd 2$, and the Quotients $\surd 25$ and $\surd 36$, that is, 5 and 6, are Rational numbers. Therefore $\surd\frac{2}{3}$ and $\surd\frac{24}{25}$ first proposed are Commensurable, and the former hath such proportion to the latter as 5 to 6. For,

$$\text{As } \tfrac{50}{75} \,.\, \tfrac{72}{75} :: 50 \,.\, 72 :: 25 \,.\, 36,$$
$$\text{Therefore, } \surd\tfrac{50}{75} \,.\, \surd\tfrac{72}{75} :: \surd 50 \,.\, \surd 72 :: 5 \,.\, 6.$$
$$\text{And because } \surd\tfrac{2}{3} = \surd\tfrac{50}{75}, \text{ and } \surd\tfrac{24}{25} = \surd\tfrac{72}{75};$$
$$\text{Therefore, } \surd\tfrac{2}{3} \,.\, \surd\tfrac{24}{25} :: 5 \,.\, 6.$$

But if either the Numerators or Denominators of two Surd Fractions or mixt numbers standing fraction-wise, (the Radical sign being neglected,) be Squares or Cubes, *&c.* viz. Powers of that kind which is denoted by the Radical sign, then you need not reduce the surd Fractions to a common Denominator, but try whether their Numerators or Denominators be Commensurable or not; for if these be Commensurable, the surd Fractions proposed

ſhall be alſo Commenſurable. As, if $\sqrt{\frac{50}{16}}$ and $\sqrt{\frac{72}{25}}$ be propoſed ; becauſe the Denominators (the Radical ſign being neglected) are Squares, (to wit, Powers of that kind which the Radical ſign denotes,) and the Numerators $\sqrt{50}$ and $\sqrt{72}$ are Commenſurable ; (for if theſe be divided by their common Diviſor $\sqrt{2}$, the Quotients are Rational, to wit, 5 and 6.) Therefore the Surd Fractions propoſed are alſo Commenſurable, and have ſuch proportion as $\frac{5}{4}$ to $\frac{6}{5}$, (whoſe Denominators 4 and 5, to wit, $\sqrt{16}$ and $\sqrt{25}$ are the given Denominators,) or as 25 to 24 ; and, (according to the preceding *Sect.* 6.) the Surd Fractions propoſed may be expreſt thus, $\frac{5}{4}\sqrt{2}$ and $\frac{6}{5}\sqrt{2}$.

When two Surd Roots propoſed be of different kinds, they muſt firſt of all be reduced to a common Radical ſign, (by the preceding *Sect.* 3. of this *Chapt.*) before the Rules aforeſaid be uſed, to try whether they be Commenſurable or not. As, if $\sqrt{(6)}64$ and $\sqrt{(3)}27$ be given ; they may be reduced to $\sqrt{(6)}64$ and $\sqrt{(6)}729$, which divided by their greateſt common Diviſor $\sqrt{(6)}1$, the Quotients will be the ſame with the Dividends. Now if $\sqrt{(6)}64$ and $\sqrt{(6)}729$ be Rational, then the Surds firſt given are Commenſurable ; but $\sqrt{(6)}64$ is 2, and $\sqrt{(6)}729$ is 3. Therefore the Surd Roots propoſed are Commenſurable, and have ſuch Proportion as 2 to 3.

But if the Quotients ariſing by the diviſion of two ſurd Roots by their greateſt common Diviſor as aforeſaid, happen to be Irrational or Surd, then the Roots propoſed are Incommenſurable ; ſuch are $\sqrt{48}$ and $\sqrt{8}$, for if they be divided ſeverally by their greateſt common Diviſor $\sqrt{8}$, the Quotients are $\sqrt{6}$ and 1 ; but $\sqrt{6}$ is Irrational, therefore the Proportion which $\sqrt{48}$ hath to $\sqrt{8}$ is not as a Rational number to a Rational number, and conſequently $\sqrt{48}$ and $\sqrt{8}$ are Incommenſurable ; and ſo are all other Surd Roots whoſe Proportion cannot be expreſt by Rational numbers.

I ſhall now ſhew how by the help of the preceding Rules we may diſcover whether two Surd quantities expreſt by letters be Commenſurable or not. As, if $\sqrt{27aa}$ and $\sqrt{12aa}$ be propoſed, they will be found Commenſurable ; for if they be ſeverally divided by their greateſt common Diviſor $\sqrt{3aa}$, the Quotients $\sqrt{9}$ and $\sqrt{4}$, that is, 3 and 2, are Rational numbers, and ſhew that $\sqrt{27aa}$ is to $\sqrt{12aa}$ as 3 to 2, to wit, as a Rational number to a Rational number ; wherefore $\sqrt{27aa}$ and $\sqrt{12aa}$ are Commenſurable, and may be expreſt thus, $3\sqrt{3aa}$ and $2\sqrt{3aa}$.

Note. If two Surd quantities be divided by ſome common Diviſor, though it be not the greateſt, yet if there come forth Rational Quotients, we may thence conclude thoſe Surd quantities to be Commenſurable, and oftentimes expreſs them various wayes. As, if $\sqrt{27aa}$ and $\sqrt{12aa}$ be again propoſed ; by dividing them ſeverally by their common Diviſor $\sqrt{3}$, there will come forth the Quotients $\sqrt{9aa}$ and $\sqrt{4aa}$, that is, $3a$ and $2a$; whence it is evident that $\sqrt{27aa}$ is to $\sqrt{12aa}$ as $3a$ to $2a$, to wit, as a Rational quantity to a Rational quantity, and conſequently $\sqrt{27aa}$ and $\sqrt{12aa}$ are Commenſurable. Moreover, according to this latter Diviſion, we may write $3a\sqrt{3}$ for $\sqrt{27aa}$, and $2a\sqrt{3}$ for $\sqrt{12aa}$.

Again, $\sqrt{:\overline{aaaa+aabb}:}$ and $\sqrt{:\overline{aabb+bbbb}:}$ are Commenſurable ; for each of them being divided by $\sqrt{:\overline{aa+bb}:}$ there ariſe $\sqrt{aa}$ and $\sqrt{bb}$, that is, a and b, which are Rational quantities, each of which being multiplied into the common Diviſor $\sqrt{\overline{aa+bb}}:$ will give, inſtead of the Surds propoſed, $a\sqrt{\overline{aa+bb}}:$ and $b\sqrt{\overline{aa+bb}}$, which have the ſame Proportion to one another as there is between a and b.

Likewiſe, $\sqrt{\frac{oozz+4mpzz}{aa}}$ and $\sqrt{\frac{aaoomm+4aammmp}{ppzz}}$ are Commenſurable, for each of them being divided by their common Diviſor $\sqrt{:\overline{oo+4mp}}:$ there will ariſe $\sqrt{\frac{zz}{aa}}$ and $\sqrt{\frac{aamm}{ppzz}}$, that is, $\frac{z}{a}$ and $\frac{am}{pz}$, (to wit, Rational quantities,) each of which multiplied into the common Diviſor $\sqrt{:\overline{oo+4mp}}:$ will produce $\frac{z}{a}\sqrt{:\overline{oo+4mp}}:$ and $\frac{am}{pz}\sqrt{:\overline{oo+4mp}}:$ which are equal to, but more ſimply expreſt than the Surd Quantities propoſed, and have that Proportion to one another as is between $\frac{z}{a}$ and $\frac{am}{pz}$.

So alſo $\sqrt{:\overline{aaaa+6aaa+21aa+72a+108}}:$ & $\sqrt{:\overline{aaaa-10aaa+37aa-120a+300}}:$ are Commenſurable, for if they be ſeverally divided by their common Divior $\sqrt{:\overline{aa+12}}:$ there will ariſe $\sqrt{:\overline{aa+6a+9}}:$ and $\sqrt{:\overline{aa-10a+25}}:$ that is, $a+3$ and $a \backsim 5$, each of which

which multiplied into the common Divisor $\sqrt{aa+12}$ will produce $a+3\ \sqrt{:aa+12}$ and $a \backsim 5\ \sqrt{:aa+12}$: which have the same Proportion between themselves as that of $a+3$ to $a \backsim 5$, and are of the same value with the Surd Quantities first proposed.

Again, $\sqrt{(3)}81abbb$ and $\sqrt{(3)}24abbb$ are Commensurable; for if each of them be divided by their common Divisor $\sqrt{(3)}3a$ there will arise $\sqrt{(3)}27bbb$ and $\sqrt{(3)}8bbb$, that is, $3b$ and $2b$; therefore the Surds proposed may be reduced to $3b\sqrt{(3)}3a$ and $2b\sqrt{(3)}3a$, the former of which is to the latter as $3b$ to $2b$: and so of others.

Sect. VIII. *Addition and Subtraction in simple Surd quantities.*

When two or more equal Surd Roots are to be added together, multiply one of them by the number which expresseth the multitude of the Roots proposed, and the Product shall be their Summ: as, the summ of $\sqrt{6}$ and $\sqrt{6}$ is $\sqrt{24}$; for $\sqrt{6}$ multiplied by 2, that is, by $\sqrt{4}$, produceth $\sqrt{24}$: also $\sqrt{(3)}6$, $\sqrt{(3)}6$ and $\sqrt{(3)}6$ added into one, make $\sqrt{(3)}162$; for $\sqrt{(3)}6$ multiplied by 3, that is, by $\sqrt{(3)}27$, makes $\sqrt{(3)}162$.

But when two unequal Surd Roots of the same kind, that is, such as have the same Radical sign prefixt before each of them, be to be added together; also when the lesser is to be subtracted from the greater, observe this Rule; *viz.* First (by the preceding *Sect.* 7. of this *Chapt.*) you must try whether they be Commensurable or not, then if they be Commensurable, that is, if after they have been severally divided by their greatest common Divisor the Quotients be Rational quantities, multiply the summ of those Rational quantities, by the said common Divisor, and the Product shall be the summ of the surd Roots proposed; but if the Difference of those Rational Quotients be multiplied by the said common Divisor, the Product shall be the Difference of the Roots proposed.

As, for example, if the Summ and Difference of $\sqrt{50}$ and $\sqrt{8}$ be desired; first, I divide each of them by their greatest common Divisor $\sqrt{2}$, and the Quotients are $\sqrt{25}$ and $\sqrt{4}$, that is, 5 and 2, (which are Rational numbers, expressing the Proportion of the given Roots one to the other;) whose summ 7 multiplied by the common Divisor $\sqrt{2}$, produceth $7\sqrt{2}$, or if you please, $\sqrt{98}$, (for 7, to wit, $\sqrt{49}$ into $\sqrt{2}$ makes $\sqrt{98}$;) which is the desired Summ of the given Roots $\sqrt{50}$ and $\sqrt{8}$. And if $5-2$, that is 3, (the Difference of the Rational Quotients before found) be multiplied by the said common Divisor $\sqrt{2}$, the Product will be $3\sqrt{2}$, that is, $\sqrt{18}$; which is the desired Difference of $\sqrt{50}$ and $\sqrt{8}$, the Roots first proposed.

Likewise, the Summ of $\sqrt{(3)}500$ and $\sqrt{(3)}108$ will be found $8\sqrt{(3)}4$, that is, $\sqrt{(3)}2048$; and their Difference $2\sqrt{(3)}4$, that is, $\sqrt{(3)}32$, as will appear by the following Work: *viz.* First, I divide each of the given Roots $\sqrt{(3)}500$ and $\sqrt{(3)}108$ by their greatest common Divisor $\sqrt{(3)}4$, and the Quotients are $\sqrt{(3)}125$ and $\sqrt{(3)}27$, that is, 5 and 3; then by multiplying 8 (to wit, $5+3$ the summ of the Rational Quotients,) by the common Divisor $\sqrt{(3)}4$, the Product is $8\sqrt{(3)}4$, that is, $\sqrt{(3)}2048$; (for 8, to wit, $\sqrt{(3)}512$ into $\sqrt{(3)}4$, makes $\sqrt{(3)}2048$;) which is the Summ of $\sqrt{(3)}500$ and $\sqrt{(3)}108$, the Roots proposed.

And by multiplying 2, (that is, $5-3$, the Difference of the Rational Quotients) by the said common Divisor $\sqrt{(3)}4$, the Product is $2\sqrt{(3)}4$, that is, $\sqrt{(3)}32$; (for 2, to wit, $\sqrt{(3)}8$ into $\sqrt{(3)}4$, makes $\sqrt{(3)}32$,) which is the Difference of $\sqrt{(3)}500$ and $\sqrt{(3)}108$, the Roots proposed.

Here follow Contractions of the Work in the two last preceding Examples; with others of like nature, to illustrate the Rule before given for the Addition and Subtraction of such simple Surd Roots as are Commensurable.

Example 1.

What is the Summ and Difference of . . . . $\sqrt{50}$ and $\sqrt{8}$?

The Operation.

$\sqrt{2})\ \sqrt{50}\ (\sqrt{25}$, that is, 5; Therefore $5\sqrt{2} = \sqrt{50}$.

$\sqrt{2})\ \sqrt{8}\ (\sqrt{4}$, that is, 2; Therefore $2\sqrt{2} = \sqrt{8}$.

The Summ, $7\sqrt{2} = \sqrt{50}+\sqrt{8}$;

Or, $\sqrt{98} = \sqrt{50}+\sqrt{8}$.

The Difference, $3\sqrt{2} = \sqrt{50}-\sqrt{8}$;

Or, $\sqrt{18} = \sqrt{50}-\sqrt{8}$.

 Example 2.

Example 2.

What is the Summ and Difference of . . . $\sqrt[3]{500}$ and $\sqrt[3]{108}$?

The Operation.

I. $\sqrt[3]{4}\ )\ \sqrt[3]{500}\ (\ \sqrt[3]{125}$, that is, 5.
II. $\sqrt[3]{4}\ )\ \sqrt[3]{108}\ (\ \sqrt[3]{27}$, that is, 3.

From Division I. $5\sqrt[3]{4} = \sqrt[3]{500}$.
From Division II. $3\sqrt[3]{4} = \sqrt[3]{108}$.

The Summ, $8\sqrt[3]{4} = \sqrt[3]{500} + \sqrt[3]{108}$;
Or, $\sqrt[3]{2048} = \sqrt[3]{500} + \sqrt[3]{108}$.

The Difference, $2\sqrt[3]{4} = \sqrt[3]{500} - \sqrt[3]{108}$;
Or, $\sqrt[3]{32} = \sqrt[3]{500} - \sqrt[3]{108}$.

Example 3.

What is the Summ and Difference of . . . . $\sqrt{147}$ and $\sqrt{12}$?

The Operation.

$\sqrt{3}\ )\ \sqrt{147}\ (\ \sqrt{49}$, that is, 7; Therefore $7\sqrt{3} = \sqrt{147}$.
$\sqrt{3}\ )\ \sqrt{12}\ (\ \sqrt{4}$, that is, 2; Therefore $2\sqrt{3} = \sqrt{12}$.

The Summ, $9\sqrt{3} = \sqrt{147} + \sqrt{12}$;
Or, $\sqrt{243} = \sqrt{147} + \sqrt{12}$.

The Difference, $5\sqrt{3} = \sqrt{147} - \sqrt{12}$;
Or, $\sqrt{75} = \sqrt{147} - \sqrt{12}$.

Example 4.

What is the Summ and Difference of . . . $\sqrt[3]{1715}$ and $\sqrt[3]{40}$?

The Operation.

I. $\sqrt[3]{5}\ )\ \sqrt[3]{1715}\ (\ \sqrt[3]{343}$, that is, 7.
II. $\sqrt[3]{5}\ )\ \sqrt[3]{40}\ (\ \sqrt[3]{8}$, that is, 2.

From Division I. $7\sqrt[3]{5} = \sqrt[3]{1715}$.
From Division II. $2\sqrt[3]{5} = \sqrt[3]{40}$.

The Summ, $9\sqrt[3]{5} = \sqrt[3]{1715} + \sqrt[3]{40}$;
Or, $\sqrt[3]{3645} = \sqrt[3]{1715} + \sqrt[3]{40}$.

The Difference; $5\sqrt[3]{5} = \sqrt[3]{1715} - \sqrt[3]{40}$;
Or, $\sqrt[3]{625} = \sqrt[3]{1715} - \sqrt[3]{40}$.

Note. When two Commensurable Surd Roots proposed to be added or subtracted are Fractions, or mixt numbers reduced into the form of Fractions, if they have not a Common Denominator reduce them into others which may have a Common Denominator in the least Terms; then to find out the Rational Quotients, divide only the two new Numerators severally by their greatest Common Divisor, and continue the process as before. The Practice of this Note will be evident in the two following Examples.

Example 5.

What is the Summ and Difference of . . . $\left\{ \begin{array}{l} \sqrt{\frac{24}{25}} \text{ and } \sqrt{\frac{2}{3}}, \\ \text{Or, } \sqrt{\frac{72}{75}} \text{ and } \sqrt{\frac{50}{75}}? \end{array} \right.$

The Operation.

$\sqrt{\frac{2}{75}}\ )\ \sqrt{\frac{72}{75}}\ (\ \sqrt{36}$, that is, 6; Therefore $6\sqrt{\frac{2}{75}} = \sqrt{\frac{72}{75}}$.
$\sqrt{\frac{2}{75}}\ )\ \sqrt{\frac{50}{75}}\ (\ \sqrt{25}$, that is, 5; Therefore $5\sqrt{\frac{2}{75}} = \sqrt{\frac{50}{75}}$.

The Summ, $11\sqrt{\frac{2}{75}} = \sqrt{\frac{72}{75}} + \sqrt{\frac{50}{75}}$;
Or, $\sqrt{\frac{242}{75}} = \sqrt{\frac{72}{75}} + \sqrt{\frac{50}{75}}$.

The Difference, $\sqrt{\frac{2}{75}} = \sqrt{\frac{72}{75}} - \sqrt{\frac{50}{75}}$.

Example 6.

Example 6.

What is the Summ and Difference of . . . $\left\{ \begin{array}{l} \sqrt{12} \text{ and } \sqrt{\tfrac{27}{4}}? \\ \text{Or, } \sqrt{\tfrac{48}{4}} \text{ and } \sqrt{\tfrac{27}{4}}? \end{array} \right.$

The Operation.

$\sqrt{\tfrac{1}{4}}$) $\sqrt{\tfrac{48}{4}}$ ($\sqrt{16}$, that is, 4; Therefore $4\sqrt{\tfrac{1}{4}} = \sqrt{\tfrac{48}{4}}$.

$\sqrt{\tfrac{1}{4}}$) $\sqrt{\tfrac{27}{4}}$ ($\sqrt{9}$, that is, 3; Therefore $3\sqrt{\tfrac{1}{4}} = \sqrt{\tfrac{27}{4}}$.

The Summ, $7\sqrt{\tfrac{1}{4}} = \sqrt{\tfrac{48}{4}} + \sqrt{\tfrac{27}{4}}$;

Or, $\sqrt{\tfrac{147}{4}} = \sqrt{\tfrac{48}{4}} + \sqrt{\tfrac{27}{4}}$.

The Difference, $\sqrt{\tfrac{1}{4}} = \sqrt{\tfrac{48}{4}} - \sqrt{\tfrac{27}{4}}$.

When two simple Surd Roots given to be added or subtracted be Incommensurable, neither their Summ nor their Difference can be exprest by any simple Root, but they are to be added by $+$, and to be subtracted by $-$. As, to add $\sqrt{5}$ and $\sqrt{3}$, I write $\sqrt{5} + \sqrt{3}$ for the Summ; but to subtract $\sqrt{3}$ from $\sqrt{5}$, I write $\sqrt{5} - \sqrt{3}$ for the Remainder: So also, the Summ of $\sqrt{(3)}40$ and $\sqrt{(3)}12$ is $\sqrt{(3)}40 + \sqrt{(3)}12$, and their Difference is $\sqrt{(3)}40 - \sqrt{(3)}12$.

But Incommensurable square Roots may be added or subtracted by this following Rule, (which is deduced from *Prop.* 4, & 7. *lib.* 2. Euclid.)

To the summ of the Squares of the given Surd square Roots, add the double Product of the multiplication of those Roots one into another; so shall the square Root of the summ be the Summ of the Roots proposed to be added: But if the said double Product be subtracted from the said summ of the Squares, the square Root of the Remainder shall be the Difference of the given Surd square Roots. As, if the Summ and Difference of $\sqrt{6}$ and $\sqrt{3}$ be desired, their Summ shall be $\sqrt{:\overline{9 + \sqrt{72}}:}$ and their Difference $\sqrt{:\overline{9 - \sqrt{72}}:}$ for the summ of the Squares of the given square Roots $\sqrt{6}$ and $\sqrt{3}$ is 9, and the double Product of their multiplication is $\sqrt{72}$, which I add to and subtract from 9; so the square Root of the summ, to wit, $\sqrt{:\overline{9 + \sqrt{72}}:}$ is the Summ desired; and the square Root of the Remainder, to wit, $\sqrt{:9 - \sqrt{72}:}$ is the Difference.

After the same manner the Addition and Subtraction of simple Surd Quantities exprest by Letters may be performed: As, to add $\sqrt{75aa}$ and $\sqrt{27aa}$, first, (by the preceding *Sect.* 7.) I find them to be Commensurable; for, if $\sqrt{75aa}$ and $\sqrt{27aa}$ be severally divided by their greatest common Divisor $\sqrt{3aa}$, the Quotients are $\sqrt{25}$ and $\sqrt{9}$, that is, 5 and 3, whose summ 8 multiplied into the common Divisor $\sqrt{3aa}$ makes $8\sqrt{3aa}$; (that is, $\sqrt{192aa}$) for the Summ of $\sqrt{75aa}$ and $\sqrt{27aa}$. But if the Difference of the same Rational Quotients 5 and 3, to wit, 2, be multiplied into the said common Divisor $\sqrt{3aa}$, it makes $2\sqrt{3aa}$ (that is, $\sqrt{12aa}$) for the Difference of $\sqrt{75aa}$ and $\sqrt{27aa}$, the Roots first proposed.

Or, we may write $8a\sqrt{3}$ (instead of $8\sqrt{3aa}$) for the Summ, and $2a\sqrt{3}$ (instead of $2\sqrt{3aa}$) for the Difference of $\sqrt{75aa}$ and $\sqrt{27aa}$ before proposed; for these divided severally by their common Divisor $\sqrt{3}$, give Rational Quotients, to wit, $\sqrt{25aa}$ and $\sqrt{9aa}$, that is, $5a$ and $3a$; whose summ $8a$ multiplied into the common Divisor $\sqrt{3}$, gives $8a\sqrt{3}$ for the Summ of $\sqrt{75aa}$ and $\sqrt{27aa}$; but if the Difference of the said Rational Quotients $5a$ and $3a$, to wit, $2a$, be multiplied into the said common Divisor $\sqrt{3}$, the Product $2a\sqrt{3}$ is the Difference of the said $\sqrt{75aa}$ and $\sqrt{27aa}$.

Again, to add $\sqrt{(3)}256aaa$ and $\sqrt{(3)}32aaa$; first, (by *Sect.* 7.) I find them to be Commensurable, for if each of them be divided by their common Divisor $\sqrt{(3)}4$, the Quotients are Rational, to wit, $\sqrt{(3)}64aaa$ and $\sqrt{(3)}8aaa$, that is, $4a$ and $2a$; these added together make $6a$, which multiplied into the common Divisor $\sqrt{(3)}4$, makes $6a\sqrt{(3)}4$ (that is, $\sqrt{(3)}864aaa$) for the desired Summ of $\sqrt{(3)}256aaa$ and $\sqrt{(3)}32aaa$; but if $2a$, the Difference of the same Rational Quotients $4a$ and $2a$, be multiplied into the said common Divisor $\sqrt{(3)}4$, the Product $2a\sqrt{(3)}4$ (that is, $\sqrt{(3)}32aaa$) shall be the Difference of $\sqrt{(3)}256aaa$ and $\sqrt{(3)}32aaa$ first proposed.

More

More Examples of the Addition and Subtraction of Commensurable simple Surd Quantities exprest by Letters.

Example 1.

What is the Summ and Difference of . . . . $\sqrt{28aa}$ and $\sqrt{7aa}$?

The Operation.

I. $\sqrt{7}$) $\sqrt{28aa}$ ($\sqrt{4aa}$, that is, $2a$.
II. $\sqrt{7}$) $\sqrt{7aa}$ ($\sqrt{aa}$, that is, a.
From Division I. $2a\sqrt{7} = \sqrt{28aa}$.
From Division II. $a\sqrt{7} = \sqrt{7aa}$.

The Summ, $3a\sqrt{7} = \sqrt{28aa} + \sqrt{7aa}$.
The Difference, $a\sqrt{7} = \sqrt{28aa} - \sqrt{7aa}$.

Example 2.

What is the Summ and Difference of . . . $\sqrt{45aabc}$ and $\sqrt{20aabc}$?

The Operation.

I. $\sqrt{5bc}$) $\sqrt{45aabc}$ ($\sqrt{9aa}$, that is, $3a$.
II. $\sqrt{5bc}$) $\sqrt{20aabc}$ ($\sqrt{4aa}$, that is, $2a$.
From Division I. $3a\sqrt{5bc} = \sqrt{45aabc}$.
From Division II. $2a\sqrt{5bc} = \sqrt{20aabc}$.

The Summ, $5a\sqrt{5bc} = \sqrt{45aabc} + \sqrt{20aabc}$.
The Difference, $a\sqrt{5bc} = \sqrt{45aabc} - \sqrt{20aabc}$.

Example 3.

What is the Summ and Difference of . . $\sqrt{(3)81abbb}$ and $\sqrt{(3)24abbb}$?

The Operation.

I. $\sqrt{(3)3a}$) $\sqrt{(3)81abbb}$ ($\sqrt{(3)27bbb}$, that is, $3b$.
II. $\sqrt{(3)3a}$) $\sqrt{(3)24abbb}$ ($\sqrt{(3)8bbb}$, that is, $2b$.
From Division I. $3b\sqrt{(3)3a} = \sqrt{(3)81abbb}$.
From Division II. $2b\sqrt{(3)3a} = \sqrt{(3)24abbb}$.

The Summ, $5b\sqrt{(3)3a} = \sqrt{(3)81abbb} + \sqrt{(3)24abbb}$.
The Difference, $b\sqrt{(3)3a} = \sqrt{(3)81abbb} - \sqrt{(3)24abbb}$.

Example 4.

What is the Summ and Difference of . . $\left\{\begin{array}{l} \sqrt{\tfrac{15}{4}aad} \text{ and } \sqrt{\tfrac{125}{48}aad}, \\ \text{Or, } \sqrt{\tfrac{180}{48}aad} \text{ and } \sqrt{\tfrac{125}{48}aad}? \end{array}\right.$

The Operation.

I. $\sqrt{\tfrac{5}{48}d}$) $\sqrt{\tfrac{180}{48}aad}$ ($\sqrt{36aa}$, that is, $6a$.
II. $\sqrt{\tfrac{5}{48}d}$) $\sqrt{\tfrac{125}{48}aad}$ ($\sqrt{25aa}$, that is, $5a$.
From Division I. $6a\sqrt{\tfrac{5}{48}d} = \sqrt{\tfrac{180}{48}aad}$.
From Division II. $5a\sqrt{\tfrac{5}{48}d} = \sqrt{\tfrac{125}{48}aad}$.

The Summ, $11a\sqrt{\tfrac{5}{48}d} = \sqrt{\tfrac{180}{48}aad} + \sqrt{\tfrac{125}{48}aad}$.
The Difference, $a\sqrt{\tfrac{5}{48}d} = \sqrt{\tfrac{180}{48}aad} - \sqrt{\tfrac{125}{48}aad}$.

If two Surd Quantities exprest by letters be Incommensurable, their Summ is given by $+$, and their Difference by $-$; as, to add $\sqrt{5a}$ and $\sqrt{3a}$, I write $\sqrt{5a} + \sqrt{3a}$ for the Summ: and to subtract $\sqrt{3a}$ from $\sqrt{5a}$, I write $\sqrt{5a} - \sqrt{3a}$ for the Remainder or Difference.

Sect. IX. *Addition and Subtraction in Compound Surd Quantities.*

The Arithmetick of Compound Surds depends upon the Rules of the Simple, and the Rules of $+$ and $-$ in Algebraical Addition, Subtraction, Multiplication and Division; but how those Rules are applied to the Arithmetick of Compound Surds, I shall shew in this and the following tenth and eleventh Sections, by Examples both in Surd Numbers and Surd Quantities exprest by Letters.

Examples

Examples of Addition and Subtraction in Commensurable simple Surd numbers connected to Rational numbers by + or —, as also in compound Surd numbers composed of Commensurable simple Surds.

To and from	$6 + \sqrt{18}\ (3\sqrt{2})$	$\sqrt{192}\ (8\sqrt{3}) + 3$
Add and Subtr.	$4 + \sqrt{8}\ (2\sqrt{2})$	$\sqrt{75}\ (5\sqrt{3}) - 3$
Summ,	$10 + \sqrt{50}\ (5\sqrt{2})$	$\sqrt{507}\ (13\sqrt{3}) + 0$
Difference,	$2 - \sqrt{2}$	$\sqrt{27}\ (3\sqrt{3}) + 6$

To and from	$+ \sqrt{242}\ (11\sqrt{2}) - 12$	$15 - 2\sqrt{2}\ (\sqrt{8})$
Add and Subtr.	$- \sqrt{50}\ (-5\sqrt{2}) + 8$	$7 + \sqrt{2}$
Summ,	$+ \sqrt{72}\ (6\sqrt{2}) - 4$	$22 - \sqrt{2}$
Difference,	$+ \sqrt{512}\ (16\sqrt{2}) - 20$	$8 - 3\sqrt{2}\ (\sqrt{18})$

To and from	$\sqrt{242} + \sqrt{192}$	that is,	$11\sqrt{2} + 8\sqrt{3}$
Add and Subtr.	$\sqrt{50} + \sqrt{75}$		$5\sqrt{2} + 5\sqrt{3}$
Summ,	$\sqrt{512} + \sqrt{507}$	that is,	$16\sqrt{2} + 13\sqrt{3}$
Difference,	$\sqrt{72} + \sqrt{27}$		$6\sqrt{2} + 3\sqrt{3}$
To and from	$\sqrt{320} - \sqrt{108}$	that is,	$8\sqrt{5} - 6\sqrt{3}$
Add and Subtr.	$\sqrt{80} - \sqrt{27}$		$4\sqrt{5} - 3\sqrt{3}$
Summ,	$\sqrt{720} - \sqrt{243}$	that is,	$12\sqrt{5} - 9\sqrt{3}$
Difference,	$\sqrt{80} - \sqrt{27}$		$4\sqrt{5} - 3\sqrt{3}$
To and from	$\sqrt{320} + \sqrt{108}$	that is,	$8\sqrt{5} + 6\sqrt{3}$
Add and Subtr.	$\sqrt{80} - \sqrt{27}$		$4\sqrt{5} - 3\sqrt{3}$
Summ,	$\sqrt{720} + \sqrt{27}$	that is,	$12\sqrt{5} + 3\sqrt{3}$
Difference,	$\sqrt{80} + \sqrt{243}$		$4\sqrt{5} + 9\sqrt{3}$
To and from	$\sqrt{(3)}2058 + \sqrt{(3)}54$	that is,	$7\sqrt{(3)}6 + 3\sqrt{(3)}2$
Add and Subtr.	$\sqrt{(3)}162 + \sqrt{(3)}16$		$3\sqrt{(3)}6 + 2\sqrt{(3)}2$
Summ,	$\sqrt{(3)}6000 + \sqrt{(3)}250$	that is,	$10\sqrt{(3)}6 + 5\sqrt{(3)}2$
Difference,	$\sqrt{(3)}384 - \sqrt{(3)}2$		$4\sqrt{(3)}6 - \sqrt{(3)}2$
To and from	$\sqrt{(4)}1875 + \sqrt{(3)}250$	that is,	$5\sqrt{(4)}3 + 5\sqrt{(3)}2$
Add and Subtr.	$\sqrt{(4)}48 - \sqrt{(3)}16$		$2\sqrt{(4)}3 - 2\sqrt{(3)}2$
Summ,	$\sqrt{(4)}7203 + \sqrt{(3)}54$	that is,	$7\sqrt{(4)}3 + 3\sqrt{(3)}2$
Difference,	$\sqrt{(4)}243 + \sqrt{(3)}686$		$3\sqrt{(4)}3 + 7\sqrt{(3)}2$

EXPLICATION.

In the first Example, the Rational numbers 6 and 4 added together make 10, and their Difference is 2; then forasmuch as $\sqrt{18}$ and $\sqrt{8}$ (that is, $3\sqrt{2}$ and $2\sqrt{2}$) are Commensurable, (for the former is to the latter as 3 to 2,) their Summ is $\sqrt{50}$, (that is, $5\sqrt{2}$,) and their Difference $\sqrt{2}$, (by *Sect.* 8.) Wherefore $10 + \sqrt{50}(5\sqrt{2})$ is the Summ, and $2 - \sqrt{2}$ the Difference of the two Binomials $6 + \sqrt{18}$ and $4 + \sqrt{8}$, proposed in the first Example.

Likewise in the second Example, the two Commensurable surd Roots $\sqrt{192}$ and $\sqrt{75}$, (that is, $8\sqrt{3}$ and $5\sqrt{3}$) added into one simple Surd make $\sqrt{507}$, (that is, $13\sqrt{3}$,) but their Difference is $\sqrt{27}$), that is, $3\sqrt{3}$;) also, $+3$ and -3 added together make 0, but -3 subtracted from $+3$ makes $+6$. Wherefore $\sqrt{507}$ (that is, $13\sqrt{3}$) is the Summ, and $\sqrt{27}$ (that is, $3\sqrt{3}$) $+6$ is the Difference of the Binomial $\sqrt{192} + 3$, and the Residual $\sqrt{75} - 3$ proposed in the second Example.

Again, in the third Example, where $-\sqrt{50} + 8$ is proposed to be added to $\sqrt{242} - 12$, and also to be subtracted from the same; first, $-\sqrt{50}$ added to $+\sqrt{242}$, (that is, $-5\sqrt{2}$ to $+11\sqrt{2}$) makes $+\sqrt{72}$ (that is, $6\sqrt{2}$;) but $-\sqrt{50}$ subtracted from

from $+\sqrt{242}$ (that is, $-5\sqrt{2}$ from $+11\sqrt{2}$) leaves the Remainder or Difference $+\sqrt{512}$, (that is, $16\sqrt{2}$;) also, $+8$ added to -12 makes -4, but $+8$ subtracted from -12, leaves the Remainder or Difference -20. Wherefore $\sqrt{72}$ (that is $6\sqrt{2}$) -4 is the Summ, and $\sqrt{512}$ (that is, $16\sqrt{2}$) -20 is the Difference of the two Residuals proposed in the third Example. The Operation in the rest of the preceding Examples is after the same manner.

Examples of Addition and Subtraction in compound Surd numbers, partly Commensurable, and partly Incommensurable.

To and from	$\sqrt{27}\,(3\sqrt{3})+\sqrt{8}$	$\sqrt{10}+\sqrt{8}\,(2\sqrt{2})$
Add and Subtr.	$\sqrt{12}\,(2\sqrt{3})+\sqrt{5}$	$\sqrt{3}-\sqrt{2}$
The Summ,	$\sqrt{75}\,(5\sqrt{3})+\sqrt{8}+\sqrt{5}$	$\sqrt{10}+\sqrt{3},+\sqrt{2}$
Or,	$\sqrt{75}\,(5\sqrt{3})+\surd{:}13+\sqrt{160}{:}$	$\surd{:}13+\sqrt{120}{:}+\sqrt{2}$
The Difference,	$\sqrt{3}+\sqrt{8}-\sqrt{5}$	$\sqrt{10}-\sqrt{3},+\sqrt{18}\,(3\sqrt{2})$
Or,	$\sqrt{3}+\surd{:}13-\sqrt{160}{:}$	$\surd{:}13-\sqrt{120}{:}+\sqrt{18}\,(3\sqrt{2})$

To and from	$\surd(3)56+\surd(3)16$	$\surd(4)405-\surd(3)2$
Add and Subtr.	$\surd(3)\,7-\surd(3)12$	$\surd(4)\,80+\surd(3)5$
Summ,	$3\surd(3)\,7+\surd(3)16-\surd(3)12$	$5\surd(4)\,5+\surd(3)5-\surd(3)2$
Difference,	$\surd(3)\,7+\surd(3)16+\surd(3)12$	$\surd(4)\,5-\surd(3)5-\surd(3)2$

EXPLICATION.

In the first of the four last preceding Examples, the Summ of the two Commensurable surd Roots $\sqrt{27}$ and $\sqrt{12}$ (that is, $3\sqrt{3}$ and $2\sqrt{3}$) is $\sqrt{75}$, (that is, $5\sqrt{3}$;) but their Difference is $\sqrt{3}$: and the Summ of the two Incommensurable Roots $\sqrt{8}$ and $\sqrt{5}$ is $\sqrt{8}+\sqrt{5}$, or, $\surd{:}13+\sqrt{160}{:}$ but their Difference is $\sqrt{8}-\sqrt{5}$, or, $\surd{:}13-\sqrt{160}{:}$ (according to the Rule before given in *Sect.* 8. for adding and subtracting two Incommensurable square Roots.) Therefore $5\sqrt{3}+\sqrt{8}+\sqrt{5}$, or, $5\sqrt{3}+\surd{:}13+\sqrt{160}{:}$ is the Summ, and $\sqrt{3}+\sqrt{8}-\sqrt{5}$, or, $\sqrt{3}+\surd{:}13-\sqrt{160}{:}$ is the Difference of the two Binomials $\sqrt{27}+\sqrt{8}$ and $\sqrt{12}+\sqrt{5}$, proposed in the said first Example.

Again, in the third of the said four Examples, where $\surd(3)56+\surd(3)16$ and $\surd(3)7-\surd(3)12$ are proposed to be added and subtracted; the Summ of the two Commensurable surd Cubick Roots $\surd(3)56$ and $\surd(3)7$ is $3\surd(3)7$, and their Difference is $\surd(3)7$: also, the Summ of the two Incommensurable cubick Roots $\surd(3)16$ and $-\surd(3)12$ is $\surd(3)16-\surd(3)12$; but $-\surd(3)12$ subtracted from $\surd(3)16$ leaves $\surd(3)16+\surd(3)12$. Wherefore $3\surd(3)7+\surd(3)16-\surd(3)12$ is the Summ, and $\surd(3)7+\surd(3)16+\surd(3)12$ is the Differeence of the said Binomial and Residual proposed in the third Example.

Examples of Addition and Subtraction in Compound Surd quantities exprest by Letters.

Example 1.

To and from	$\sqrt{75aa}+\sqrt{8bb}$	viz.	$5a\sqrt{3}+2b\sqrt{2}$
Add and Subtr.	$\sqrt{12aa}+\sqrt{2bb}$		$2a\sqrt{3}+b\sqrt{2}$
The Summ is	. . . .		$7a\sqrt{3}+3b\sqrt{2}$
The Difference is	. . . .		$3a\sqrt{3}+b\sqrt{2}$.

EXPLICATION.

First, (by *Sect.* 7.) I find that $\sqrt{75aa}$ and $\sqrt{12aa}$ are Commensurable, and may be reduced to $5a\sqrt{3}$ and $2a\sqrt{3}$; likewise $\sqrt{8bb}$ and $\sqrt{2bb}$ are Commensurable, and may be reduced to $2b\sqrt{2}$ and $b\sqrt{2}$: then the Summ of $5a\sqrt{3}$ and $2a\sqrt{3}$ is $7a\sqrt{3}$; also, the Summ of $2b\sqrt{2}$ and $b\sqrt{2}$ is $3b\sqrt{2}$: therefore the Summ of the two Binomials proposed in the Example is $7a\sqrt{3}+3b\sqrt{2}$. But by subtracting $2a\sqrt{3}$ from $5a\sqrt{3}$, the Remainder is $3a\sqrt{3}$; and by subtracting $b\sqrt{2}$ from $2b\sqrt{2}$, the Remainder is $b\sqrt{2}$. Therefore the the Difference of the two Binomials proposed is $3a\sqrt{3}+b\sqrt{2}$.

Example 2.

Example 2.

What is the Summ and Difference of this Binomial and Residual, ... $\left\{\begin{array}{l} \sqrt{(3)}1715a^3b^3 + \sqrt{(3)}bcd, \\ \sqrt{(3)}40a^3b^3 - \sqrt{(3)}bcd? \end{array}\right.$

Those reduced give these, to wit; . . . . $\left\{\begin{array}{l} 7ab\sqrt{(3)}5 + \sqrt{(3)}bcd, \\ 2ab\sqrt{(3)}5 - \sqrt{(3)}bcd. \end{array}\right.$

The Summ, ... $9ab\sqrt{(3)}5$

The Difference; . . . . . $5ab\sqrt{(3)}5 + 2\sqrt{(3)}bcd.$

Examples of Addition and Subtraction in compound Surd numbers altogether Incommensurable.

To and from $\sqrt{10} + \sqrt{7}$

Add and Subtr. $\sqrt{3} + \sqrt{2}$

Summ, $\sqrt{10} + \sqrt{7} + \sqrt{3} + \sqrt{2}$

Or, $\sqrt{:17 + \sqrt{280}:} + \sqrt{:5 + \sqrt{24}:}$

Difference; $\sqrt{10} + \sqrt{7} - \sqrt{3} - \sqrt{2}$

Or, $\sqrt{:17 + \sqrt{280}:} - \sqrt{:5 + \sqrt{24}:}$

To and from $\sqrt{(3)}10 + \sqrt{(3)}7$

Add and Subtr. $\sqrt{(3)}3 - \sqrt{(3)}2$

Summ, $\sqrt{(3)}10 + \sqrt{(3)}7 + \sqrt{(3)}3 - \sqrt{(3)}2$

Difference; $\sqrt{(3)}10 + \sqrt{(3)}7 - \sqrt{(3)}3 + \sqrt{(3)}2.$

Sect. X. *Of Multiplication in Compound Surds.*

Example 1.

Multiplicand, $\sqrt{180} + \sqrt{48}$
Multiplicator, $\sqrt{125} + \sqrt{12}$ } that is, $\left\{\begin{array}{l} 6\sqrt{5} + 4\sqrt{3} \\ 5\sqrt{5} + 2\sqrt{3} \end{array}\right.$

$150 + 20\sqrt{15}$
$\quad + 12\sqrt{15} + 24$

Product, $150 + 32\sqrt{15} + 24$

That is, $174 + 32\sqrt{15}.$

Example 2.

Multiplicand, $6 - \sqrt{20}$
Multiplicator, $8 - \sqrt{45}$ } that is, $\left\{\begin{array}{l} 6 - 2\sqrt{5} \\ 8 - 3\sqrt{5} \end{array}\right.$

$48 - 16\sqrt{5}$
$\quad - 18\sqrt{5} + 30$

Product; $78 - 34\sqrt{5}.$

Example 3.

Multiplicand, $\sqrt{18} - 3$
Multiplicator, $\sqrt{8} + 2$ } that is, $\left\{\begin{array}{l} 3\sqrt{2} - 3 \\ 2\sqrt{2} + 2 \end{array}\right.$

$12 - 6\sqrt{2}$
$\quad + 6\sqrt{2} - 6$

Product, $12 \quad - 6$

That is, $6.$

Example 4.

Multiplicand, $4\sqrt{5} + 3\sqrt{5}$
Multiplicator, $4\sqrt{5} + 3\sqrt{5}$ } that is, $\left\{\begin{array}{l} 7\sqrt{5} \\ 7\sqrt{5} \end{array}\right.$

Product, $245.$

EXPLICATION.

In the first Example, the two Compound Surd numbers propos'd to be multiplied are $\surd 180 + \surd 48$ and $\surd 125 + \surd 12$, which are reduced to $6\surd 5 + 4\surd 3$ and $5\surd 5 + 2\surd 3$; (by *Sect.* 6. of this *Chapt.*) then $6\surd 5$ multiplied by $5\surd 5$ (according to Rule 5. in *Sect.* 4. of this *Chapt.*) produceth 150; also, $4\surd 3$ multiplied by $5\surd 5$ (according to Rule 6. in *Sect* 4.) produceth $20\surd 15$; again, $6\surd 5$ into $2\surd 3$ makes $12\surd 15$; and $4\surd 3$ into $2\surd 3$, produceth 24; lastly, those Products added together make $174 + 32\surd 15$, the Product sought. The rest of the Examples are wrought in like manner.

When the Multiplicand hath not the same Radical sign with the Multiplier, they must first be reduced to the same Radical sign, (by *Sect.* 3. of this *Chapt.*) and then the Multiplication is to be made by some of the Rules in *Sect.* 4. as will be manifest in the following Example.

Multiplicand,	$\surd(5)6 + \surd(3)7 + 5$
Multiplicator,	$\surd 3$
Product,	$\surd(10)8748 + \surd(6)1323 + 5\surd 3.$

EXPLICATION.

1. $\surd(5)6$ and $\surd 3$ are reduced to these having a common Radical sign, to wit, $\surd(10)36$ and $\surd(10)243$, which multiplied one into the other, produce $\surd(10)8748$.

2. $\surd(3)7$ and $\surd 3$ are reduced to $\surd(6)49$ and $\surd(6)27$, which multiplied one by the other, produce $\surd(6)1323$.

3. The Rational number 5 multiplied into $\surd 3$ makes $5\surd 3$, or, $\surd 75$.

Lastly, those three simple Products added together give the Product sought, to wit, $\surd(10)8748 + \surd(6)1323 + 5\surd 3$ ($\surd 75$).

Three compendious Rules, very useful in the Multiplication of Binomials and Residuals.

1. Because $a+e$ multiplied by $a+e$ produceth $aa + 2ae + ee$; it is evident that the summ of the Squares of the parts (or Names) of any Binomial, together with twice the Product of the parts multiplied one into the other is equal to the Square of the Summ of the parts. Therefore, to multiply any Binomial by it self, (or to square it,) take the Squares of the parts, and twice the Product of the parts for the Square sought.

2. Because $a-e$ multiplied by $a-e$ produceth $aa - 2ae + ee$; it is manifest that the summ of the Squares of the parts of any Residual, less by the double Product of the parts, is equal to the Square of the difference of the parts. Therefore, to square any Residual, from the summ of the Squares of the parts subtract twice the Product of the parts, and take the Remainder for the Square sought.

3. Because $a+e$ multiplied by $a-e$ produceth $aa - ee$; it is evident that the difference of the Squares of the parts of any Binomial, is equal to the Product made by the multiplication of the summ of the parts into their difference. Therefore, if a Binomial be to be multiplied by its correspondent Residual, that is, by the difference of the parts of the Binomial, take the difference of the Squares of the parts for the Product sought. These three Rules will be exercis'd by the six Examples next following, and by divers other Examples in this and the following Sections of this Chapter.

Multiplicand,	$3 + \surd 5$	$3 - \surd 5$
Multiplicator,	$3 + \surd 5$	$3 - \surd 5$
Product,	$9 + 6\surd 5 + 5$	$9 - 6\surd 5 + 5$
That is,	$14 + 6\surd 5$	$14 - 6\surd 5$
Multiplicand,	$3 + \surd 5$	$\surd(3)\,27 + \surd(3)\,8$
Multiplicator,	$3 - \surd 5$	$\surd(3)\,27 - \surd(3)\,8$
Product,	$9 - 5$	$\surd(3)729 - \surd(3)64$
That is,	4	5
Multiplicand,	$\surd(6)7 - \surd(6)5$	$\surd(10)7 + \surd(10)3$
Multiplicator,	$\surd(6)7 + \surd(6)5$	$\surd(10)7 - \surd(10)3$
Product,	$\surd(3)7 - \surd(3)5$	$\surd(5)7 - \surd(5)3$

EXPLI-

EXPLICATION.

In the firſt of the ſix laſt Examples, the Binomial $3 + \sqrt{5}$ multiplied into it ſelf or ſquared, produceth $14 + 6\sqrt{5}$. For the Squares of the parts 3 and $\sqrt{5}$ are 9 and 5, and twice the Product of 3 into $\sqrt{5}$ makes $6\sqrt{5}$, to wit, $\sqrt{180}$; therefore (by the ſecond of the three preceding Rules,) $9 + 5 + 6\sqrt{5}$; that is, $14 + 6\sqrt{5}$ is the Square of the given Binomial $3 + \sqrt{5}$.

In the ſecond Example, the Reſidual $3 - \sqrt{5}$ ſquared or multiplied by it ſelf produceth $14 - 6\sqrt{5}$, (by the ſecond of the ſaid three Rules.)

In the third Example, the Binomial $3 + \sqrt{5}$ multiplied by its correſpondent Reſidual $3 - \sqrt{5}$, produceth 4; which (by the laſt of the ſaid three Rules) is equal to the difference of the Squares of the parts 3 and $\sqrt{5}$.

Likewiſe in the fourth Example, the Binomial $\sqrt{(3)}27 + \sqrt{(3)}8$, multiplied by its correſpondent Reſidual $\sqrt{(3)}27 - \sqrt{(3)}8$, produceth $\sqrt{(3)}729 - \sqrt{(3)}64$; to wit, the difference of the Squares of the parts of the given Binomial or Reſidual.

And in the fifth Example, the Reſidual $\sqrt{(6)}7 - \sqrt{(6)}5$ multiplied by its correſpondent Binomial $\sqrt{(6)}7 + \sqrt{(6)}5$, produceth $\sqrt{(3)}7 - \sqrt{(3)}5$; which is equal to the difference of the Squares of the parts of the given Reſidual or Binomial. For (by the ſeventh Rule in *Sect.* 4. of this *Chapt.*) the Square of $\sqrt{(6)}7$ is $\sqrt{(3)}7$, and the Square of $\sqrt{(6)}5$ is $\sqrt{(3)}5$.

Examples of Multiplication in Compound Surd quantities expreſt by Letters.

Multiplicand,	$\sqrt{abb} + \sqrt{cff}$	that is,	$b\sqrt{a} + f\sqrt{c}$
Multiplicator,	$\sqrt{add} + \sqrt{caa}$		$d\sqrt{a} + a\sqrt{c}$
			$bda + fd\sqrt{ca}$
			$+ ba\sqrt{ca} + fac$
		Product,	$bda + \overline{fd + ba} \times \sqrt{ca}, + fac.$

Multiplicand,	$2a + 3a\sqrt{d}$	$\sqrt{bc} + a$
Multiplicator,	$3c - 2c\sqrt{d}$	$\sqrt{bc} - a$
	$6ac + 9ac\sqrt{d}$	$bc + a\sqrt{bc}$
	$- 4ac\sqrt{d} - 6acd$	$- a\sqrt{bc} - aa$
Product,	$6ac + 5ac\sqrt{d} - 6acd$	$bc - aa$

Multiplicand,	$a + \sqrt{b}$	$\sqrt{ab} + \sqrt{c}$
Multiplicator,	$a + \sqrt{b}$	$\sqrt{ac} + \sqrt{d}$
Product,	$aa + 2a\sqrt{b} + b$	$a\sqrt{bc} + c\sqrt{a} + \sqrt{abd} + \sqrt{cd}$

Multiplicand,	$3bb\sqrt{d} + d\sqrt{d}$	that is,	$\overline{3bb + d} \times \sqrt{d}$
Multiplicator,	$3bb\sqrt{d} + d\sqrt{d}$		$\overline{3bb + d} \times \sqrt{d}$
Product,	$9bbbbd + 6bbdd + ddd$;	or,	$\overline{9bbbb + 6bbd + dd} \times d.$

The Operation in theſe ſix laſt Examples will be familiar to him that underſtands the Rules and Examples before delivered concerning the Multiplication of Surd numbers, and Surd quantities expreſt by Letters.

Sect. XI. *Diviſion in Compound Surds.*

Examples of Diviſion, where the Dividend is a Compound quantity, and the Diviſor a Simple quantity.

Dividend,	$\sqrt{21} + \sqrt{15}$	$\sqrt{(3)}14 - \sqrt{(3)}28$
Diviſor,	$\sqrt{3}$	$\sqrt{(3)}7$
Quotient,	$\sqrt{7} + \sqrt{5}$	$\sqrt{(3)}2 - \sqrt{(3)}4$

Dividend,	$12\sqrt{6} + 6\sqrt{18} - 2\sqrt{12}$	$\sqrt{20} - \surd(3)10$
Divisor,	$3\sqrt{6}$	3
Quotient,	$4 + 2\sqrt{3} - \frac{2}{3}\sqrt{2}$	$\sqrt{\frac{20}{9}} - \surd(3)\frac{10}{27}$

Dividend,	$\surd(4)8 + \surd(5)3$	$\surd(4)23328 - \surd(4)10368$
Divisor,	$\sqrt{2}$	6
Quotient,	$\surd(4)2 + \surd(10)\frac{9}{32}$	$\surd(4)18 - \surd(4)8$

EXPLICATION.

The first Example is wrought according to Rule 1. in *Sect.* 5. of this *Chapt.* For first, $\sqrt{21}$ divided by $\sqrt{3}$ gives the Quotient $\sqrt{7}$, then $\sqrt{15}$ divided by $\sqrt{3}$ gives the Quotient $\sqrt{5}$. Therefore $\sqrt{21} + \sqrt{15}$ divided by $\sqrt{3}$, gives $\sqrt{7} + \sqrt{5}$, the Quotient sought in the first Example.

The second Example is wrought like the first; for $\surd(3)14$ divided by $\surd(3)7$ gives $\surd(3)2$, and $- \surd(3)28$ divided by $\surd(3)7$ gives $- \surd(3)4$. Therefore, $\surd(3)14 - \surd(3)28$ divided by $\surd(3)7$, gives $\surd(3)2 - \surd(3)4$, the Quotient sought in the second Example.

The third Example is wrought according to the fifth and sixth Rules of *Sect.* 5. of this *Chapt.* For first, $12\sqrt{6}$ divided by $3\sqrt{6}$ gives the Quotient 4, (by the said fifth Rule;) then $6\sqrt{18}$ divided by $3\sqrt{6}$ gives $2\sqrt{3}$, (by the said sixth Rule;) likewise, $-2\sqrt{12}$ divided by $3\sqrt{6}$ gives $-\frac{2}{3}\sqrt{2}$; (for 2 divided by 3 gives $\frac{2}{3}$, and $\sqrt{12}$ divided by $\sqrt{6}$ gives $\sqrt{2}$.) Therefore, $12\sqrt{6} + 6\sqrt{18} - 2\sqrt{12}$ divided by $3\sqrt{6}$, gives $4 + 2\sqrt{3} - \frac{2}{3}\sqrt{2}$, the Quotient sought in the third Example.

In the fourth Example, $\sqrt{20}$ divided by 3, (that is, by $\sqrt{9}$,) gives $\sqrt{\frac{20}{9}}$, or $\sqrt{2\frac{2}{9}}$; and $- \surd(3)10$ divided by 3, (that is, by $\surd(3)27$,) gives $- \surd(3)\frac{10}{27}$.

In the fifth Example, $\surd(4)8$ and $\sqrt{2}$ are first reduced to $\surd(4)8$ and $\surd(4)4$; then $\surd(4)8$ divided by $\surd(4)4$, gives $\surd(4)2$; likewise, $\surd(5)3$ and $\sqrt{2}$ are reduced to $\surd(10)9$ and $\surd(10)32$; then $\surd(10)9$ divided by $\surd(10)32$ gives the Quotient $\surd(10)\frac{9}{32}$. Therefore, $\surd(4)8 + \surd(5)3$ divided by $\sqrt{2}$, gives $\surd(4)2 + \surd(10)\frac{9}{32}$, the Quotient sought in the fifth Example. The sixth Example is wrought in like manner; and the Proof in these or the like Examples of Division may be made by Multiplication.

Propositions concerning Division in Surd Quantities, when the Divisor is a Binomial or Trinomial, &c.

When the Divisor is a Binomial or Residual consisting of two square Roots or biquadratick Roots, or of one square Root or biquadratick Root, and of a Rational number; as also when the Divisor is a Trinomial, or Quadrinomial, and none of its Radical signs exceeds that of the square Root, the work of Division in those cases is grounded upon the five following Propositions, *viz.*

1. If a Binomial consisting of two simple square Roots connected by $+$, be multiplied by its correspondent Residual, that is, by the difference of those Roots; or if a Residual consisting of two simple square Roots connected by $-$, be multiplied by its correspondent Binomial, that is, by the summ of the same Roots, the Product will be entirely Rational. So the Binomial $\sqrt{5} + \sqrt{3}$ multiplied by $\sqrt{5} - \sqrt{3}$, (or, the Residual $\sqrt{5} - \sqrt{3}$ by $\sqrt{5} + \sqrt{3}$,) gives the Rational Product 2; (by the last of the three Rules before delivered in *Sect.* 10. of this *Chapt.*)

Likewise, $\sqrt{a} + \sqrt{b}$ multiplied by $\sqrt{a} - \sqrt{b}$, gives the Rational Product $a - b$.

2. If a Binomial consisting of two Biquadratick simple Roots connected by $+$, be multiplied by its correspondent Residual, to wit, by the difference of those Roots, the Product will be also a Residual consisting of two square Roots connected by $-$, and if this Residual be multiplied by the summ of its Names, (or Parts,) it will give a Product entirely Rational.

As, for example, the Binomial $\surd(4)5 + \surd(4)3$ multiplied by $\surd(4)5 - \surd(4)3$ makes $\sqrt{5} - \sqrt{3}$, which multiplied by $\sqrt{5} + \sqrt{3}$ gives the Rational Product 2.

Likewise $\surd(4)81 - 2$, or $\surd(4)81 - \surd(4)16$, multiplied by $\surd(4)81 + \surd(4)16$ makes $\sqrt{81} - \sqrt{16}$, which multiplied by $\sqrt{81} + \sqrt{16}$ gives the Rational Product 65.

3. If

3. If a Trinomial confifting of three fimple fquare Roots connected by $+$, or by $+$ and $-$, be multiplied by the fame Trinomial, after any one Sign $+$ is changed into $-$; or any one Sign $-$ into $+$, the Product will confift of two Names, (or Parts;) and then if this Product be multiplied by its correfpondent Binomial or Refidual, (according to the preceding *Prop.* 1.) the laft Product will be entirely Rational.

As, for example, the Trinomial $\sqrt{5}+\sqrt{3}+\sqrt{2}$ multiplied by $\sqrt{5}+\sqrt{3}-\sqrt{2}$ gives $2\sqrt{15}+6$, and this multiplied by $2\sqrt{15}-6$ gives the Rational Product 24.

Likewife, $\sqrt{30}-\sqrt{5}-\sqrt{3}$ multiplied by $\sqrt{30}+\sqrt{5}-\sqrt{3}$ produceth $28-2\sqrt{90}$, and this multiplied by $28+2\sqrt{90}$ gives the Rational Product 424.

After the fame manner, $\sqrt{a}+\sqrt{b}-\sqrt{c}$ multiplied by $\sqrt{a}+\sqrt{b}+\sqrt{c}$ gives the Product $2\sqrt{ab}+a+b-c$, whofe Rational Part $a+b-c$ we may fuppofe to be equal to fome fingle Quantity d, and then the faid Product will be a Binomial $2\sqrt{ab}+d$; which multiplied by its correfpondent Refidual $2\sqrt{ab} \backsim d$ gives a Product entirely Rational, to wit, $4ab \backsim dd$. And fo of other Trinomials that are qualified as before is fuppofed.

4. If a Quadrinomial confifting of four fimple fquare Roots connected by $+$, or by $+$ and $-$, be multiplied by the fame Quadrinomial after two Signs $+$ are changed into $-$, or two Signs $-$ into $+$, the Product will confift of three Names, (or Parts;) then if this Product be multiplied by its correfpondent Trinomial (according to *Prop.* 3.) there will come forth a Binomial or Refidual; and laftly, this Binomial or Refidual multiplied by its correfpondent Refidual or Binomial will give a Rational Product.

As, for example, the Quadrinomial $\sqrt{6}+\sqrt{5}+\sqrt{3}+\sqrt{2}$ multiplied by $\sqrt{6}+\sqrt{5}-\sqrt{3}-\sqrt{2}$ produceth the Trinomial $6+2\sqrt{30}-2\sqrt{6}$; which multiplied by its correfpondent Trinomial $6+2\sqrt{30}+2\sqrt{6}$, (according to the precedent *Prop.* 3.) gives the Binomial $132+24\sqrt{30}$; and this multiplied by its correfpondent Refidual $132-24\sqrt{30}$, gives the Rational Product 144.

After the fame manner, the Quadrinomial $\sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{d}$ multiplied by $\sqrt{a}-\sqrt{b}-\sqrt{c}-\sqrt{d}$ gives the Product $a+d-b-c-2\sqrt{ad}-2\sqrt{bc}$, whofe Rational part $a+d-b-c$ we may fuppofe to be equal to fome fingle Quantity f, and then the faid Product will be a Trinomial, to wit, $f-2\sqrt{ad}-2\sqrt{bc}$; this multiplied by it felf after one of its figns $-$ is changed into $+$, (according to *Prop.* 3.) will produce a Refidual of two Names (or Parts,) and this Refidual multiplied by its correfpondent Binomial will give a Rational Product.

5. If two numbers be given for a Dividend and Divifor, and each be multiplied by fome number, the firft Product divided by the latter will give the fame Quotient that arifeth by dividing the given Dividend by the given Divifor. As, if 6 be to be divided by 2, if you multiply each by 4, and divide the firft Product 24 by the latter 8, the Quotient 3 is the fame that arifeth by dividing 6 by 2. For (by 17 *Prop.* 7. *Elem.* Euclid.) if a number a multiplying two numbers b, c, produce two other numbers ab and ac, the numbers produced fhall be in the fame Proportion that the numbers multiplied are, *viz.* as, $b \,.\, c :: ab \,.\, ac$, and therefore $\frac{ab}{ac} = \frac{b}{c}$; alfo, $\frac{ac}{ab} = \frac{c}{b}$. From the foregoing five Propofitions the following Rule is deduced, *viz.*

6. *A Rule for Divifion in Surd Quantities when the Divifor is a Binomial, Trinomial or Quadrinomial of fuch kind as before is declared.*

Reduce the given Divifor to a new Divifor that may be a fimple Rational quantity; reduce alfo the given Dividend to a new Dividend, by multiplying the former by the fame quantity or quantities that were Multiplicators in reducing the given Divifor to a Rational quantity; then divide the new Dividend by the new Divifor, (according to the method in the Examples at the beginning of this *Sect.* 11.) fo the Quotient fhall be the fame with that which would arife by dividing the given Dividend by the given Divifor.

As, for example, to divide $\sqrt{8}+\sqrt{6}$ by $\sqrt{4}+\sqrt{2}$, I firft multiply the Divifor $\sqrt{4}+\sqrt{2}$ by its correfpondent Refidual $\sqrt{4}-\sqrt{2}$, and it produceth 2 for a new Divifor; alfo I multiply the Dividend $\sqrt{8}+\sqrt{6}$ by the faid $\sqrt{4}-\sqrt{2}$, and it gives the Product $\sqrt{32}+\sqrt{24}-\sqrt{16}-\sqrt{12}$ for a new Dividend, this divided by 2 (the Divifor before found,) gives $\sqrt{8}+\sqrt{6}-2-\sqrt{3}$ the Quotient fought, being equal to that which would arife by dividing $\sqrt{8}+\sqrt{6}$ by $\sqrt{4}+\sqrt{2}$, as will be evident by the Proof; for

for if the said Quotient $\sqrt{8} + \sqrt{6} - 2 - \sqrt{3}$ be multiplied by the given Divisor $\sqrt{4} + \sqrt{2}$, it will produce the given Dividend $\sqrt{8} + \sqrt{6}$.

Likewise, to divide $ab + b\sqrt{bc}$ by $a + \sqrt{bc}$, I multiply each by $a - \sqrt{bc}$, (the Residual correspondent to the Divisor,) and it produceth $aa - bc$ for a new Divisor, and $aab - bbc$ for a new Dividend, this divided by that gives b for the Quotient sought; for b multiplied into the given Divisor $a + \sqrt{bc}$ makes the given Dividend $ab + b\sqrt{bc}$. Another way of finding out the Quotient in this last Example, is shewn in the first of the six Examples at the latter end of this *Sect.* 11.

Again, to divide 10 by $\sqrt{(4)5} + \sqrt{(4)3}$, I multiply each by $\sqrt{(4)5} - \sqrt{(4)3}$; and there comes forth a new Dividend $\sqrt{(4)50000} - \sqrt{(4)30000}$, and a new Divisor $\sqrt{5} - \sqrt{3}$; but this Divisor not being a Rational number, I multiply again both the said new Dividend and Divisor by $\sqrt{5} + \sqrt{3}$, and it produceth another new Dividend $\sqrt{(4)1250000} - \sqrt{(4)750000} + \sqrt{(4)450000} - \sqrt{(4)270000}$, and another new Divisor 2; by this I divide the last Dividend and there ariseth $\sqrt{(4)78125} - \sqrt{(4)46875} + \sqrt{(4)28125} - \sqrt{(4)16875}$ the Quotient sought; for if it be multiplied by the proposed Divisor $\sqrt{(4)5} + \sqrt{(4)3}$ it will produce the given Dividend 10.

Again, to divide $\sqrt{8}$ by $\sqrt{3} + \sqrt{2} + 1$, I first multiply the Divisor by $\sqrt{3} + \sqrt{2} - 1$ and it makes $\sqrt{24} + 4$, this multiplied by its correspondent Residual $\sqrt{24} - 4$ gives the Product 8 for a new Divisor: Now because the given Divisor was first multiplied by $\sqrt{3} + \sqrt{2} - 1$ and the Product by $\sqrt{24} - 4$, the given Dividend must likewise be multiplied first by $\sqrt{3} + \sqrt{2} - 1$, and the Product $\sqrt{24} + 4 - \sqrt{8}$ by $\sqrt{24} - 4$, and there will be produced $8 + \sqrt{128} - \sqrt{192}$ for a new Dividend; so instead of the given Dividend and Divisor we have other numbers in the same Proportion, *viz.* $8 + \sqrt{128} - \sqrt{192}$ and 8. Therefore (by *Prop.* 5.) the former divided by the latter will give the Quotient sought, to wit, $1 + \sqrt{2} - \sqrt{3}$; but that this is the true Quotient will appear by Multiplication, for if $1 + \sqrt{2} - \sqrt{3}$ be multiplied by the proposed Divisor $\sqrt{3} + \sqrt{2} + 1$, it will produce the given Dividend $\sqrt{8}$.

Note. Although the new Divisor and Dividend found out as aforesaid, may sometimes happen to be Negative quantities, (that is, such whose values are less than nothing,) yet Division being made by them with respect to the Rules of + and —, they will give the true Quotient sought. As, for example, suppose 30 be to be divided by $2 + \sqrt{9}$, (that is, 30 by 5;) first, the Divisor $2 + \sqrt{9}$ being multiplied by $2 - \sqrt{9}$ gives $4 - 9$, that is, -5 for a new Divisor, and the Dividend 30 multiplied by the said $2 - \sqrt{9}$ gives $60 - \sqrt{8100}$ for a new Dividend, which divided by -5 gives $+6$; which is the same with the Quotient that ariseth by dividing 30 by $2 + \sqrt{9}$, that is, by 5.

Again, let $4 + \sqrt{25}$ be to be divided by $1 + \sqrt{9}$, (that is, 9 by 4, where the Quotient is manifestly $2\frac{1}{4}$;) first, the Divisor $1 + \sqrt{9}$ multiplied by $1 - \sqrt{9}$ produceth $1 - 9$, that is, -8 for a new Divisor; and the Dividend $4 + \sqrt{25}$ multiplied by the said $1 - \sqrt{9}$ makes $4 + \sqrt{25} - 4\sqrt{9} - \sqrt{225}$ for a new Dividend, which divided by -8, (according to the Examples at the beginning of this *Sect.* 11.) gives $-\frac{1}{2} - \sqrt{\frac{25}{64}} + \frac{1}{2}\sqrt{9} + \sqrt{\frac{225}{64}}$ the Quotient sought, which after due contraction makes $2\frac{1}{4}$. For $\frac{1}{2}\sqrt{9}$, that is, $\sqrt{\frac{144}{64}}$ is equal to $\frac{12}{8}$, and $\sqrt{\frac{225}{64}}$ is $\frac{15}{8}$, which added to the said $\frac{12}{8}$ makes $\frac{27}{8}$; also $-\sqrt{\frac{25}{64}}$ is $-\frac{5}{8}$, which added to $-\frac{1}{2}$, (or $-\frac{4}{8}$,) makes $-\frac{9}{8}$, this added to $\frac{27}{8}$ gives $\frac{18}{8}$ (or $2\frac{1}{4}$,) the Quotient before found.

7. When the Divisor is a Binomial, or a Residual consisting of two simple Cubick or Biquadratick, &c. Roots, it may be reduced to a Rational Divisor by this following Proposition, *viz.*

If in the Proportion of the Names (or Parts) of a Binomial or Residual, there be found so many continual Proportionals in multitude as there be Units in the Index of the Radical sign, and that the Radical signs of the Parts of the Binomial or Residual, and also of the Proportionals be the same, but connected in the Binomial by +, and in the Proportionals by + and — alternately; or contrarily, in the Proportionals by +, and in the Residual by + and —; the Product made by the multiplication of the Proportionals by the Binomial or Residual shall be Rational.

As, for example, if there be proposed the Binomial $\sqrt{(3)7} + \sqrt{(3)5}$; find three continual Proportionals, that the first may be to the second, and the second to the third, as $\sqrt{(3)7}$ to $\sqrt{(3)5}$, which may be done by the help of *Sect.* 8. *Chap.* 5. of this Book; where it hath been shewn, that aa, ae and ee are continual Proportionals in the Reason of a to e. Therefore if we suppose $\sqrt{(3)7}$ to be a, and $\sqrt{(3)5}$ to be e, then the Square of $\sqrt{(3)7}$,

of $\sqrt{(3)}7$, to wit, $\sqrt{(3)}49$, shall be the first Proportional (aa); the Product of $\sqrt{(3)}7$ into $\sqrt{(3)}5$, to wit, $\sqrt{(3)}35$, shall be the second Proportional (ae); and the Square of $\sqrt{(3)}5$, to wit, $\sqrt{(3)}25$, shall be the third Proportional (ee): so that these three Cubick Roots, to wit, $\sqrt{(3)}49$, $\sqrt{(3)}35$ and $\sqrt{(3)}25$ are continual Proportionals in the Reason of $\sqrt{(3)}7$ and $\sqrt{(3)}5$. Now I say (according to the Proposition,) If $\sqrt{(3)}49 - \sqrt{(3)}35 + \sqrt{(3)}25$ be multiplied by $\sqrt{(3)}7 + \sqrt{(3)}5$, the Product shall be Rational; also, if $\sqrt{(3)}49 + \sqrt{(3)}35 + \sqrt{(3)}25$ be multiplied by $\sqrt{(3)}7 - \sqrt{(3)}5$ the Product shall be Rational, as will appear by the following Operation.

$$\begin{array}{lrl}
\text{Multiplicand,} & & \sqrt{(3)}49 - \sqrt{(3)}35 + \sqrt{(3)}25 \\
\text{Multiplicator,} & & \sqrt{(3)}7 + \sqrt{(3)}5 \\
\hline
 & 7 & - \sqrt{(3)}245 + \sqrt{(3)}175 \\
 & & + \sqrt{(3)}245 - \sqrt{(3)}175 + 5 \\
\hline
\text{The Product} & 12 & \text{is Rational.}
\end{array}$$

$$\begin{array}{lrl}
\text{Multiplicand,} & & \sqrt{(3)}49 + \sqrt{(3)}35 + \sqrt{(3)}25 \\
\text{Multiplicator,} & & \sqrt{(3)}7 - \sqrt{(3)}5 \\
\hline
 & 7 & + \sqrt{(3)}245 + \sqrt{(3)}175 \\
 & & - \sqrt{(3)}245 - \sqrt{(3)}175 - 5 \\
\hline
\text{The Product} & 2 & \text{is Rational.}
\end{array}$$

But for the greater evidence of the certainty of this Proposition in a Binomial and Residual consisting of any two simple Cubick Roots whatever, let there be proposed this Binomial $\sqrt{(3)}b + \sqrt{(3)}d$, and suppose b greater than d; then three continual Proportionals in the proportion of $\sqrt{(3)}b$ to $\sqrt{(3)}d$ will be found $\sqrt{(3)}bb$, $\sqrt{(3)}bd$ and $\sqrt{(3)}dd$; then multiply as before, *viz.*

$$\begin{array}{lrl}
\text{Multiplicand,} & & \sqrt{(3)}bb - \sqrt{(3)}bd + \sqrt{(3)}dd \\
\text{Multiplicator,} & & \sqrt{(3)}b + \sqrt{(3)}d \\
\hline
 & b & - \sqrt{(3)}bbd + \sqrt{(3)}bdd \\
 & & + \sqrt{(3)}bbd - \sqrt{(3)}bdd + d \\
\hline
\text{The Product} & b + d & \text{is Rational.}
\end{array}$$

Again,

$$\begin{array}{lrl}
\text{Multiplicand,} & & \sqrt{(3)}bb + \sqrt{(3)}bd + \sqrt{(3)}dd \\
\text{Multiplicator,} & & \sqrt{(3)}b - \sqrt{(3)}d \\
\hline
 & b & + \sqrt{(3)}bbd + \sqrt{(3)}bdd \\
 & & - \sqrt{(3)}bbd - \sqrt{(3)}bdd - d \\
\hline
\text{The Product} & b - d & \text{is Rational.}
\end{array}$$

Whence you may observe, that the first Rational Product is the summ of the Names (or Parts,) omitting the Radical signs, of the cubick Binomial proposed; and the latter Rational Product is the difference of the Parts, omitting the Radical signs, of the Cubick Residual proposed: so that the Rational Product made by the multiplication of the said Proportionals and Binomial or Residual may be discovered without any multiplication.

8. Now, that the Use of the last preceding Proposition may appear, let it be required to divide 10 by $\sqrt{(3)}7 - \sqrt{(3)}5$; First, because the Index of the Radical sign is 3, I seek three continual Proportionals in the proportion of $\sqrt{(3)}7$ to $\sqrt{(3)}5$; which Proportionals (as before hath been shewn) are $\sqrt{(3)}49$, $\sqrt{(3)}35$ and $\sqrt{(3)}25$; these I connect by $+$, because the Parts of the given Divisor are connected by $-$, and there ariseth $\sqrt{(3)}49 + \sqrt{(3)}35 + \sqrt{(3)}25$: then by this common Multiplicator I multiply as well the Dividend 10, as the Divisor $\sqrt{(3)}7 - \sqrt{(3)}5$, and it produceth $\sqrt{(3)}49000 + \sqrt{(3)}35000 + \sqrt{(3)}25000$ for a new Dividend, and 2 for a new Divisor: lastly, by dividing the said new Dividend by the new Divisor, there ariseth $\sqrt{(3)}6125 + \sqrt{(3)}4375 + \sqrt{(3)}3125$ the Quotient sought; for if it be multiplied by the given Divisor $\sqrt{(3)}7 - \sqrt{(3)}5$, it will produce the given Dividend 10.

In

In like manner, to divide 10 by this Binomial $\sqrt{(3)}5 + \sqrt{(3)}3$; first I seek three continual Proportionals in the Reason of $\sqrt{(3)}5$ to $\sqrt{(3)}3$, which Proportionals will be found $\sqrt{(3)}25$, $\sqrt{(3)}15$ and $\sqrt{(3)}9$; these I connect by + and − alternately, because the Parts of the given Divisor are connected by +, *viz.* to the first Proportional I prefix +, to the second −, and to the third +, so they make $\sqrt{(3)}25 - \sqrt{(3)}15 + \sqrt{(3)}9$; by this, as a common Multiplicator, I multiply as well the Dividend 10 as the Divisor $\sqrt{(3)}5 + \sqrt{(3)}3$, and there ariseth a new Dividend $\sqrt{(3)}25000 - \sqrt{(3)}15000 + \sqrt{(3)}9000$, and a new Divisor 8; by which I divide the said new Dividend, and there comes forth $\sqrt{(3)}\frac{3125}{64} - \sqrt{(3)}\frac{1875}{64} + \sqrt{(3)}\frac{1125}{64}$, the Quotient sought.

The same method is to be observed when the Divisor is a Binomial or a Residual consisting of two simple Biquadratick Roots.

As, for example, to divide 10 by $\sqrt{(4)}5 + \sqrt{(4)}3$, (which hath already been done after another manner in the third Example of the Rule in the sixth step of this Section;) First, because the Index of the Radical sign is 4, I search out four continual Proportionals in the Reason of $\sqrt{(4)}5$ to $\sqrt{(4)}3$ in this manner, *viz.* For as much as (by *Sect.* 8. *Chapt.* 5. of this Book,) these are continual Proportionals, to wit, aaa, aae, aee and eee, I suppose $\sqrt{(4)}5$ to be a, and $\sqrt{(4)}3$ to be e; then I multiply $\sqrt{(4)}5$ into it self cubically, and it gives the first Proportional $\sqrt{(4)}125$, (to wit, aaa;) also I multiply the Square of $\sqrt{(4)}5$ into $\sqrt{(4)}3$, and it gives the second Proportional $\sqrt{(4)}75$, (to wit, aae;) again, I multiply $\sqrt{(4)}5$ into the Square of $\sqrt{(4)}3$, and it gives the third Proportional $\sqrt{(4)}45$, (to wit, aee;) lastly, I multiply $\sqrt{(4)}3$ into it self cubically, and it gives the fourth Proportional $\sqrt{(4)}27$, (to wit, eee:) Then because the two Parts of the given Divisor are connected by +, I connect those four Proportionals by + and − alternately; so there ariseth this Compound number $\sqrt{(4)}125 - \sqrt{(4)}75 + \sqrt{(4)}45 - \sqrt{(4)}27$, by which, as a common Multiplicator, I multiply as well the given Dividend 10, as the given Divisor $\sqrt{(4)}5 + \sqrt{(4)}3$, and there ariseth a new Dividend $\sqrt{(4)}1250000 - \sqrt{(4)}750000 + \sqrt{(4)}450000 - \sqrt{(4)}270000$, and a new Divisor 2; which are the same in every respect with those found in the place before cited.

After the same manner, when the Divisor is a Binomial or a Residual having 5 or 6, &c. for the Index of the common Radical sign of the Roots, it may be reduced to a new Divisor that shall be Rational. But it must be remembred, that when the Roots are of different kinds they must first be reduced to a common Radical sign.

But when the Divisor cannot be reduced to a simple Rational number by any of the foregoing Rules, (which are all that I have met with in Algebraical Authors,) the Dividend may be set as a Numerator over the Divisor as a Denominator, and the Fraction so constituted shall be equal to the Quotient. As, for example, if $\sqrt{48} + \sqrt{(3)}3$ be to be divided by $\sqrt{15} + \sqrt{(3)}6 - \sqrt{3}$, the Quotient may be represented by this Fraction, to wit,

$$\frac{\sqrt{48} + \sqrt{(3)}3}{\sqrt{15} + \sqrt{(3)}6 - \sqrt{3}}.$$

Examples of Division in Compound Surd quantities exprest by Letters.

Division in Compound Surd quantities exprest by Letters depends upon the Rules of Simple Surds before delivered; as also upon the General method of Division in *Sect.* 9. *Chapt.* 5. *Book* 1. as will appear by the following Examples, some of which I shall afterwards explain.

Divisor.	Dividend.		Quotient.
$a + \sqrt{bc}$)	$ab + b\sqrt{bc}$	(b,	Quotient.
	$ab + b\sqrt{bc}$		
	0 0		

$a + \sqrt{bc}$)	$aa - bc$	($a - \sqrt{bc}$
	$aa + a\sqrt{bc}$	
	$- bc - a\sqrt{bc}$	
	$- bc - a\sqrt{bc}$	
	0 0	

$\sqrt{ab} -$

$$\sqrt{ab} - \sqrt{cd}\ )\ ab - cd\ (\ \sqrt{ab} + \sqrt{cd}$$
$$ab - \sqrt{abcd}$$
$$- cd + \sqrt{abcd}$$
$$- cd + \sqrt{abcd}$$
$$0 \quad 0$$

$$a + \sqrt{bc}\ )\ aaa + bc\sqrt{bc}\ (\ aa + bc - a\sqrt{bc}$$
$$aaa + aa\sqrt{bc}$$
$$+ bc\sqrt{bc} - aa\sqrt{bc}$$
$$+ bc\sqrt{bc} + abc$$
$$- aa\sqrt{bc} - abc$$
$$- aa\sqrt{bc} - abc$$
$$0 \quad 0$$

$$aa + a\sqrt{bc}\ )\ aaab - abbc\ (\ ab - b\sqrt{bc}$$
$$aaab + aab\sqrt{bc}$$
$$- abbc - aab\sqrt{bc}$$
$$- abbc - aab\sqrt{bc}$$
$$0 \quad 0$$

$$a - \sqrt{bc}\ )\ aab - bbc - ab\sqrt{bc} + \frac{bbc}{a}\sqrt{bc}\ (\ ab - \frac{bbc}{a}$$
$$aab - ab\sqrt{bc}$$
$$- bbc + \frac{bbc}{a}\sqrt{bc}$$
$$- bbc + \frac{bbc}{a}\sqrt{bc}$$
$$0 \quad 0$$

EXPLICATION.

In the firſt Example; firſt, ab divided by a gives the Quotient b, by which I multiply the whole Diviſor $a + \sqrt{bc}$, and it makes $ab + b\sqrt{bc}$, this ſubtracted from the given Dividend $ab + b\sqrt{bc}$, there remains 0; ſo the Quotient ſought is b.

In the third Example; firſt, ab divided by $\sqrt{ab}$ gives the Quotient $\sqrt{ab}$, by which I multiply the whole Diviſor $\sqrt{ab} - \sqrt{cd}$ and the Product is $ab - \sqrt{abcd}$, this ſubtracted from the given Dividend $ab - cd$, there remains to be yet divided $- cd + \sqrt{abcd}$; then I divide $-cd$ by $-\sqrt{cd}$ and it gives the Quotient $+\sqrt{cd}$, by which I multiply the whole Diviſor $\sqrt{ab} - \sqrt{cd}$ and it makes $-cd + \sqrt{abcd}$, this ſubtracted from the remaining Dividend $- cd + \sqrt{abcd}$ leaves 0; ſo the Diviſion is finiſht, and the Quotient ſought is $\sqrt{ab} + \sqrt{cd}$.

In the ſixth and laſt Example; firſt, aab divided by a gives the Quotient ab, this multiplying the whole Diviſor $a - \sqrt{bc}$ produceth $aab - ab\sqrt{bc}$, which ſubtracted from the given Dividend leaves to be yet divided $-bbc + \frac{bbc}{a}\sqrt{bc}$; then I divide $+\frac{bbc}{a}\sqrt{bc}$ by $-\sqrt{bc}$ and it gives the Quotient $-\frac{bbc}{a}$, by which I multiply the whole Diviſor $a - \sqrt{bc}$ and it produceth $-bbc + \frac{bbc}{a}\sqrt{bc}$, which ſubtracted from the remaining Dividend $-bbc + \frac{bbc}{a}\sqrt{bc}$ leaves nothing; ſo the Quotient ſought is $ab - \frac{bbc}{a}$.

 The

The Arithmetick of Univerſal Surd Roots, both in Numbers and Quantities expreſt by Letters.

Sect. XII. *Multiplication in Univerſal Surds.*

Univerſal Roots are the Roots of Compound Numbers or Quantities; how to expreſs Univerſal Roots, and to find out their values, hath already been ſhewn in *Sect.* 28. *Chap.* 1. *Book* 1. I ſhall therefore proceed to their Multiplication.

1. If the ſquare Root of any Compound number be to be ſquared, or multiplied into it ſelf, caſt away the univerſal Radical ſign $\sqrt{}$ or $\sqrt{(2)}$, as alſo the Line that is drawn over the Compound number, and the Compound number it ſelf ſhall be the Square of the Univerſal Root propoſed. Alſo, the Cube of the cubick Root of any Compound number is the Compound number it ſelf, the Line drawn over it and the univerſal Radical ſign $\sqrt{(3)}$ being caſt away; and ſo of others.

As, for example, the Square of this univerſal ſquare Root, $\sqrt{:\overline{12+\sqrt{3}}:}$ is $12+\sqrt{3}$; likewiſe, the Square of $\sqrt{:\overline{12-\sqrt{3}}:}$ is $12-\sqrt{3}$; alſo, the Square of $\sqrt{:\overline{15+\sqrt{3}+\sqrt{2}}:}$ is $15+\sqrt{3}+\sqrt{2}$; and the Square of $\sqrt{:\overline{15-\sqrt{3}-\sqrt{2}}:}$ is $15-\sqrt{3}-\sqrt{2}$.

After the ſame manner, the Cube of this univerſal cubick Root, $\sqrt{(3)}:\overline{\sqrt{25}+\sqrt{9}}:$ is $\sqrt{25}+\sqrt{9}$, that is, 8.

Likewiſe, the Square of $\sqrt{:\overline{aa+bb}:}$ is $aa+bb$; and the Cube of $\sqrt{(3)}:\overline{bbb+ccc}:$ is $bbb+ccc$; alſo, the Square of $\sqrt{:\overline{\tfrac{1}{2}c+\sqrt{\tfrac{1}{4}cc-n}}:}$ is $\frac{1}{2}c+\sqrt{\frac{1}{4}cc-n}$: and ſo of others.

2. When an univerſal Root is to be multiplied by a rational Quantity, or by a ſimple or compound Surd, or by an univerſal Root; multiply the Square of the Multiplicand by the Square of the Multiplier, when the univerſal Radical ſign is Quadratick; or, the Cube of the one by the Cube of the other, when the univerſal Radical ſign is Cubick, *&c.* then before that Product prefix the given univerſal Radical ſign; ſo ſhall this new univerſal Root be the Product ſought.

As, for example, if it be deſired to double or multiply by 2, this univerſal ſquare Root, $\sqrt{:\overline{10+\sqrt{40}}:}$ I take the Square of 2, which is 4, and the Square of $\sqrt{:\overline{10+\sqrt{40}}:}$ which (by the foregoing firſt Rule of this *Sect.*) is $10+\sqrt{40}$; then I multiply $10+\sqrt{40}$ by 4, and it makes $40+4\sqrt{40}$, or, $40+\sqrt{640}$; whoſe univerſal ſquare Root, to wit, $\sqrt{:\overline{40+4\sqrt{40}}:}$ or, $\sqrt{:\overline{40+\sqrt{640}}:}$ is the Product of $\sqrt{:\overline{10+\sqrt{40}}:}$ multiplied by 2; or the ſaid Product may be expreſt thus, $2\sqrt{:\overline{10+\sqrt{40}}:}$

Likewiſe, if $\sqrt{(3)}:\overline{\sqrt{(3)}64+\sqrt{(3)}27}:$ be to be doubled, or multiplied by 2; I firſt multiply each of thoſe numbers cubically, becauſe the Radical ſign of the given univerſal Root is $\sqrt{(3)}$, and their Cubes will be $\sqrt{(3)}64+\sqrt{(3)}27$ and 8; which multiplied one into the other make $8\sqrt{(3)}64+8\sqrt{(3)}27$, to which Product I prefix the univerſal Radical ſign $\sqrt{(3)}$, and it gives $\sqrt{(3)}:\overline{8\sqrt{(3)}64+8\sqrt{(3)}27}:$ that is, $\sqrt{(3)}:\overline{32+24}:$ or $\sqrt{(3)}56$; which is the Product ſought, to wit, the double of $\sqrt{(3)}:\overline{\sqrt{(3)}64+\sqrt{(3)}27}:$

After the ſame manner, if $\sqrt{(3)}:\overline{\sqrt{(3)}64+\sqrt{(2)}36+3}:$ be to be multiplied by 5, or $\sqrt{(3)}125$, the Product will be $\sqrt{(3)}:\overline{125\sqrt{(3)}64+125\sqrt{(2)}36+375}:$ that is, $\sqrt{(3)}1625$.

Again, to multiply $\sqrt{:\overline{\sqrt{10}+\sqrt{3}}:}$ by $\sqrt{5}$; their Squares are $\sqrt{10}+\sqrt{3}$ and 5, which multiplied one into another make $5\sqrt{10}+5\sqrt{3}$, (that is, $\sqrt{250}+\sqrt{75}$;) whoſe univerſal ſquare Root, to wit, $\sqrt{:\overline{5\sqrt{10}+5\sqrt{3}}:}$ (or, $\sqrt{:\overline{\sqrt{250}+\sqrt{75}}:}$) is the Product of $\sqrt{:\overline{\sqrt{10}+\sqrt{3}}:}$ multiplied by $\sqrt{5}$.

Likewiſe, to multiply $\sqrt{:\overline{13+\sqrt{9}}:}$ by $\sqrt{:\overline{5+\sqrt{16}}:}$ (that is, 4 by 3, where the Product is manifeſtly 12;) the Squares of the univerſal Roots propoſed are $13+\sqrt{9}$ and $5+\sqrt{16}$, which multiplied one into another make $65+5\sqrt{9}+13\sqrt{16}+\sqrt{144}$; whoſe univerſal ſquare Root, to wit, $\sqrt{:\overline{65+5\sqrt{9}+13\sqrt{16}+\sqrt{144}}:}$ that is, $\sqrt{144}$, or 12 is the Product ſought.

Again, to multiply $\sqrt{:\overline{\tfrac{7}{2}+\sqrt{\tfrac{29}{4}}}:}$ into $\sqrt{:\overline{\tfrac{7}{2}-\sqrt{\tfrac{29}{4}}}:}$ I multiply their Squares $\frac{7}{2}+\sqrt{\frac{29}{4}}$ and $\frac{7}{2}-\sqrt{\frac{29}{4}}$ one into another according to the laſt of the three compendious Rules in *Sect.* 10. of this *Chapt.* and there comes forth $\frac{49}{4}-\frac{29}{4}$, that is, 5; (to wit, the

difference

difference between the Squares of $\frac{7}{2}$ and $\sqrt{\frac{29}{4}}$:) laſtly, the ſquare Root of the ſaid 5 is $\sqrt{5}$ for the Product ſought.

So alſo, to multiply $\sqrt{:}\,5+\sqrt{2}\,:$ by $\sqrt{5}+\sqrt{2}$; their Squares $5+\sqrt{2}$ and $7+2\sqrt{10}$ multiplied one into another give $35+10\sqrt{10}+7\sqrt{2}+2\sqrt{20}$; whoſe univerſal ſquare Root, to wit, $\sqrt{:}\,35+10\sqrt{10}+7\sqrt{2}+2\sqrt{20}\,:$ is the Product ſought.

Moreover, to multiply $\sqrt{:}\,\sqrt{144}+4\,:-\sqrt{:}\,\sqrt{4}+2\,:$ by $\sqrt{:}\,\sqrt{100}-1\,:$ (that is, 2 by 3, which will produce 6;) I firſt multiply the Square of $\sqrt{:}\,\sqrt{144}+4\,:$ by the Square of $\sqrt{:}\,\sqrt{100}-1\,:$ *viz.* $\sqrt{144}+4$ by $\sqrt{100}-1$ and it makes $\sqrt{14400}+4\sqrt{100}-\sqrt{144}-4$, before which I prefix the univerſal Radical ſign $\sqrt{}$, and it gives $\sqrt{:}\,\sqrt{14400}+4\sqrt{100}-\sqrt{144}-4\,:$ which is one of the members of the Product ſought; then I multiply in like manner $-\sqrt{:}\,\sqrt{4}+2\,:$ by $\sqrt{:}\,\sqrt{100}-1\,:$ and it makes $-\sqrt{:}\,\sqrt{400}+2\sqrt{100}-\sqrt{4}-2\,:$ for the latter member of the Product ſought; laſtly, both thoſe members being joyned together give $\sqrt{:}\,\sqrt{14400}+4\sqrt{100}-\sqrt{144}-4\,:-\sqrt{:}\,\sqrt{400}+2\sqrt{100}-\sqrt{4}-2\,:$ that is, $\sqrt{144}-\sqrt{36}$; that is, $12-6$, or 6, for the Product required.

3. Sometimes the fourth, fifth and ſixth Rules in *Sect.* 4. of this *Chapt.* will be uſeful in the Multiplication of univerſal Surds: As, if it be deſired to multiply $3\sqrt{:}\,2+\sqrt{5}$ by $4\sqrt{:}\,2+\sqrt{5}\,:$ (which are commenſurable Roots, for they are in proportion one to the other as 3 to 4,) I multiply 3 by 4, and the Product 12 into $2+\sqrt{5}$; ſo there is produced $24+12\sqrt{5}$ (that is, $24+\sqrt{720}$) for the Product ſought.

Likewiſe, $5\sqrt{:}\,6+\sqrt{9}\,:$ multiplied by $2\sqrt{:}\,6+\sqrt{9}\,:$ (that is, 15 by 6,) produceth $60+10\sqrt{9}$, (that is, 90.)

Moreover, if $5\sqrt{:}\,6+\sqrt{9}\,:$ be to be multiplied by $3\sqrt{:}\,19-\sqrt{9}\,:$ (that is, 15 by 12;) I firſt multiply 5 by 3 and it makes 15; then I multiply $\sqrt{:}\,6+\sqrt{9}\,:$ by $\sqrt{:}\,19-\sqrt{9}\,:$ and it produceth $\sqrt{:}\,105+13\sqrt{9}\,:$ which latter Product multiplied into the former Product 15 makes $15\sqrt{:}\,105+13\sqrt{9}\,:$ (that is, 180;) the Product ſought.

4. Sometimes alſo the three Rules before delivered in *Sect.* 10. of this *Chapt.* concerning the multiplying of Binomials and Reſiduals will be uſeful in the Multiplication of Univerſal ſurd Roots: As, if this Binomial Root $\sqrt{:}\,12+\sqrt{6}\,:+\sqrt{:}\,12-\sqrt{6}\,:$ be to be ſquared, or multiplied into it ſelf, the Squares of the Parts are $12+\sqrt{6}$ and $12-\sqrt{6}$, whoſe ſumm is 24: then the Product made by the multiplication of the Parts one into the other, *viz.* $\sqrt{:}\,12+\sqrt{6}\,:$ into $\sqrt{:}\,12-\sqrt{6}\,:$ is $\sqrt{138}$, (for the difference of the Squares of 12 and $\sqrt{6}$ is 138, whoſe ſquare Root is $\sqrt{138}$;) and the double of the ſaid Product is $2\sqrt{138}$, which added to 24 (the ſumm of the Squares of the Parts) makes $24+2\sqrt{138}$, which is the Square of $\sqrt{:}\,12+\sqrt{6}\,:+\sqrt{:}\,12-\sqrt{6}$. Moreover, the ſquare Root of the ſaid $24+2\sqrt{138}$, to wit, $\sqrt{:}\,24+2\sqrt{138}\,:$ is the ſumm of the two Parts $\sqrt{:}\,12+\sqrt{6}\,:$ and $\sqrt{:}\,12-\sqrt{6}\,:$ For when the ſumm of two numbers is multiplied into it ſelf, the ſquare Root of the Product is equal to the ſame ſumm.

Likewiſe if $\sqrt{:}\,10+\sqrt{36}\,:-\sqrt{:}\,10-\sqrt{36}$, that is, 2, be to be ſquared, or multiplied into it ſelf, the Product will be found $20-2\sqrt{64}$, that is, 4; and the ſquare Root of this 4; to wit, 2 is the difference of the two Roots or Parts $\sqrt{:}\,10+\sqrt{36}\,:$ and $\sqrt{:}\,10-\sqrt{36}$. For when the difference of two numbers is multiplied into it ſelf, the ſquare Root of the Product is equal to the ſaid difference.

Again, if $6+\sqrt{:}\,20-\sqrt{16}\,:$ be to be multiplied into $6-\sqrt{:}\,20-\sqrt{16}\,:$ the Product will be found 20. For (according to Rule 3. in *Sect.* 10. of this Chapter,) if $20-\sqrt{16}$, which is the Square of $\sqrt{:}\,20-\sqrt{16}\,:$ be ſubtracted from 36 the Square of 6, there will remain $16+\sqrt{16}$, that is, 20, the Product ſought.

Likewiſe, if $\sqrt{20}+\sqrt{:}\,20-\sqrt{5}\,:$ be to be multiplied into $\sqrt{20}-\sqrt{:}\,20-\sqrt{5}\,:$ the Product will be $\sqrt{5}$.

So alſo, if $\sqrt{:}\,5+\sqrt{:}\,20-\sqrt{16}\,:$ be to be multiplied by $\sqrt{:}\,5-\sqrt{:}\,20-\sqrt{16}\,:$ (that is, 3 by 1;) firſt, the Squares of the univerſal Roots propoſed are $5+\sqrt{:}\,20-\sqrt{16}\,:$ and $5-\sqrt{:}\,20-\sqrt{16}\,:$ theſe multiplied one by the other, by taking the difference of the Squares of 5 and $\sqrt{:}\,20-\sqrt{16}\,:$ give the Product $5+\sqrt{16}$, whoſe univerſal

ſquare Root, to wit, $\surd:\overline{5+\surd 16}:$ that is, 3, is the Product of the two univerſal ſquare Roots propos'd to be multiplied.

5. The four preceding Rules of this Section are alſo to be obſerved in the multiplication of univerſal ſurd Roots expreſt by Letters: As, if it be deſired to multiply $\surd:\overline{aa+bb}:$ by a, I multiply their Squares $aa+bb$ and aa one into the other, and there comes forth $aaaa+aabb$, whoſe univerſal ſquare Root $\surd:\overline{aaaa+aabb}:$ is the Product ſought; which may more compendiouſly be expreſt thus, $a\surd:\overline{aa+bb}:$

Likewiſe, to multiply $\surd:\overline{oo+4mp}:$ into $\frac{z}{a}$, I write $\surd\frac{oozz+4mpzz}{aa}$, or $\frac{z}{a}\surd:\overline{oo+4mp}:$ for the Product.

Again, if $\surd:\overline{aa+12}:$ be to be multiplied by $a+3$, the Product may be ſignified by $\overline{a+3}$ into $\surd:\overline{aa+12}:$ Or, after the Squares of the quantities propoſed are multiplied one into the other, and the univerſal Radical ſign prefixed, the product may be expreſt thus, $\surd:\overline{aaaa+6aaa+21aa+72a+108}:$

So alſo, $\surd bc$ multiplied into $\surd:\overline{aa+bb}:$ produceth $\surd:\overline{aabc+bbbc}:$ and $\surd:\overline{\surd bc+\surd a}:$ muliplied by $\surd:\overline{\surd ba-\surd bc}:$ produceth $\surd:\overline{b\surd ca+a\surd b-bc-\surd abc}:$ that is, $\surd:\overline{\surd bbca+\surd aab-bc-\surd abc}:$

Again, after the manner of the preceding third Rule of this Section, $a\surd:\overline{bb-cc}:$ multiplied by $d\surd:\overline{bb-cc}:$ produceth $adbb-adcc$.

And $a\surd:\overline{b+c}:$ into $d\surd:\overline{b-c}:$ produceth $ad\surd:\overline{bb-cc}:$

Moreover, if this Binomial Root $\surd:\overline{\surd a+\surd bc}:+\surd:\overline{\surd a-\surd bc}:$ be to be ſquared or multiplied into it ſelf; firſt, the Squares of the Names or Parts of the Binomial are $\surd a+\surd bc$ and $\surd a-\surd bc$, which added together make $2\surd a$; then the double Product of the Parts is $2\surd:\overline{a-bc}:$ (for the difference of the Squares of $\surd a$ and $\surd bc$ is $a-bc$, whoſe univerſal ſquare Root doubled is $2\surd:\overline{a-bc}:$) which double Product added to $2\surd a$, (to wit, the ſumm of the Squares of the Parts firſt found,) makes $2\surd a+2\surd:\overline{a-bc}:$ which is the Square or Product deſired; and if the ſquare Root of this Product be extracted, it gives $\surd:\overline{2\surd a+2\surd:a-bc}:$ which is equal to the ſumm of the Parts of the Binomial Roots firſt propoſed to be ſquared.

Sect. XIII. *Diviſion in Univerſal Surds.*

1. Divide the Square of the Dividend by the Square of the Diviſor, when the univerſal Radical ſign is quadratick; or the Cube of the one by the Cube of the other, when the univerſal Radical ſign is cubick, *&c.* then prefix the given univerſal Sign to the Quotient, ſo ſhall this new Root be the Quotient ſought.

As, for example, if it be deſired to divide $\surd:\overline{40+4\surd 40}:$ by 2; I divide $40+4\surd 40$, which is the Square of the Dividend, by 4, the Square of the Diviſor; (according to *Sect.* 11. of this *Chapt.*) and there ariſeth $10+\surd 40$, whoſe ſquare Root univerſal, to wit, $\surd:\overline{10+\surd 40}:$ is the Quotient ſought.

Again, if it be deſired to divide $\surd:\overline{40+4\surd 40}:$ by $\surd:\overline{10+\surd 40}:$ firſt, I take their Squares, to wit, $40+4\surd 40$ and $10+\surd 40$ as a Dividend and Diviſor, then becauſe the Diviſor is a Compound number, a new Dividend and Diviſor muſt be found out, ſuch that the new Diviſor may be a Rational number; ſo (according to the Rule in the ſixth branch of *Sect.* 11. of this *Chapt.*) there will be produced 240 and 60 for a new Dividend and Diviſor, which give the Quotient 4, whoſe ſquare Root is 2 the Quotient ſought, to wit, the Quotient of $\surd:\overline{40+4\surd 40}:$ divided by $\surd:\overline{10+\surd 40}:$

Likewiſe, to divide 20 by $\surd:\overline{10-\surd 5}:$ firſt, I reduce their Squares 400 and $10-\surd 5$ to a new Dividend and Diviſor, to wit, $4000+400\surd 5$ and 95; then I divide $4000+400\surd 5$ by 95, and there ariſeth $42\frac{2}{19}+\frac{80}{19}\surd 5$, whoſe univerſal ſquare Root, to wit, $\surd:\overline{42\frac{2}{19}+\frac{80}{19}\surd 5}:$ is the Quotient ſought.

Another Example (in Rational numbers expreſt Surd-wiſe) may be this, *viz.* ſuppoſe it be deſired to divide $\surd:\overline{4+\surd 25}:$ by $\surd:\overline{1+\surd 9}:$ (that is, 3 by 2, which gives the Quotient $1\frac{1}{2}$;) firſt, I reduce $4+\surd 25$ and $1+\surd 9$ the Squares of the given Dividend and

and Divisor, to a new Dividend and Divisor, to wit, $4+\surd 25-4\surd 9-\surd 225$ and -8; these give the Quotient $\frac{9}{4}$, (as hath been proved in the latter Example of the Note in the preceding *Sect.* 11.) the square Root whereof, to wit, $\frac{3}{2}$, is the Quotient sought; for if the given Divisor $\surd\!:\overline{1+\surd 9}:$ be mutiplied by the Quotient $\frac{3}{2}$ it will produce 3, which is equal to the given Dividend $\surd\!:\overline{4+\surd 25}:$

Again, to divide $\surd(3)\!:\overline{8\surd(3)64+8\surd(2)27}:$ by 2; I divide the Cube of the one by the Cube of the other, *viz.* $8\surd(3)64+8\surd(2)27$ by 8, and there ariseth $\surd(3)64+\surd(2)27$; whose universal cubick Root, to wit, $\surd(3)\!:\overline{\surd(3)64+\surd(2)27}:$ is the Quotient sought, to wit, the half of the Dividend proposed.

2. If the given universal Roots, to wit, the Dividend and Divisor be commensurable, then (according to the fifth Rule of *Sect.* 5. of this *Chapt.*) divide the Rational part of the Dividend by the Rational part of the Divisor, and what ariseth is the Quotient sought: As, to divide $21\surd\!:\overline{6+\surd 9}:$ by $3\surd\!:\overline{6+\surd 9}:$ I divide 21 by 3, and there ariseth 7 for the Quotient sought.

Likewise, $183\surd\!:\overline{\surd 3-\surd 2}:$ divided by $\frac{61}{8}\surd\!:\overline{\surd 3-\surd 2}:$ gives the Quotient 24.

3. Division in universal Surds exprest by Letters depends upon the Rules before given: As, to divide $\surd\!:\overline{aaaa+aabb}:$ by a; I divide the Square of the Dividend by the Square of the Divisor, *viz.* $aaaa+aabb$ by aa, and there ariseth $aa+bb$, whose square Root universal, to wit, $\surd\!:\overline{aa+bb}:$ is the Quotient sought.

Again, if it be desired to divide $\surd\!:\overline{\surd bbca+\surd aab-bc-\surd abc}:$ by $\surd\!:\overline{\surd bc+\surd a}:$ I divide the Square of the Dividend by the Square of the Divisor, *viz.* $\surd bbca+\surd aab-bc-\surd abc$ by $\surd bc+\surd a$, (according to the method in the Examples at the latter end of *Sect.* 11. of this *Chapt.*) and there ariseth $\surd ba-\surd bc$, whose universal square Root, to wit, $\surd\!:\overline{\surd ba-\surd bc}:$ is the Quotient sought.

Moreover, to divide $d\surd\!:\overline{bb+cc}:$ by $3a\surd\!:\overline{bb+cc}:$ because they are commensurable, I divide only the Rational part by the Rational, and there ariseth $\frac{d}{3a}$ for the Quotient.

4. Lastly, when the work of Division in universal Surds according to the foregoing Rules and Examples in this Section, happens to be intricate, or will not work off just without a Remainder, you may set the Power of the Dividend (the universal Radical sign being omitted) as a Numerator, over the Power of the Divisor as a Denominator, and prefix the universal Radical sign before the line that seperates the Numerator from the Denominator; then shall the universal Root so denoted signifie the Quotient sought.

As, if it be desired to divide $\surd\!:\overline{\surd 5+\surd 8-3}:$ by $\surd\!:\overline{\surd 7-\surd 2+1}:$ the Quotient may be represented by this Fraction, $\surd\!:\frac{\surd 5+\surd 8-3}{\surd 7-\surd 2+1}:$

Likewise, if $\surd\!:\overline{\surd abb+bcd}:$ be to be divided by $\surd\!:\overline{\surd ac-dd}:$ you may write $\surd\!:\frac{\surd abb+bcd}{\surd ac-dd}:$ to signifie the Quotient.

Sect. XIV. *Addition and Subtraction in Universal Surds.*

1. When two universal Surds proposed to be added or subtracted are Commensurable, they may be added or subtracted like simple Surds; (according to the Rule in *Sect.* 8. of this *Chapt.*) As, for example, if the Summ and Difference of $\surd\!:\overline{8+4\surd 3}:$ and $\surd\!:\overline{2+\surd 3}:$ be desired; because each of them divided by their common Divisor $\surd\!:\overline{2+\surd 3}:$ gives $\surd 4$ and $\surd 1$, that is, 2 and 1, which are Rational numbers expressing the Proportion of the Surds proposed. Therefore the summ of 2 and 1, to wit, 3 multiplied into the said common Divisor gives $3\surd\!:\overline{2+\surd 3}:$ for the Summ required, (which may also be exprest thus, $\surd\!:\overline{18+\surd 243}:$) and the difference of the said 2 and 1, to wit, 1 multiplied into the said common Divisor $\surd\!:\overline{2+\surd 3}:$ makes $\surd\!:\overline{2+\surd 3}:$ for the Difference of the two Roots first proposed.

Another Example in Rational numbers exprest Surd-wise, *viz.* let it be required to find out the Summ and Difference of $\surd\!:\overline{99+9\surd 25}:$ and $\surd\!:\overline{44+4\surd 25}:$ (that is, 12 and 8;) First, those universal Roots being severally divided by the common Divisor

$\surd\!:\overline{11+\surd 25}:$

$\sqrt{:11+\sqrt{25}:}$ give the Quotients $\sqrt{9}$ and $\sqrt{4}$, to wit, 3 and 2, which are Rational numbers expressing the Proportion which the given Roots have one to another. Therefore, $3+2$, to wit, 5, multiplied into the common Divisor $\sqrt{:11+\sqrt{25}:}$ gives $5\sqrt{:11+\sqrt{25}:}$ that is, $\sqrt{:275+\sqrt{15625}:}$ (to wit, 20,) which is the Summ of the Roots proposed; and $3-2$, that is, 1, multiplied into the said $\sqrt{:11+\sqrt{25}:}$ gives $\sqrt{:11+\sqrt{25}:}$ (that is, 4,) for the Difference of the given Roots.

Here follow Contractions of the work of Addition and Subtraction in the two last Examples, with others of like nature in Surd quantities exprest by Letters.

Example 1.

What is the Summ and Difference of . . $\sqrt{:8+4\sqrt{3}:}$ and $\sqrt{:2+\sqrt{3}:}$?

The Operation.

I. $\sqrt{:2+\sqrt{3}:}$) $\sqrt{:8+4\sqrt{3}:}$ ($\sqrt{4}$, that is, 2.
II. $\sqrt{:2+\sqrt{3}:}$) $\sqrt{:2+\sqrt{3}:}$ ($\sqrt{1}$, that is, 1.

Therefore from I. $2\sqrt{:2+\sqrt{3}:} = \sqrt{:8+4\sqrt{3}:}$
And from II. . $1\sqrt{:2+\sqrt{3}:} = \sqrt{:2+\sqrt{3}:}$

The Summ, $3\sqrt{:2+\sqrt{3}:} = \sqrt{:8+4\sqrt{3}:} + \sqrt{:2+\sqrt{3}:}$
The Difference, $1\sqrt{:2+\sqrt{3}:} = \sqrt{:8+4\sqrt{3}:} - \sqrt{:2+\sqrt{3}:}$

Example 2.

What is the Summ and Difference of $\sqrt{:99+9\sqrt{25}:}$ and $\sqrt{:44+4\sqrt{25}:}$?

The Operation.

I. $\sqrt{:11+\sqrt{25}:}$) $\sqrt{:99+9\sqrt{25}:}$ ($\sqrt{9}$, that is, 3.
II. $\sqrt{:11+\sqrt{25}:}$) $\sqrt{:44+4\sqrt{25}:}$ ($\sqrt{4}$, that is, 2.

Therefore from I. $3\sqrt{:11+\sqrt{25}:} = \sqrt{:99+9\sqrt{25}:}$
And from II. . $2\sqrt{:11+\sqrt{25}:} = \sqrt{:44+4\sqrt{25}:}$

The Summ, $5\sqrt{:11+\sqrt{25}:} = \sqrt{:99+9\sqrt{25}:} + \sqrt{:44+4\sqrt{25}:}$
The Difference, $1\sqrt{:11+\sqrt{25}:} = \sqrt{:99+9\sqrt{25}:} - \sqrt{:44+4\sqrt{25}:}$

Example 3.

What is the Summ and Difference of $\sqrt{:aaaa+aabb:}$ and $\sqrt{:aabb+bbbb:}$?
Those reduced (by *Sect.6.* of this *Chapt.*) give $a\sqrt{:aa+bb:}$ and $b\sqrt{:aa+bb:}$
Therefore their Summ is . . . . $\overline{a+b}$ into $\sqrt{:aa+bb:}$
And their Difference is . . . . . $\overline{a \backsim b}$ into $\sqrt{:aa+bb:}$

Example 4.

What is the Summ and Difference of $\sqrt{\dfrac{oozz+4mpzz}{aa}}$ and $\sqrt{\dfrac{aaoomm+4aammmp}{ppzz}}$

By dividing each of them by their common Divisor $\sqrt{:oo+4mp:}$ there will arise Rational quotients, to wit, ... $\dfrac{z}{a}$ and $\dfrac{am}{pz}$

Therefore the Surds proposed are Commmensurable, and instead of them we may write . . . . $\dfrac{z}{a}\sqrt{:oo+4mp:}$ and $\dfrac{am}{pz}\sqrt{:oo+4mp:}$

Therefore their Summ shall be . $\dfrac{z}{a}+\dfrac{am}{pz}$ into $\sqrt{:oo+4mp:}$

That is, ... $\dfrac{pzz+aam}{apz}$ into $\sqrt{:oo+4mp:}$

And

And the Difference of the given Surds ſhall be $\left\{ \dfrac{pzz \backsim aam}{apz} \right.$ into $\sqrt{:\,oo + 4mp\,:}$

Example 5.

What is the Summ and Difference of theſe two Univerſal Roots?

$$\sqrt{:\,aaaa + 6aaa + 21aa + 72a + 108\,:} \quad \text{and,}$$
$$\sqrt{:\,aaaa + 10aaa + 37aa - 120a + 300\,:}$$

The Operation.

The given Roots are Commenſurable, (as hath been ſhewn in the laſt Example but one in *Sect.* 7. of this *Chapt.*) and may be expreſt thus,

$$a + 3\sqrt{:\,aa + 12\,:} \quad \text{and} \quad a \backsim 5\sqrt{:\,aa + 12\,:}$$

Therefore their Summ, ſuppoſing a to be greater than 5, ſhall be

$$2a - 2 \text{ into } \sqrt{:\,aa + 12\,:}$$

And their Difference ſhall be, $8\sqrt{:\,aa + 12\,:}$

But if we ſuppoſe a to be leſs than 5, then the Summ of the given Surds will be $8\sqrt{:\,aa+12\,:}$ and their Difference $2a \backsim 2\sqrt{:\,aa+12\,:}$ that is, $2a \backsim 2$ into $\sqrt{:\,aa+12\,:}$

2. When the Root of a Reſidual is to be added unto, or ſubtracted from the Root of its correſpondent Binomial, thoſe Roots may be connected together by $+$ or $-$; and then the whole being multiplied into it ſelf, the univerſal Root of the Product ſhall be the Summ or Difference of the Roots given to be added or ſubtracted, as before hath been ſhewn in Rule 4. *Sect.* 12. of this *Chapt.*

As, if theſe two Roots be propoſed to be added, to wit, $\sqrt{:\,12+\sqrt{6}\,:}$ and $\sqrt{:\,12-\sqrt{6}\,:}$ we may multiply this compoſed number $\sqrt{:\,12+\sqrt{6}\,:} + \sqrt{:\,12-\sqrt{6}\,:}$ into it ſelf, and there will be produced $24 + 2\sqrt{138}$; whoſe univerſal ſquare Root, to wit, $\sqrt{:\,24 + 2\sqrt{138}\,:}$ ſhall be the ſumm of the two Roots propoſed to be added.

Likewiſe, if $\sqrt{:\,12+\sqrt{6}\,:} - \sqrt{:\,12-\sqrt{6}\,:}$ be multiplied into it ſelf, the Product will be $24 - 2\sqrt{138}$, whoſe univerſal ſquare Root, to wit, $\sqrt{:\,24 - 2\sqrt{138}\,:}$ is the difference of the two Roots propoſed.

After the ſame manner, the ſumm of theſe two Roots, $\sqrt{:\,10+\sqrt{36}\,:}$ and $\sqrt{:\,10-\sqrt{36}\,:}$ will be found $\sqrt{:\,20+2\sqrt{64}\,:}$ (that is, $\sqrt{36}$, to wit, 6;) but their difference $\sqrt{:\,20-2\sqrt{64}\,:}$ (that is, $\sqrt{4}$, to wit, 2.)

Likewiſe the ſumm of theſe Binomial Roots, $\sqrt{:\,\sqrt{a}+\sqrt{bc}\,:}$ and $\sqrt{:\,\sqrt{a}-\sqrt{bc}\,:}$ will be found $\sqrt{:\,2\sqrt{a} + 2\sqrt{:\,a-bc\,:}}$ and their difference $\sqrt{:\,2\sqrt{a} - 2\sqrt{:\,a-bc\,:}}$

3. But if the univerſal Roots propoſed be not commenſurable, nor ſuch Binomials and Reſiduals as are mentioned in the laſt preceding Rule, then they are to be added by $+$, and ſubtracted by $-$.

As, if $\sqrt{:\,5+\sqrt{2}\,:}$ and $\sqrt{:\,5-\sqrt{3}\,:}$ be to be added, I write $\sqrt{:\,5+\sqrt{2}\,:} + \sqrt{:\,5-\sqrt{3}\,:}$ for the Summ; and to ſubtract $\sqrt{:\,5-\sqrt{3}\,:}$ from $\sqrt{:\,5+\sqrt{2}\,:}$ I write $\sqrt{:\,5+\sqrt{2}\,:} - \sqrt{:\,5-\sqrt{3}\,:}$ for the Remainder.

Likewiſe the Summ of $\sqrt{:\,aa+bb\,:}$ and $\sqrt{:\,aa-cc\,:}$ is $\sqrt{:\,aa+bb\,:} + \sqrt{:\,aa-cc\,:}$ and their Difference is $\sqrt{:\,aa+bb\,:} - \sqrt{:\,aa-cc\,:}$

Sect. XV. *Concerning the Conſtitution and Invention of ſix Binomials in Numbers, agreeable to thoſe expounded in* Prop. 49, 50, 51, 52, 53, 54, Elem. 10. *Euclid.*

By way of preparation to the Conſtruction of the ſix Binomials in Numbers, I ſhall premiſe this

QUESTION.

To find two Square numbers whoſe difference may be equal to a given Rational number?

CANON.

Take any two numbers, which multiplied one by the other will produce the given number;

number; then half the summ of those two numbers and half their difference shall be the Sides or Roots of the two Squares sought.

As, if 5 be given for the difference of two Squares sought, I take 5 and 1; for the Product of their multiplication is 5; then the half of their summ is 3, and the half of their difference is 2; lastly, the Squares of the said 3 and 2 are 9 and 4, the Squares sought: for their difference is 5, as was prescribed.

Again, the same number 5 being given for the difference of two Squares, take a number at pleasure, as 2, by this divide the given number 5, the Quotient is $\frac{5}{2}$, therefore the Product of the multiplication of the Divisor 2 by the Quotient $\frac{5}{2}$ is 5; then according to the Canon, half the summ and half the difference of the said 2 and $\frac{5}{2}$, to wit, $\frac{9}{4}$ and $\frac{1}{4}$ shall be the sides of the Squares sought; and consequently the Squares themselves are $\frac{81}{16}$ and $\frac{1}{16}$, whose difference is 5, as was desired.

After the same manner innumerable pairs of Squares may be found out in Rational numbers, and the difference of each pair shall be equal to one and the same given number.

The reason of the Canon may be made manifest by this

Theorem.

The Product made by the multiplication of any two unequal numbers is equal to the difference of two Squares, to wit, of the Square of half the summ, and the Square of half the difference of the same two unequal numbers.

As, if c be the greater, and b the lesser of two numbers, then

The Square of $\frac{1}{2}c + \frac{1}{2}b$ is . . . . . $\frac{1}{4}cc + \frac{1}{2}cb + \frac{1}{4}bb$,
The Square of $\frac{1}{2}c - \frac{1}{2}b$ is . . . . . $\frac{1}{4}cc - \frac{1}{2}cb + \frac{1}{4}bb$;
The difference of those two Squares is . . . . . cb.

Which difference is manifestly the Product of the multiplication of the two proposed numbers c and b. Wherefore the Theorem, and consequently the Canon first given is manifest.

The Definition of Binomial I.

When the greater Name (or part) of a Binomial is a Rational number, and the lesser part is a Surd square Root of some Rational number, and the square Root of the difference of the Squares of the parts is a Rational number, the summ of the two parts is called a First Binomial.

Explication.

Let this Binomial be proposed, ...	$3 + \sqrt{5}$
The Squares of the Names, or parts, are	$\left\{ \begin{matrix} 9 \\ 5 \end{matrix} \right.$
The difference of those Squares is	4
The square Root of that difference is	2

Because the greater part 3 is a Rational number, and the lesser part $\sqrt{5}$ is a Surd square Root of a Rational number 5, and the difference of the Squares of the parts, *viz.* 4, is a Square whose Root 2 is a Rational number; the Binomial proposed, to wit, $3 + \sqrt{5}$ is called a First Binomial.

How to find out two such numbers as may constitute a First Binomial.

1. By the Canon of the preceding Question at the beginning of this 15. *Sect.* find out two Square numbers whose difference may be some Rational number not a Square, such are these Squares, ...	9 4
2. Their difference is	5
3. Take some Rational number at pleasure for the greater part of the Binomial sought, as	6
4. Then say, by the Rule of Three, If 9 the greater of the two Squares found out in the first step, give 5 the difference in the second, what shall 36 the Square of the number taken in the third give? whence the fourth Proportional will be found 20, the square Root whereof is the lesser part, to wit, ...	$\sqrt{20}$
5. I say, the summ of the two numbers found out in the third and fourth steps, is a first Binomial, to wit, ...	$6 + \sqrt{20}$

The

The Definition of Binomial II.

When the lesser part of a Binomial is a Rational number, and the greater part is a Surd square Root of a Rational number, and the square Root of the Difference of the Squares of the parts is Commensurable to the greater part; the summ of the two Parts is called a Second Binomial.

Explication.

Let this Binomial be proposed, ... $\sqrt{18} + 4$

The Squares of the Parts are . . . . $\left\{\begin{matrix} 18 \\ 16 \end{matrix}\right.$

The Difference of those Squares is 2

The square Root of that Difference is . . . . . . $\sqrt{2}$

Because the lesser Part 4 is a Rational number, and the greater Part $\sqrt{18}$ is the Surd square Root of a Rational number 18, and the square Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{2}$, is Commensurable to the greater Part $\sqrt{18}$; (for according to the Definition in *Sect.* 7. of this *Chapt.* $\sqrt{2} . \sqrt{18} :: 1 . 3$, that is, as a Rational number to a Rational number;) the proposed number $\sqrt{18} + 4$ is a Second Binomial.

How to find out two such numbers as may constitute a Second Binomial.

1. By the foregoing Canon find out two square numbers whose Difference may be some Rational number not a Square; such are these Squares, ...	9 4
2. Their Difference is	5
3. Take some Rational number at pleasure for the lesser Part of the Binomial sought, as, ...	10
4. Then say, If 5 the Difference in the third step, gives 9 the greater of the two Squares in the first; what shall 100 the Square of the number taken in the third give? whence you will find 180, whose square Root shall be the greater Part, *viz.*	$\sqrt{180}$
5. I say the summ of the two numbers found out in the third and fourth steps is a Second Binomial, *viz.*	$\sqrt{180} + 10$

The Definition of Binomial III.

When each of the two Parts of a Binomial is a Surd square Root of a Rational number, and the square Root of the Difference of the Squares of the Parts is Commensurable to the greater Part; the summ of the two Parts is called a Third Binomial.

Explication.

Let this Binomial be proposed, ... $\sqrt{50} + \sqrt{32}$

The Squares of the Parts are . . . . $\left\{\begin{matrix} 50 \\ 32 \end{matrix}\right.$

The Difference of those Squares is 18

The square Root of that Difference is . . . . . $\sqrt{18}$

Because the two Parts $\sqrt{50}$ and $\sqrt{32}$ are Surd square Roots of two Rational numbers 50 and 32, and the square Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{18}$, is Commensurable to the greater Part $\sqrt{50}$; (for $\sqrt{18} . \sqrt{50} :: 3 . 5$; that is, as a Rational number to a Rational number;) the proposed number $\sqrt{50} + \sqrt{32}$ is a Third Binomial.

How to find out two such numbers as may constitute a Third Binomial.

1. Find out two Square numbers whose Difference may be some Rational number not a Square; such are these Squares, ...	9 4
2. Their Difference is	5
3. Take some Rational number not a Square, which may exceed the said Difference 5 by an Unit or two, *viz.* by 1, when the said Difference increased with 1 makes not a Square; but by 2, when the Difference increased with 1 makes a Square: so in this Example, I take 6, because 5 + 1 makes not a Square, ...	6
4. Again, take some Rational number at pleasure, as	12

5. The

5. The Square thereof is 144

6. Then say, If 6 the number taken in the third step, gives 9 the greater of the two Squares in the first; what shall 144 the square number in the fifth give? whence the fourth Proportional is 216, whose square Root, to wit $\sqrt{216}$ shall be the greater Part; . . $\sqrt{216}$

7. Say again, If the said Square 9 gives 5 the Difference in the second step; what shall 216 the fourth Proportional found out in the sixth give? whence you will find 120, whose square Root, to wit, $\sqrt{120}$ shall be the lesser Part; . . . . $\sqrt{120}$

8. I say, the summ of the two numbers found out in the sixth and seventh steps is a third Binomial, to wit, ... $\sqrt{216} + \sqrt{120}$

The Definition of Binomial IV.

When the greater Part of a Binomial is a Rational number, and the lesser Part is a Surd square Root of a Rational number, and the square Root of the Difference of the Squares of the Parts is Incommensurable to the greater Part; the summ of the two Parts is called a Fourth Binomial.

Explication.

Let this Binomial be proposed, ... $5 + \sqrt{12}$

The Squares of the Parts are 25, 12

The Difference of those Squares is 13

The square Root of that Difference is . . . . . $\sqrt{13}$

Because the greater Part 5 is a Rational number, and the lesser Part $\sqrt{12}$ is a Surd square Root of a Rational number 12; and the square Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{13}$, is Incommensurable to the greater Part 5; (for $\sqrt{13}$ hath not such proportion to 5 as a Rational number to a Rational number;) the number $5 + \sqrt{12}$ above proposed is a Fourth Binomial.

How to find out two such numbers as may constitute a Fourth Binomial.

1. Take any square number, as 9
2. Divide that square number 9 into two numbers not Squares, as into 6 and 3
3. Take a Rational number at pleasure for the greater Part of the Binomial sought, as 6
4. Then say, If 9 the square number in the first step, give 6 the greater of the two numbers in the second; what shall 36 the Square of the number taken in the third give? so the fourth Proportional will be found 24, whose square Root, to wit $\sqrt{24}$, shall be the lesser Part; $\sqrt{24}$
5. I say, the summ of the two numbers found out in the third and fourth steps, is a Fourth Binomial, *viz.* . . . . $6 + \sqrt{24}$

The Definition of Binomial V.

When the lesser Part of a Binomial is a Rational number, and the greater Part is a Surd square Root of some Rational number, and the square Root of the Difference of the Squares of the Parts is Incommensurable to the greater Part; the summ of the two Parts is called a Fifth Binomial.

Explication.

Let this Binomial be proposed, ... $\sqrt{6} + 2$

The Squares of the Parts are 6, 4

The Difference of those Squares is 2

The square Root of that Difference is . . . . . . $\sqrt{2}$

Because the lesser Part 2 is a Rational number, and the greater Part $\sqrt{6}$ is a Surd square Root of a Rational number 6, and the square Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{2}$, is Incommensurable to the greater Part $\sqrt{6}$; (for $\sqrt{2} \cdot \sqrt{6} :: 1 \cdot \sqrt{3}$, not as a Rational number to a Rational number;) the proposed number $\sqrt{6} + 2$ is a Fifth Binomial.

How

How to find out two such numbers as may constitute a Fifth Binomial.

1. Take any square number, as	9
2. Divide that square number 9 into two numbers not Squares, as into	6 and 3
3. Take a Rational number at pleasure for the lesser Part of the Binomial sought, as	2
4. Then say, If 6 the greater of the two numbers in the second step, gives 9 the square number in the first; what shall 4 the Square of the Rational number taken in the third give? whence you will find the fourth Proportional 6, whose square Root, to wit, $\sqrt{6}$, shall be the greater Part sought;	$\sqrt{6}$
I say, the summ of the two numbers found out in the third and fourth steps is a Fifth Binomial, *viz.*	$2 + \sqrt{6}$

The Definition of Binomial VI.

When each of the two Parts of a Binomial is a Surd square Root of some Rational number, and the square Root of the Difference of the Squares of the Parts is Incommensurable to the greater Part; the summ of the two Parts is called a Sixth Binomial.

Explication.

Let this Binomial be proposed,	$\sqrt{5} + \sqrt{3}$
The Squares of the Parts are	5 3
The Difference of the Squares of the Parts is	2
The square Root of that Difference is	$\sqrt{2}$

Because the two Parts $\sqrt{5}$ and $\sqrt{3}$ are Surd square Roots of two Rational numbers 5 and 3; and the square Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{2}$, is Incommensurable to the greater Part $\sqrt{5}$; (for $\sqrt{2}$ hath not such proportion to $\sqrt{5}$, as a Rational number to a Rational number;) the number $\sqrt{5} + \sqrt{3}$ above proposed is a Sixth Binomial.

How to find out two such numbers as may constitute a Sixth Binomial.

1. Take two such Prime numbers that their summ may not be a Square, as	7 and 5
2. Their summ is	12
3. Take also any square number, as	9
4. Take again some Rational number at pleasure, as	6
5. The Square thereof is	36
6. Then say, If 9 the square number taken in the third step, gives 12 the summ of the two Prime numbers in the first; what shall 36 the Square in the fifth step give? whence you will find 48, whose square Root, to wit, $\sqrt{48}$, shall be the greater Part;	$\sqrt{48}$
7. Say again, If 12 the summ of the two Prime numbers in the first step, gives 7 the greater of those Prime numbers; what shall 48 the fourth Proportional found out in the sixth step give? whence you will find 28, whose square Root, *viz.* $\sqrt{28}$ shall be the lesser Part;	$\sqrt{28}$
I say, the summ of the two numbers found out in the sixth and seventh steps is a Sixth Binomial, *viz.*	$\sqrt{48} + \sqrt{28}$

If of every one of those six Binomials the lesser Part be subtracted from the greater, by interposing the sign —, the six Remainders answer to the six Lines which *Euclid* in *Prop.* 86, 87, 88, 89, 90, 91. of his Tenth *Elem.* calls *Apotomes* or *Residual lines*; as,

Out of Binomial	I.	$3 + \sqrt{5}$	By changing + into —, is made Residual.	I.	$3 - \sqrt{5}$
	II.	$\sqrt{18} + 4$		II.	$\sqrt{18} - 4$
	III.	$\sqrt{50} + \sqrt{32}$		III.	$\sqrt{50} - \sqrt{32}$
	IV.	$5 + \sqrt{12}$		IV.	$5 - \sqrt{12}$
	V.	$\sqrt{6} + 2$		V.	$\sqrt{6} - 2$
	VI.	$\sqrt{5} + \sqrt{3}$		VI.	$\sqrt{5} - \sqrt{3}$

The precedent Constructions of the said six Binomials are demonstrated in *Prop.* 49, 50, 51, 52, 53, 54. of 10. *Elem.* Euclid.

Now if any Binomial or Residual be given, we may easily find out another of the same kind in this manner, *viz.* For the first and fourth Binomials, if it be made as the greater Name or Part to the lesser, so any Rational number assumed for the greater Part of a new first or fourth Binomial, to a fourth Proportional number, this number shall be the lesser Part of the new first or fourth Binomial. But for the second and fifth, if it be made as the lesser Part to the greater, so any Rational number taken for the lesser Part of a new second or fifth Binomial to a fourth Proportional, the number so produced shall be the greater Part of the new second or fifth Binomial. And lastly, for the third and sixth Binomials, if it be made as the greater Part to the lesser, (each of which is a Surd square Root,) so any Surd square Root assumed for the greater Part of a new third or sixth Binomial, to a fourth Proportional, there will come forth the lesser Part of a new third or sixth Binomial. (The reason of this Operation is manifest, *per Prop.* 15. *Elem.* 10. *Euclid.*) And, after a new Binomial is found out, its correspondent Residual is also made, by changing the sign $+$ into $-$, as before hath been said.

As, for example, if a first Binomial $3 + \sqrt{5}$ be proposed, to find another like to it; I take a Rational number at pleasure, as 8, for the greater Part of the Binomial sought; then by the Rule of Three, as 3 is to $\sqrt{5}$, so 8 to a fourth Proportional, to wit, $\sqrt{\frac{320}{9}}$, for the lesser Part sought, therefore $8 + \sqrt{\frac{320}{9}}$ shall be a new first Binomial, and $8 - \sqrt{\frac{320}{9}}$ a new first Residual; and so of the rest.

Sect. XVI. *Concerning the extraction of the Square Root out of Binomials and Residuals constituted in such manner as hath been shewn in the preceding* Sect. 15.

Every one of the Binomials and Residuals whose Construction hath been shewn in the preceding *Sect.* 15. hath a square Root, that is, such a Binomial or Residual that if it be multiplied into it self will produce the given Binomial or Residual; as may be evidently collected out of *Prop.* 55, 56, 57, 58, 59, and 60. Also out of *Prop.* 92, 93, 94, 95, 96, and 97. of the Tenth Book of *Euclid*'s Elements.

As, for example, a Binomial of the first kind, suppose $7 + \sqrt{48}$ hath a square Root, to wit, $2 + \sqrt{3}$; for this being squared (or multiplied into it self) produceth that Binomial $7 + 48$; whose greater Part 7 is composed of 4 and 3 the Squares of the Parts of the Root $2 + \sqrt{3}$; and the lesser Part $\sqrt{48}$ is the double of the Product made by the multiplication of 2 into $\sqrt{3}$, the Parts of the Root $2 + \sqrt{3}$: all which is evident by the multiplication of $2 + \sqrt{3}$ into it self. The like effect will be found in every one of the rest of the Binomials constituted in the preceding *Sect.* 15. Therefore if a Binomial be proposed, and its square Root desired, there is given the summ of the Squares of the Parts of the Root; (which summ is the greater Part of the Binomial proposed;) and the double of the Product of the Parts of the Root, (which double Product is the lesser Part of the Binomial proposed,) to find out the two Parts of the Root severally. And therefore in order to the Extraction of the square Root of a Binomial, it will be requisite to search out a Canon for the solving of this following

QUEST.

The summ (b) of the Squares of two numbers being given; as also (c) the double Product of the multiplication of the same two numbers; to find the numbers severally.

RESOLUTION.

1. For one of the two numbers sought put	a
2. Then for as much as the double of the Product of their multiplication is given c, therefore the Product it self is . .	$\frac{c}{2}$
3. Which Product divided by the first number a gives the other number	$\frac{c}{2a}$
4. Therefore the Square of the first number is	aa
5. And the Square of the other number is	$\frac{cc}{4aa}$
6. Therefore the summ of the Squares of the two numbers is .	$aa + \frac{cc}{4aa}$

7. Which

7. Which summ must be equal to b, the given summ of the Squares hence this Equation, ... $\Big\}$ $aa + \frac{cc}{4aa} = b$
8. From which Equation, after due Reduction, there will arise $\succ$ $baa - aaaa = \frac{1}{4}cc$.
9. And from the last Equation (*per* Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) there will arise this following Canon, to find out the two numbers sought, *viz.*

CANON 1.

$$\left\{\begin{array}{l} \sqrt{:\tfrac{1}{2}b + \sqrt{.\tfrac{1}{4}bb - \tfrac{1}{4}cc.}:} = \text{the greater number,} \\ \sqrt{:\tfrac{1}{2}b - \sqrt{.\tfrac{1}{4}bb - \tfrac{1}{4}cc.}:} = \text{the lesser number.} \end{array}\right.$$

That is, in words,

From a quarter of the Square of the given summ of the Squares, subtract a quarter of the Square of the double Product given; then add and subtract the square Root of that Remainder to and from half the given summ of the Squares: so shall the square Roots of the Summ and Remainder of that Addition and Subtraction be the two numbers sought.

10. Moreover, because $\frac{b + \sqrt{bb - cc}}{2} = \frac{1}{2}b + \sqrt{:\tfrac{1}{4}bb - \tfrac{1}{4}cc:}$

11. Therefore, . . $\sqrt{:\frac{b + \sqrt{bb - cc}}{2}:} = \sqrt{:\tfrac{1}{2}b + \sqrt{.\tfrac{1}{4}bb - \tfrac{1}{4}cc:}}.$

12. Likewise because $\frac{b - \sqrt{bb - cc}}{2} = \frac{1}{2}b - \sqrt{:\tfrac{1}{4}bb - \tfrac{1}{4}cc:}$

13. Therefore, . . $\sqrt{:\frac{b - \sqrt{bb - cc}}{2}:} = \sqrt{\tfrac{1}{2}:b - \sqrt{:\tfrac{1}{4}bb - \tfrac{1}{4}cc:}}$

14. Therefore from the eleventh and thirteenth steps another Canon ariseth to solve the Question; *viz.*

CANON 2.

$$\left\{\begin{array}{l} \sqrt{:\frac{b + \sqrt{bb - cc}}{2}:} = \text{the greater number,} \\ \sqrt{:\frac{b - \sqrt{bb - cc}}{2}:} = \text{the lesser number.} \end{array}\right.$$

That is, in words,

From the Square of the given summ of the Squares subtract the Square of the double Product given; then add and subtract the square Root of the Remainder to and from the given summ of the Squares: so shall the square Root of half the Summ and Remainder of that Addition and Subtraction be the two numbers sought.

By the help of either of those Canons we may extract the square Root of a Binomial or Residual, but I shall use the latter only, whence ariseth

A General Rule for the Extraction of the Square Root out of Binomials and Residuals.

From the Square of the greater part of a given Binomial or Residual, subtract the Square of the lesser; then add the square Root of the Remainder to the greater part, and subtract it also from the same; lastly, connect the square Roots of the half of that Summ and Remainder by the sign + if a Binomial be proposed, but by — if a Residual: so you have the desired square Root of the given Binomial or Residual.

The practice of this Rule will be shewn at large in the following Examples.

Example 1.

Let it be required to extract the square Root of this first Binomial, $\succ$ $27 + \sqrt{704}$

The Operation.

1. From the Square of the greater part 27, *viz.* from . . . . $\succ$ 729
2. Subtract the Square of the lesser part $\sqrt{704}$, to wit, ... $\succ$ 704
3. The Remainder is . . . . $\succ$ 25
4. The square Root of that Remainder is . . . . $\succ$ 5

5. To

5. To which square Root add the greater part	27
6. The summ is	32
7. The half of that summ is	16
8. The square Root of the said half summ is the greater part of the Root sought, to wit,	4
9. Then from the greater part of the given Binomial, *viz.* from	27
10. Subtract the square Root before found in the fourth step, to wit,	5
11. The Remainder is	22
12. The half of which Remainder is	11
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit,	$\sqrt{11}$
14. I say, the two Names or parts in the eighth and thirteenth steps being connected by + shall be the Square Root sought, to wit,	$4 + \sqrt{11}$

But if — instead of + be prefixt to the lesser part of the said Root, it will give $4 - \sqrt{11}$, which is the square Root of the first Residual or Apotome $27 - \sqrt{704}$.

The former of those two Roots answers to the Irrational line called (in *prop.* 37, and 55. *lib.* 10. *Elem* Euclid.) a *Binomial* line; and the latter answers to the Irrational line called (in *prop.* 74, and 92.) an *Apotome* or *Residual* line.

The Proof of the Root *above extracted out of the first* Binomial, *is made by multiplying the* Root *into it self; thus,*

The summ of the Squares of the parts of $4 + \sqrt{11}$, the Root found out is	$16 + 11$, that is,	27
The Product of the same parts multiplied one into the other is	$4\sqrt{11}$, that is,	$\sqrt{176}$
The double of the said Product is	$8\sqrt{11}$, that is,	$\sqrt{704}$
The summ of the said summ of the Squares of the parts and the double Product is	$27 + \sqrt{704}$	

Whence it is manifest that $27 + \sqrt{704}$ is the Square of $4 + \sqrt{11}$, therefore this is the true square Root of that first Binomial: which was to be proved. Moreover, if the said double Product be subtracted from the said summ of the Squares of the Parts, the Remainder $27 - \sqrt{704}$ is the Square of $4 - \sqrt{11}$; therefore this is the square Root of that first Residual.

Example. 2.

Let it be required to extract the square Root out of this second Binomial $\sqrt{\frac{147}{4}} + 6$

The Operation.

1. From the Square of the greater part $\sqrt{\frac{147}{4}}$, *viz.* from	$\frac{147}{4}$
2. Subtract the Square of the lesser part 6, to wit,	36
3. The Remainder is	$\frac{3}{4}$
4. The square Root of that Remainder is	$\sqrt{\frac{3}{4}}$
5. To which square Root add the greater part, (by the Rule in *Sect.* 8. of this *Chapt.*)	$\sqrt{\frac{147}{4}}$
6. The Summ is	$\sqrt{48}$
7. The half of which Summ is	$\sqrt{12}$
8. The square Root of that half Summ is the greater part of the Root sought, to wit,	$\sqrt{(4)}12$
9. Again, from the greater part of the given Binomial, *viz.* from	$\sqrt{\frac{147}{4}}$
10. Subtract the square Root before found in the fourth step, (by the said Rule in *Sect.* 8.) *viz.*	$\sqrt{\frac{3}{4}}$
11. The Remainder is	$\sqrt{27}$
12. The half of which Remainder is,	$\sqrt{\frac{27}{4}}$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit,	$\sqrt{(4)}\frac{27}{4}$
14. I say, the two parts in the eighth and thirteenth steps, being connected by the sign + shall be the Root sought, to wit,	$\sqrt{(4)}12 + \sqrt{(4)}\frac{27}{4}$

And if — instead of + be prefixt to the lesser part of the said Root, it will give $\sqrt{(4)}12 - \sqrt{(4)}\frac{27}{4}$, which is the square Root of the second Residual $\sqrt{\frac{147}{4}} - 6$.

The

The former of those two Roots answers to the irrational line called (in *Prop.* 38, & 56. *lib.* 10. *Elem.* Euclid.) a *first Bimedial*; and the latter answers to the irrational line called (in *Prop.* 75, & 93.) a *first Medial Residual.*

The Proof of the Root *above extracted out of the second* Binomial.

The Squares of the Parts of $\surd(4)12 + \surd(4)\frac{27}{4}$ the Root found out, are	$\surd 12$ and $\surd\frac{27}{4}$
Which Squares added together, (as in Example 6. *Sect.* 8. of this *Chapt.* is manifest,) makes the summ	$7\surd\frac{3}{4}$, that is, $\surd\frac{147}{4}$
The Product of the Parts, *viz.* $\surd(4)12$ into $\surd(4)\frac{27}{4}$ is .	$\surd(4)81$, that is, 3
The double of the said Product is	6
Therefore the summ of the summ of the Squares of the Parts and the said double Product is	$\surd\frac{147}{4} + 6$

Whence it is manifest that $\surd\frac{147}{4} + 6$ is the Square of $\surd(4)12 + \surd(4)\frac{27}{4}$, therefore this is the true square Root of that second Binomial; which was to be proved. Moreover, if the said double Product be subtracted from the said summ of the Squares of the Parts, the Remainder $\surd\frac{147}{4} - 6$ is the Square of $\surd(4)12 - \surd(4)\frac{27}{4}$; therefore this is the square Root of that second Residual.

Example 3.

Let it be required to extract the square Root of this third Binomial . $\surd\frac{245}{3} + \surd 80$.

The Operation.

1. From the Square of the greater part $\surd\frac{245}{3}$, *viz.* from . .	. . $\frac{245}{3}$
2. Subtract the Square of the lesser part, to wit, ...	. . 80
3. The Remainder is	. . . $\frac{5}{3}$
4. The square Root of that Remainder is	. . $\surd\frac{5}{3}$
5. To which square Root add the greater part	. $\surd\frac{245}{3}$
6. The summ is	. $\surd\frac{320}{3}$
7. The half of which summ is	. $\surd\frac{80}{3}$
8. The square Root of that half summ is the greater part of the Root sought, to wit, ...	$\surd(4)\frac{80}{3}$
9. Again, from the greater part of the given Binomial, *viz.* from	. $\surd\frac{245}{3}$
10. Subtract the square Root before found in the fourth step, to wit, ...	. . $\surd\frac{5}{3}$
11. The Remainder is	. $\surd 60$
12. The half of which Remainder is	. . $\surd 15$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, ...	$\surd(4)15$
14. I say, the two parts in the eighth and thirteenth steps, being connected by +, shall be the square Root sought; to wit,	$\surd(4)\frac{80}{3} + \surd(4)15$

And if — instead of + be prefixt to the lesser part of the said Root, it gives $\surd(4)\frac{80}{3} - \surd(4)15$, which is the square Root of the third Residual $\surd\frac{245}{3} - \surd 80$.

The former of those two Roots answers to the irrational line called (in *Prop.* 39, & 57. *lib.* 10. *Elem.* Euclid.) a *second Bimedial*; and the latter answers to the irrational line called (in *Prop.* 76, & 94.) a *second Medial Residual.*

The Proof of the Root *above extracted out of the third* Binomial.

The Squares of the parts of $\surd(4)\frac{80}{3} + \surd(4)15$, the Root found out, are	$\surd\frac{80}{3}$ and $\surd 15$
Which Squares added together, make	$7\surd\frac{5}{3}$, that is, $\surd\frac{245}{3}$
The Product of the parts, *viz.* $\surd(4)\frac{80}{3}$ into $\surd(4)15$, is	$\surd(4)400$, that is, $\surd 20$
The double of the said Product is	$\surd 80$
Therefore the summ of the summ of the Squares of the parts and the said double Product is	$\surd\frac{245}{3} + \surd 80$.

Whence it is manifest, that $\surd\frac{245}{3} + \surd 80$ is the Square of $\surd(4)\frac{80}{3} + \surd(4)15$; therefore this is the square Root of that third Binomial: which was to be proved.

Moreover,

Moreover, if the ſaid double Product be ſubtracted from the ſaid ſumm of the Squares of the parts, the Remainder $\sqrt{24\frac{5}{3}} - \sqrt{80}$ is the Square of $\sqrt{(4)}\,\frac{80}{3} - \sqrt{(4)}\,15$; therefore this is the ſquare Root of that third Reſidual.

Example 4.

Let it be required to extract the ſquare Root of this fourth Binomial . $7 + \sqrt{20}$.

The Operation.

1. From the Square of the greater part 7, *viz.* from	49
2. Subtract the Square of the leſſer part $\sqrt{20}$, to wit,	20
3. The Remainder is	29
4. The ſquare Root of that Remainder is . . .	$\sqrt{29}$
5. To which ſquare Root add the greater part . . .	7
6. The Summ is	$7 + \sqrt{29}$
7. The half of which Summ is	$\frac{7}{2} + \sqrt{\frac{29}{4}}$
8. The ſquare Root of that half Summ is the greater part of the Root ſought, to wit, ...	$\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:}$
9. Again, from the greater part of the given Binomial, *viz.* from	7
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, ...	$\sqrt{29}$
11. The Remainder is	$7 - \sqrt{29}$
12. The half of which Remainder is	$\frac{7}{2} - \sqrt{\frac{29}{4}}$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ...	$\sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$
14 I ſay, the two parts in the eighth and thirteenth ſteps, (the former of which is a Binomial, and the latter a Reſidual) being connected by +, ſhall be the ſquare Root ſought, to wit, ...	$\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:} + \sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$

Which Root anſwers to the irrational line called (in *Prop.* 40, & 58. *lib.* 10. *Elem. Euclid.*) a *Major line.*

And if the leſſer Name of the ſaid Root be ſubtracted from the greater, by interpoſing the ſign —, it gives $\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:} - \sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$ which is the Root of the fourth Reſidual $7 - \sqrt{20}$, and anſwers to the irrational line called (in *Prop.* 77, & 95. *lib.* 10. *Elem.* Euclid.) a *Minor line.*

The Proof of the Root *above extracted out of the fourth* Binomial.

The Squares of the parts of the Root found out are . .	$\frac{7}{2} + \sqrt{\frac{29}{4}}$ and $\frac{7}{2} - \sqrt{\frac{29}{4}}$
Therefore the ſumm of the Squares of the parts is . . .	$\frac{7}{2} + \frac{7}{2}$, that is, 7
The Product of the parts will be found (by Rule 2. *Sect.* 12. of this *Chapt.*)	$\sqrt{:\frac{49}{4} - \frac{29}{4}:}$ that is, $\sqrt{5}$
The double of the ſaid Product is	$\sqrt{20}$
Therefore the ſumm of the ſaid ſumm of the Squares of the parts and the double Product is	$7 + \sqrt{20}$.

Whence it is manifeſt that $7 + \sqrt{20}$ is the Square of $\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:} + \sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$ therefore this is the ſquare Root of that fourth Binomial: which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid ſumm of the Squares of the Parts, the Remainder $7 - \sqrt{20}$ is the Square of $\sqrt{:\frac{7}{2} + \sqrt{\frac{29}{4}}:} - \sqrt{:\frac{7}{2} - \sqrt{\frac{29}{4}}:}$ therefore this is the ſquare Root of that fourth Reſidual $7 - \sqrt{20}$.

Example 5.

Let it be required to extract the ſquare Root out of this fifth Binomial, $\sqrt{20} + 4$.

The Operation.

1. From the Square of the greater part $\sqrt{20}$, *viz.* from	20
2. Subtract the Square of the leſſer part 4, to wit,	16
3. The Remainder is	4
4. The ſquare Root of that Remainder is	2
5. To which ſquare Root add the greater part . . .	$\sqrt{20}$

6. The

6. The summ is . . . . $\sqrt{20} + 2$
7. The half of that summ is . . . . $\sqrt{5} + 1$
8. The square Root of the said half summ is the greater part of the Root sought, to wit, ... $\sqrt{:\sqrt{5}+1:}$
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $\sqrt{20}$
10. Subtract the square Root before found in the fourth step; to wit, ... 2
11. The Remainder is . . . . $\sqrt{20} - 2$
12. The half of which Remainder is . . . . . . $\sqrt{5} - 1$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, ... $\sqrt{:\sqrt{5}-1:}$
14. I say, the two parts in the eighth and thirteenth steps, (the former of which parts is a Binomial, and the latter a Residual,) being connected by +, shall be the Square Root sought, to wit, ... $\sqrt{:\sqrt{5}+1:} + \sqrt{:\sqrt{5}-1:}$

Which Root answers to the irrational line called (in *Prop.* 41, & 59. *lib.* 10. *Elem* Eucl.) a line containing in Power a *Rational* and a *Medial Rectangle*: And if the lesser Name of the said Root be subtracted from the greater, by the interposition of the sign —, it gives $\sqrt{:\sqrt{5}+1:} - \sqrt{:\sqrt{5}-1:}$ which is the square Root of the fifth Residual $\sqrt{20} - 4$, and answers to the irrational line which (in *Prop.* 78, & 96. *lib.* 10.) is called a line making with a *Rational Space* the whole *Space Medial.*

The Proof of the Root *above extracted out of the fifth* Binomial.

The Squares of the parts of $\sqrt{:\sqrt{5}+1:} + \sqrt{:\sqrt{5}-1:}$ the Root found out, are . . . . $\sqrt{5}+1$ and $\sqrt{5}-1$

Therefore the summ of the said Squares of the parts is $\sqrt{5}+\sqrt{5}$, that is, $\sqrt{20}$

The Product of the parts multiplied one into the other (according to Rule 2. *Sect.* 12. of this *Chapt.*) is . . $\sqrt{:5-1:}$ that is, 2

The double of the said Product is 4

Therefore the summ of the said summ of the Squares of the parts and double Product is . . . . $\sqrt{20}+4$

Whence it is manifest that $\sqrt{20}+4$ is the Square of $\sqrt{:\sqrt{5}+1:} + \sqrt{:\sqrt{5}-1:}$ therefore this is the square Root of that fifth Binomial: which was to be proved. Moreover, if the said double Product be subtracted from the said summ of the Squares of the parts, the Remainder $\sqrt{20}-4$ is the Square of $\sqrt{:\sqrt{5}+1:} - \sqrt{:\sqrt{5}-1:}$ Therefore this is the square Root of the said fifth Residual $\sqrt{20}-4$.

Example 6.

Let it be required to extract the square Root of this sixth Binomial, $\sqrt{20} + \sqrt{8}$.

The Operation.

1. From the Square of the greater part $\sqrt{20}$, *viz.* from 20
2. Subtract the Square of the lesser part $\sqrt{8}$, to wit, . 8
3. The Remainder is 12
4. The square Root of that Remainder is . . . $\sqrt{12}$
5. To which square Root add the greater part . . . $\sqrt{20}$
6. The summ is . . . . $\sqrt{20}+\sqrt{12}$
7. The half of which summ is . . . . $\sqrt{5}+\sqrt{3}$
8. The square Root of the said half summ is the greater part of the Root sought, to wit, . . $\sqrt{:\sqrt{5}+\sqrt{3}:}$
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $\sqrt{20}$
10. Subtract the square Root before found in the fourth step, to wit, ... $\sqrt{12}$
11. The Remainder is . . . . $\sqrt{20}-\sqrt{12}$
12. The half of that Remainder is . . . . . . $\sqrt{5}-\sqrt{3}$

13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, . . . $\sqrt{:}\sqrt{5}-\sqrt{3}:$

14. I say, the two parts in the eighth and thirteenth steps, (the former of which parts is a Binomial, and the latter a Residual) being connected by $+$, shall be the square Root sought; to wit, ... $\sqrt{:}\sqrt{5}+\sqrt{3}: + \sqrt{:}\sqrt{5}-\sqrt{3}:$

Which Root answers to the irrational line which (in *Prop.* 42, & 60. *lib.* 10. *Elem* Eucl.) is called; a line containing in Power two *Medial Rectangles:* And, if the lesser part of the said Root be subtracted from the greater by the interposition of the sign $-$, it gives $\sqrt{:}\sqrt{5}+\sqrt{3}: - \sqrt{:}\sqrt{5}-\sqrt{3}:$ which is the Root of the sixth Residual $\sqrt{20}-\sqrt{8}$; and answers to the irrational line which (in *Prop.* 79, & 97. *lib.* 10. Euclid.) is called a line making with a *Medial Rectangle* a whole *Space Medial.*

The Proof of the Root *above extracted out of the sixth* Binomial.

The Squares of the parts of $\sqrt{:}\sqrt{5}+\sqrt{3}: + \sqrt{:}\sqrt{5}-\sqrt{3}:$ the Root sought, are	$\sqrt{5}+\sqrt{3}$ and $\sqrt{5}-\sqrt{3}$
Therefore the summ of the said Squares of the parts is	$\sqrt{5}+\sqrt{5}$, that is, $\sqrt{20}$
The Product of the parts multiplied one into the other is	$\sqrt{:}5-3:$ that is, $\sqrt{2}$
The double of the said Product is	$\sqrt{8}$
Therefore the summ of the said summ of the Squares of the parts and double Product is	$\sqrt{20}+\sqrt{8}$.

Whence it is manifest that $\sqrt{20}+\sqrt{8}$ is the Square of $\sqrt{:}\sqrt{5}+\sqrt{3}: + \sqrt{:}\sqrt{5}-\sqrt{3}:$ therefore this is that square Root of the sixth Binomial: which was to be proved. Moreover, if the said double Product be subtracted from the said summ of the Squares of the parts, the Remainder $\sqrt{20}-\sqrt{8}$ is the Square of $\sqrt{:}\sqrt{5}+\sqrt{3}: - \sqrt{:}\sqrt{5}-\sqrt{3}:$ therefore this is the square Root of that sixth Residual.

Note. In every Binomial and Residual constituted according to the preceding *Sect.* 15. the square Root of the difference of the Squares of the Names or parts is equal to the difference of the Squares of the parts of the Root of the Binomial or Residual.

As in the first Binomial $27+\sqrt{704}$, whose square Root hath before been found $4+\sqrt{11}$; the Square of 27, to wit, 729, exceeds 704, the Square of $\sqrt{704}$, by 25, whose square Root 5 is equal to the difference of the Squares of the parts of the Root of the Binomial proposed, to wit, the difference between 16 and 11.

This property may be demonstrated thus, let $b+\sqrt{d}$ represent a Binomial Root whose greater part is b; then the Square of that Root is $bb+2b\sqrt{d}+d$, this divided into its Names or parts makes the Binomial $bb+d$ more $2b\sqrt{d}$; then the Squares of the parts of this Binomial are $bbbb+2bbd+dd$ and $4bbd$, and the difference of these Squares is $bbbb-2bbd+dd$, whose square Root $bb-d$ is manifestly the difference of the Squares of the parts of the Root $b+\sqrt{d}$ first proposed: which was to be shewn. The like property may be demonstrated in a Residual.

How to extract the Square Root out of a Binomial design'd by Letters, if it hath a Binomial Root.

By the same general Rule which hath before been exercis'd in extracting the square Root out of Binomials exprest by Numbers, we may extract the square Root out of a Binomial design'd by Letters, when it hath a binomial Root; as will be evident by the following Examples; where for the more apparent distinction of the parts of the given Binomial, instead of $+$ I set the word [*more*] between the parts, and instead of $-$ I set the word [*less*] between the parts of a given Residual.

Example 1.

Let it be required to extract the square Root out of this Binomial, ... $bb+d$ more $2b\sqrt{d}$.

The Operation.

1. From the Square of the greater part, (which suppose to be $bb+d$,) *viz.* from . . . . $bbbb+2dbb+dd$
2. Subtract the Square of the lesser part $2b\sqrt{d}$, to wit, ... $+4dbb$

3. The

3. The Remainder is . . . . $bbbb - 2dbb + dd$
4. The square Root of that Remainder is . . . . $bb - d$
5. To which square Root add the greater part, to wit, . . . $bb + d$
6. The Summ is . . . . $2bb$
7. The half of that Summ is . . . . bb
8. The square Root of that half Summ is the greater part of the Root sought, to wit, ... b
9. Then from the greater part of the given Binomial, *viz.* from $bb + d$
10. Subtract the square Root before found in the fourth step, to wit, $bb - d$
11. The Remainder is . . . . $+ 2d$
12. The half of which Remainder is . . . . $+ d$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, ... $\sqrt{d}$
14. I say, the two parts in the eighth and thirteenth steps being connected by the sign + shall be the square Root sought, to wit, $b + \sqrt{d}$

Which Root being squared, or multiplied into it self, will evidently produce the given Binomial $bb + d$ more $2b\sqrt{d}$.

Example 2.

Let it be required to extract the square Root out of this Binomial, ... $mm + \frac{pxx}{m}$ more $x\sqrt{4pm}$.

The Operation.

1. From the Square of the greater part $mm + \frac{pxx}{m}$, *viz.* from . . . . $mmmm + 2mpxx + \frac{ppxxxx}{mm}$
2. Subtract the Square of the lesser part $x\sqrt{4pm}$, to wit, $+ 4mpxx$
3. The Remainder is . . . . $mmmm - 2mpxx + \frac{ppxxxx}{mm}$
4. The square Root of that Remainder is . . . $mm - \frac{pxx}{m}$
5. To which square Root add the greater part, to wit, $mm + \frac{pxx}{m}$
6. The Summ is . . . . $2mm$
7. The half of which Summ is . . . . mm
8. The square Root of the said half Summ is the greater part of the Root sought, to wit, ... m
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $mm + \frac{pxx}{m}$
10. Subtract the square Root before found in the fourth step, to wit, ... $mm - \frac{pxx}{m}$
11. The Remainder is . . . . $+ \frac{2pxx}{m}$
12. The half of which Remainder is . . . . . . $+ \frac{pxx}{m}$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, ... $\sqrt{\frac{pxx}{m}}$ or $x\sqrt{\frac{p}{m}}$
14. I say, the two parts in the eighth and thirteenth steps, being connected by +, shall be the square Root sought; to wit, ... $m + x\sqrt{\frac{p}{m}}$

Which Binomial Root being squared or multiplied into it self, will produce the given Binomial.

Example 3.

Let it be required to extract the square Root out of this Binomial, ... $\overline{a + b}\sqrt{ab}$ more $2ab$.

The Operation.

1. From the Square of the greater part, *viz.* from . . . $aaab + 2aabb + abbb$
2. Subtract the Square of the leſſer part, to wit, ... $+ 4aabb$
3. The Remainder is . . . . $aaab - 2aabb + abbb$
4. The ſquare Root of that Remainder is . . . . $\overline{a - b}\sqrt{ab}$
5. To which ſquare Root add the greater part, to wit, . . $\overline{a + b}\sqrt{ab}$
6. The Summ is . . . . $2a\sqrt{ab}$
7. The half of that Summ is . . . . $a\sqrt{ab}$
8. The ſquare Root of the ſaid half Summ is the greater part of the Root ſought, to wit, ... $\sqrt{:a\sqrt{ab}:}$ or $\sqrt{(4)}aaab$
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $\overline{a + b}\sqrt{ab}$
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, ... $\overline{a - b}\sqrt{ab}$
11. The Remainder is . . . . $2b\sqrt{ab}$
12. The half of which Remainder is . . . . . . $b\sqrt{ab}$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... $\sqrt{:b\sqrt{ab}:}$ or $\sqrt{(4)}abbb$
14. I ſay, the two parts in the eighth and thirteenth ſteps, being connected by +, ſhall be the ſquare Root ſought, to wit, ... $\sqrt{:a\sqrt{ab}:} + \sqrt{:b\sqrt{ab}:}$
15. Which Binomial Root may be alſo expreſt thus, . . $\sqrt{(4)}aaab + \sqrt{(4)}abbb$

The Proof may be made by multiplying the Root found out into it ſelf.

Example 4.

Again, if the ſquare Root of this Reſidual be deſired, . . $\overline{a + d}\sqrt{bc}$ leſs $2\sqrt{abcd}$
The Root being extracted by the precedent method, will be found . . . . $\sqrt{:a\sqrt{bc}:} - \sqrt{:d\sqrt{bc}:}$
Which Root may be alſo expreſt thus, ... $\sqrt{(4)}aabc - \sqrt{(4)}ddbc$

But if it happen that when the Square of the leſſer part of the given Binomial or Reſidual is ſubtracted from the Square of the greater part, the ſquare Root of the Remainder and the greater part are not commenſurable, (according to the Definition before given in *Sect.* 7. of this *Chapt.*) there is no more to be done in ſuch caſe, but to prefix before the given Binomial or Reſidual the ſign $\sqrt{\ }$, with a line drawn over both its parts, to denote the univerſal ſquare Root of the given Binomial or Reſidual. As, to extract the ſquare Root out of this Reſidaal $\sqrt{:\tfrac{1}{4}aa + bb:} - \tfrac{1}{2}a$, I write $\sqrt{:\sqrt{\tfrac{1}{4}aa + bb} - \tfrac{1}{2}a:}$ which kind of Roots are commonly called Univerſal.

Sect. 17. *Queſtions to exerciſe the foregoing Rules of this Chapter.*

QUEST. 1.

To divide 100 into two ſuch parts, that if each part be divided by the other part, the ſumm of the Quotients may make 3.

RESOLUTION.

1. For one of the parts ſought put . . . . . . a
2. Then conſequently the other part is . . . . . $100 - a$
3. Therefore, according to the import of the Queſtion, this Equation ariſeth, *viz.* . . . . $\dfrac{a}{100 - a} + \dfrac{100 - a}{a} = 3$
4. Which Equation duly reduced gives . . . . $100a - aa = 2000$
5. Wherefore by reſolving the ſaid Equation by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two values of a, which are the deſired parts of 100, will be found theſe, to wit, ... $a = \begin{cases} 50 + 10\sqrt{5} \\ 50 - 10\sqrt{5} \end{cases}$

6. The

6. The ſumm of the ſaid parts or numbers found out is manifeſtly 100; ſo it remains only to prove that

$$\frac{50 + 10\sqrt{5}}{50 - 10\sqrt{5}} + \frac{50 - 10\sqrt{5}}{50 + 10\sqrt{5}} = 3.$$

The Proof.

7. To add thoſe two ſurd Fractions in the ſixth ſtep into one ſumm; reduce them to a common Denominator, *viz.* multiply $50 + 10\sqrt{5}$ by $50 + 10\sqrt{5}$, and the Product (by the firſt of the three compendious Rules in *Sect.* 10. of this *Chapt.*) will be found — $3000 + 1000\sqrt{5}$

8. Likewiſe, multiply $50 - 10\sqrt{5}$ by $50 - 10\sqrt{5}$, and the Product (by the ſecond of the ſaid three Rules) will be . . . — $3000 - 1000\sqrt{5}$

9. Then take the ſumm of thoſe two Products for the Numerator of a Fraction, or a Dividend, to wit, ... — 6000

10. Alſo multiply the two Denominators of the ſurd Fractions in the ſixth ſtep one by the other, (according to the laſt of the three Rules above cited,) and take the Product for a Denominator, or Diviſor, *viz.* . . . . — 2000

11. Laſtly, the Numerator in the ninth ſtep being ſet over the Denominator in the tenth gives the ſumm of the two ſurd Fractions or Quotients in the ſixth ſtep, *viz.* . . . . — $\frac{6000}{2000} = 3$

Which ſumm is manifeſtly 3, as was to be proved.

Another Proof.

The Quotient that ariſeth by dividing $50 + 10\sqrt{5}$ by $50 - 10\sqrt{5}$, (according to the Rule of Diviſion in the ſixth branch of *Sect.* 11. of this *Chapt.*) is . . . . — $\frac{3}{2} + \sqrt{\frac{5}{4}}$

Likewiſe, the Quotient that ariſeth by dividing $50 - 10\sqrt{5}$ by $50 + 10\sqrt{5}$ is . . . . — $\frac{3}{2} - \sqrt{\frac{5}{4}}$

The ſumm of thoſe two Quotients is manifeſtly 3; (as before.)

QUEST. 2.

To divide a given number, ſuppoſe 6, into three ſuch unequal numbers in continual proportion, that the ſumm of the Squares of the extremes may be to the Square of the mean in a given proportion; but the firſt term of this proportion muſt exceed the double of the latter term. Let it therefore be deſired that the ſumm of the Squares of the extremes may be to the Square of the mean as 3 to 1.

RESOLUTION.

1. For the mean Proportional put . . . . — a
2. Then becauſe the ſumm of all the three Pronportionals muſt make 6; and the mean is a; the ſumm of the extremes ſhall be — $6 - a$
3. Therefore the Square of the ſumm of the extremes is . . — $36 - 12a + aa$
4. But (by *Theor.* 3. *Chap.* 6. of this Book) the Square of the ſumm of the extremes of three numbers continually proportional is equal to the Squares of the extremes, together with the double Square of the mean; therefore from the Square in the third ſtep I ſubtract $2aa$ (the double Square of the mean,) and there remains the ſumm of the Squares of the extremes, to wit, ... — $36 - 12a - aa$
5. But (according to the Queſtion) the ſumm of the Squares of the extremes muſt be equal to the triple Square of the mean; therefore from the fourth and firſt ſtep this Equation ariſeth, *viz.* . . . . — $36 - 12a - aa = 3aa$
6. From which Equation after due Reduction this ariſeth, *viz.* — $aa + 3a = 9$
7. Therefore by reſolving the laſt Equation, (according to the Canon in *Sect.* 6. *Chap.* 15.) the value of a, that is, the mean Proportional ſought will be diſcovered, *viz.* . . . — $\sqrt{\frac{45}{4}} - \frac{3}{2} =$ the mean.

8. And

8. And from the seventh and second steps the summ of the extremes will be also made known, *viz.* } $\frac{15}{2} - \sqrt{\frac{45}{4}} =$ summ of the extremes.

9. Then, (as is manifest by *Quest.* 4. *Chap.* 16. *Book* 1.) the summ of the extremes of three numbers continually proportional being given; as also the mean, the extremes shall be given severally by this following

CANON.

From the Square of half the summ of the extremes subtract the Square of the mean, and extract the square Root of the Remainder; then this square Root being added to, and subtracted from the said half summ, will give the extremes severally. Therefore,

10. From the Square of the half of $\frac{15}{2} - \sqrt{\frac{45}{4}}$, that is, from . . > $\frac{135}{8} - \frac{15}{4}\sqrt{\frac{45}{4}}$

11. Subtract the Square of $\sqrt{\frac{45}{4}} - \frac{3}{2}$, *viz.* . . . . . . > $\frac{108}{8} - \frac{12}{4}\sqrt{\frac{45}{4}}$

12. The Remainder is . . . . > $\frac{27}{8} - \frac{3}{4}\sqrt{\frac{45}{4}}$

13. Then the square Root of that Remainder being extracted, (by the General Rule before delivered in *Sect.* 16. of this *Chapt.* for extracting the square Root out of Binomials,) will be found . . } $\sqrt{\frac{45}{16}} - \frac{3}{4}$

14. Which square Root added to the half of $\frac{15}{2} - \sqrt{\frac{45}{4}}$, gives the greater extreme sought, to wit, ... } . . 3

15. But the said square Root subtracted from the half of $\frac{15}{2} - \sqrt{\frac{45}{4}}$, leaves the lesser extreme, to wit, ... } . . $\frac{9}{2} - \sqrt{\frac{45}{4}}$

16. Wherefore, (in the seventh, fourteenth and fifteenth steps,) three numbers continually proportional are found out, *viz.* 3, $\sqrt{\frac{45}{4}} - \frac{3}{2}$, and $\frac{9}{2} - \sqrt{\frac{45}{4}}$, whose summ is *6*; and the summ of the Squares of the extremes is equal to the triple of the Square of the mean, as will appear by

The Proof.

First, the Product made by the multiplication of the first and third numbers one into the other, that is, of 3 into $\frac{9}{2} - \sqrt{\frac{45}{4}}$, is $\frac{27}{2} - 3\sqrt{\frac{45}{4}}$, which is also the Square of the second number $\sqrt{\frac{45}{4}} - \frac{3}{2}$, (as will easily appear by Multiplication;) therefore the said three numbers are Proportionals.

Secondly, the summ of the said three proportional numbers is *6*; for the mean $\sqrt{\frac{45}{4}} - \frac{3}{2}$ added to $\frac{9}{2} - \sqrt{\frac{45}{4}}$ the lesser extreme, makes 3, to which adding the greater extreme 3, the summ is *6*.

Thirdly, the summ of the Squares of the extremes 3 and $\frac{9}{2} - \sqrt{\frac{45}{4}}$, is equal to the triple of the Square of the mean $\sqrt{\frac{45}{4}} - \frac{3}{2}$; for the said summ, as also the said triple Square will by Multiplication be found $\frac{81}{2} - 9\sqrt{\frac{45}{4}}$. Therefore all the conditions in the Question are satisfied.

But that the necessity of the Determination annexed to the Question may be made manifest, it remains to prove, That if three unequal numbers be in continual proportion, the summ of the Squares of the extremes is greater than the double of the Square of the mean: Therefore,

Let three unequal numbers in continual proportion be exposed, suppose these, ... } a, $\sqrt{ae}$, e ∺

Then their Squares shall be also Proportionals, (*per* 22. *Prop.* 6. *Elem.* Euclid.) *viz.* . . . . } $aa \,.\, ae :: ae \,.\, ee$

Therefore (by 25. *Prop.* 5. *Elem.* Euclid.) . . . . . > $aa + ee \sqsubset 2ae$.

But $aa + ee$ is the summ of the Squares of the extremes of the three Proportionals exposed, and $2ae$ is equal to the double Square of the mean Proportional; wherefore the Theorem is proved; and consequently the Determination is manifestly necessary to be annexed to the Question proposed, that there may be a possibility of finding out what is thereby desired. The Determination may also be easily inferr'd from the Canon in the foregoing ninth step.

QUEST. 3.

What is the Product made by the continual multiplication of these four numbers one into another, which differ by an equal excess, to wit, Unity? . . . .

$$\begin{cases} \sqrt{:\tfrac{5}{4} + \sqrt{101}:} - \tfrac{3}{2} \\ \sqrt{:\tfrac{5}{4} + \sqrt{101}:} - \tfrac{1}{2} \\ \sqrt{:\tfrac{5}{4} + \sqrt{101}:} + \tfrac{1}{2} \\ \sqrt{:\tfrac{5}{4} + \sqrt{101}:} + \tfrac{3}{2} \end{cases}$$

Answ.

Answ. The desired Product is exactly 100

For, (by the last of the three compendious Rules before delivered in *Sect.* 10. of this *Chapt.* for the multiplication of Binomials and Residuals,) the Product of the first and fourth numbers is . . . . $\sqrt{101} - 1$

Likewise, the Product of the second and third number is . . . $\sqrt{101} + 1$

Lastly, the two last preceding Products being multiplied one into another (by the same Rule) make 100

QUEST. 4.

1. If a, b, c, be such Quantities that . . . . . . $aa + ca = b$
   What is the value of a?
2. *Answ.* By the Canon in *Sect.* 6. *Ch.* 15. *Book* 1. . . . . . $a = \sqrt{:b + \frac{1}{4}cc:} - \frac{1}{2}c$

By which value of a, the Equation propos'd may be expounded, as is manifest by the following

Demonstration.

3. If . . . . $a = \sqrt{:b + \frac{1}{4}cc:} - \frac{1}{2}c$
4. Then consequently by adding $\frac{1}{2}c$ to each part, . . $a + \frac{1}{2}c = \sqrt{:b + \frac{1}{4}cc:}$
5. And by multiplying each part of the last Equation into it self, ... $aa + ca + \frac{1}{4}cc = b + \frac{1}{4}cc$
6. Wherefore, by subtracting $\frac{1}{4}cc$ from each part, there remains . . . . $aa + ca = b$

   Which was to be proved.

Note. This Demonstration is formed in the way of Composition by the steps of the Resolution of the same Question in *Sect.* 5. *Chap.* 15. *Book* 1. but in a retrograde or backward order; for the first step in the Composition, (or Demonstration) is the last in the Resolution; the second step in the Composition is the last but one in the Resolution; and so by returning backwards by the steps of the Resolution, the Demonstration ends in the Equation propos'd to be resolved. But this is largely handled in my fourth Book of Algebraical Elements.

QUEST. 5.

1. If a, b, k, be such Quantities that . . . . . . $aa - ba = k$
   What is the value of a?
2. *Answ.* By the Canon in *Sect.* 8. *Ch.* 15. *Book* 1. . . . . . $a = \frac{1}{2}b + \sqrt{:k + \frac{1}{4}bb:}$

By which value of a, the Equation propos'd may be expounded, as appears by the following

Demonstration.

3. If . . . . $a = \frac{1}{2}b + \sqrt{:k + \frac{1}{4}bb:}$
4. Then by subtracting $\frac{1}{2}b$ from each part, . . . $a - \frac{1}{2}b = \sqrt{:k + \frac{1}{4}bb:}$
5. And by multiplying each part of the last Equation into it self, ... $aa - ba + \frac{1}{4}bb = k + \frac{1}{4}bb$
6. Wherefore by subtracting $\frac{1}{4}bb$ from each part, $aa - ba = k$

   Which was to be proved.

QUEST. 6.

1. If c and n be put for such known Quantities, that . . . . n not $\sqsubset \frac{1}{4}cc$,
2. And if a be put for a Quantity unknown, and . . . $ca - aa = n$;
   What is the value of a?
3. *Answ.* By the Canon in *Sect.* 10. *Chapt.* 15. *Book* 1. these two values of a will be found out, *viz.* . . . . $a = \begin{cases} \frac{1}{2}c + \sqrt{:\frac{1}{4}cc - n:} \\ \frac{1}{2}c - \sqrt{:\frac{1}{4}cc - n:} \end{cases}$

By each of which values of a, the Equation proposed in the second step may be expounded, *viz.* if either $\frac{1}{2}c + \sqrt{:\frac{1}{4}cc - n:}$ or, $\frac{1}{2}c - \sqrt{:\frac{1}{4}cc - n:}$ be put equal to a, then $ca - aa = n$.

Demon-

Demonstration.

4. First, if . . . . $a = \frac{1}{2}c + \sqrt{:\frac{1}{4}cc - n:}$
5. Then by subtracting $\frac{1}{2}c$ from each part, . . $a - \frac{1}{2}c = \sqrt{:\frac{1}{4}cc - n:}$
6. And by multiplying each part of the last Equation into it self, ... $aa - ca + \frac{1}{4}cc = \frac{1}{4}cc - n$
7. And by adding ca to each part, ... $aa + \frac{1}{4}cc = \frac{1}{4}cc + ca - n$
8. And by subtracting $\frac{1}{4}cc$ from each part, . . $aa = ca - n$
9. And by adding n to each part, ... $aa + n = ca$
10. Wherefore by subtracting aa from each part, $n = ca - aa$
11. That is, ... $ca - aa = n$

Which was to be proved.

Again, If . . . . $a = \frac{1}{2}c - \sqrt{:\frac{1}{4}cc - n:}$
12. Then by adding $\sqrt{:\frac{1}{4}cc - n:}$ to each part, $a + \sqrt{:\frac{1}{4}cc - n:} = \frac{1}{2}c$
13. And by subtracting a from each part, . . $\sqrt{:\frac{1}{4}cc - n:} = \frac{1}{2}c - a$
14. And by multiplying each part of the last Equation into it self, ... $\frac{1}{4}cc - n = \frac{1}{4}cc - ca + aa$
15. And by adding ca to each part, ... $ca + \frac{1}{4}cc - n = \frac{1}{4}cc + aa$
16. And subtracting $\frac{1}{4}cc$ from each part, . . $ca - n = aa$
17. And by adding n to each part, ... $ca = aa + n$
18. Wherefore by subtracting aa from each part, $ca - aa = n$

Which was to be proved.

QUEST. 7.

1. If b and c be put for such known Quantities, that c is greater than b, but less than $2b$; and if a be put for a Quantity unknown;
2. And if . . . . $\sqrt{\frac{aa + 3bb}{4}} + \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{baa}{c}}$;

What is the value of a?

RESOLUTION.

3. Because the Squares of equal Quantities are also equal, by multiplying each part of the Equation in the second step into it self, this is produced, *viz.*

$$\frac{aa}{2} + \sqrt{\frac{a^4 - 9b^4}{4}} = \frac{baa}{c}.$$

4. Then to the end the Surd Quantity in the Equation in the third step may solely make one part of an Equation, let $\frac{aa}{2}$ be subtracted from each part of that Equation, and this will remain, *viz.*

$$\sqrt{\frac{a^4 - 9b^4}{4}} = \frac{baa}{c} - \frac{aa}{2} = \frac{2baa - caa}{2c}.$$

5. And to the end the Radical sign in the first part of the last Equation may vanish, let each part be multiplied by it self, so an Equation in Rational quantities will be produced, *viz.*

$$\frac{a^4 - 9b^4}{4} = \frac{4bba^4 - 4bca^4 + cca^4}{4cc}.$$

6. And by reducing the last Equation to a common Denominator $4cc$, and then by multiplying each part by the same $4cc$, this Equation in Integers will be produced, *viz.*

$$cca^4 - 9b^4cc = 4bba^4 - 4bca^4 + cca^4.$$

7. And from the Equation in the last preceding step, after due reduction is made to make those Quantities wherein a^4 is found to possess one part, this following Equation ariseth, *viz.* $4bca^4 - 4bba^4 = 9b^4cc.$

8. Then by dividing each part of the last Equation by $4bc - 4bb$, to the end that a^4 may stand alone, this Equation ariseth, *viz.*

$$a^4 = \frac{9b^4cc}{4bc - 4bb} = \frac{9b^3cc}{4c - 4b}.$$

9. But . . . . . . $\frac{9bbcc}{4}$ into $\frac{b}{c - b} = \frac{9b^3cc}{4c - 4b}.$

10. There-

10. Therefore from the two last preceding Equations, by exchanging equal Quantities, this Equation ariseth, *viz.*

$$a^4 = \frac{9bbcc}{4} \text{ into } \frac{b}{c-b}.$$

11. And by extracting the square Root out of each part of the Equation in the tenth step, this ariseth;

$$aa = \frac{3bc}{2} \text{ into } \sqrt{\frac{b}{c-b}}.$$

12. Wherefore by extracting the square Root out of each part of the Equation in the eleventh step, the desired value of a is discovered, *viz.*

$$a = \sqrt{:\frac{3bc}{2} \text{ into } \sqrt{\frac{b}{c-b}}:}$$

An Example of Quest. 7. *in Numbers.*

13. If . . . . $b = 16$;
14. And . . . . $c = 25$;
15. And . . . . $a =$ a number unknown;
16. And if . . . . . . $\sqrt{\frac{aa + 3bb}{4}} + \frac{\sqrt{aa - 3bb}}{4} = \sqrt{\frac{baa}{c}}$;

What is the number a?

17. *Answ.* From the thirteenth, fourteenth, and twelfth steps, $a = \sqrt{800}$, or $20\sqrt{2}$. By which value of a the Equation propos'd may be expounded, as will appear by

The Proof.

18. If $b = 16$, $c = 25$, and $a = \sqrt{800}$; Then it will follow, that

$$\sqrt{\frac{aa + 3bb}{4}} + \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{baa}{c}} \quad (= 8\sqrt{8}, \text{ or, } \sqrt{512}.)$$

Note. The numbers to express the values of b and c must not be taken at pleasure but such, that the number c may exceed the number b, and be less than $2b$, as is prescribed in the Question; the former part of which Determination is discovered by the Denominator $c - b$ of the surd Fraction in the twelfth step, and the latter part of the Determination is manifest by the latter part of the Equation in the fourth step, where caa is to be subtracted from $2baa$, which cannot be done so as to leave a Remainder greater than nothing, unless c be less than $2b$.

Sect. XVIII. *An Explication of* Fran. van Schooten's *General Rule, to extract what Root you please out of any Binomial in numbers; having such a Binomial Root as is desired.*

Preparation.

First, if the given Binomial hath Fractions in it, it must be freed from them, by multiplying the Binomial by their Denominator. As, for example, to extract $\sqrt{(3)}$, that is, the cubick Root, out of $\sqrt{242} + 12\frac{1}{2}$, I multiply the Binomial by 2, and it makes $\sqrt{968} + 25$; for $\sqrt{242}$ multiplied by $\sqrt{4}$, (that is, by 2,) produceth $\sqrt{968}$; and $12\frac{1}{2}$ into 2, makes 25. Likewise, if there be proposed $\sqrt{\frac{242}{5}} + \sqrt{\frac{125}{4}}$, I first multiply it by $\sqrt{5}$, and it makes $\sqrt{242} + \frac{25}{2}$; then this Binomial multiplied by 2 produceth (as before) $\sqrt{968} + 25$; and so of others.

Secondly, if neither of the two parts of the given Binomial be Rational, it must be reduced by Multiplication or Division to another Binomial that shall have one of its parts Rational; which Reduction may alwayes be done by the multiplication of either part, but often times more briefly by the multiplication or division of the lesser number. As, for example, $\sqrt{242} + \sqrt{243}$ may be multiplied by $\sqrt{242}$, and it makes $242 + \sqrt{58806}$; but more compendiously by $\sqrt{2}$, and there comes forth $22 + \sqrt{486}$. After the same manner, $\sqrt{(3)}3993 + \sqrt{(6)}17578125$ may be first multiplied by $\sqrt{(3)}3993$, and the Product again by $\sqrt{(3)}3993$, so there will be produced another Binomial whose Rational part is the absolute number 3993; but more briefly by $\sqrt{(3)}9$, and there will

be produced another Binomial whose Rational part is 33; and yet more compendiously, if the Binomial proposed be divided by $\sqrt{(3)}3$, there will arise $11 + \sqrt{125}$.

But here is to be noted, that when one part of a Binomial is Rational, whether it be of a Binomial first given, or of another deduced (as above) from that given, then also the Square of the other part ought to be Rational, otherwise no Root can be extracted out of the Binomial or the other deduced from it.

Thirdly, to extract $\sqrt{(6)}$ out of a given Binomial qualified as above is supposed, we must first extract the square Root, and then out of this the cubick Root; and to extract $\sqrt{(9)}$, we must first extract $\sqrt{(3)}$, and then out of the cubick Root found out we must again extract $\sqrt{(3)}$; and so of any other Root whose Index is a Composit number. But as to the extraction of the square Root out of a Binomial, a Rule hath been already given and exemplified in the preceding *Sect.* 16. so that here there is need only that I shew how to extract $\sqrt{(3)}$, $\sqrt{(5)}$, $\sqrt{(7)}$, $\sqrt{(11)}$, and such like, whose Indices are Prime numbers.

Fourthly, to extract $\sqrt{(3)}$, $\sqrt{(5)}$, $\sqrt{(7)}$, or the like Root whose Index is a Prime number, we must first of all try whether out of the given Binomial there can be extracted a binomial Root which hath one part Rational; but that may be discovered by subtracting the Square of the lesser part of the given Binomial from the Square of the greater, and extracting the Root out of the Remainder; to wit, the cubick Root, if $\sqrt{(3)}$ be to be extracted out of the given Binomial; or the Root of the fifth Power, if $\sqrt{(5)}$ be to be extracted; and so of others: For if the Root of the said Remainder be not a Rational number, then the Binomial Root sought will certainly want a Rational part, *viz.* each of its parts will be surd; in which case, in order to extract that Root, the given Binomial must be multiplied by the difference of the Squares of the parts, if the Question be concerning the extraction of the cubick Root; or by the Square of the said difference, if $\sqrt{(5)}$ be sought; or by the Cube of the same difference, if $\sqrt{(7)}$ be required; or by the fifth Power of the said difference, if $\sqrt{(11)}$ be sought; and so of the rest. By which multiplication another Binomial will alwayes be produced, wherein the Root of the difference of the Squares of the parts will be the same with the difference of the Squares of the parts of the former Binomial.

As, to extract the cubick Root out of $25 + \sqrt{968}$; I first subtract 625, the Square of 25, from 968, the Square of $\sqrt{968}$, and there remains 343, whose cubick Root 7 is a Rational number: which argues that the Root of the given Binomial, if there can be a Root extracted out of it, is a Binomial which hath one of its parts Rational.

Likewise, to extract the cubick Root out of $22 + \sqrt{486}$, we must subtract 484, the Square of 22, from 486, and extract the cubick Root out of the Remainder 2; but because that cannot be done exactly, it shews that the cubick Root of $22 + \sqrt{486}$ wants a Rational part; and therefore $22 + \sqrt{486}$ must be multiplied by the said Remainder 2, that there may be a Binomial $44 + \sqrt{1944}$, wherein the cubick Root of the difference of the Squares of the parts is 2.

So to extract $\sqrt{(5)}$ out of $11 + \sqrt{125}$; because 121 the Square of 11 subtracted from 125 leaves 4, which considered as a fifth Power hath not an exact Rational Root, we must multiply $11 + \sqrt{125}$ by 16 the Square of 4, that there may come forth $176 + \sqrt{32000}$, where $\sqrt{(5)}$ of the difference of the Squares of the parts is 4.

Again, to extract $\sqrt{(7)}$ out of $338 + \sqrt{114242}$, wherein the difference of the Squares of the parts is 2; because this 2 is not the seventh Power of any Rational number, the given Binomial may be multiplied by 8, that is, by the Cube of 2, and it makes $2704 + \sqrt{7311488}$, wherein the $\sqrt{(7)}$ of the difference of the Squares of the parts is 2.

The RULE.

When a Binomial given, or another deduced from it (if need be) by the precedent Preparation, is such, that one of its parts, and the Square of the other part, as also the Root of the difference of the Squares of the parts, (to wit, the cubick Root when $\sqrt{(3)}$, or $\sqrt{(5)}$ when $\sqrt{(5)}$ is sought) are Rational whole numbers; then out of a Binomial so qualified, $\sqrt{(3)}$, or $\sqrt{(5)}$, or $\sqrt{(7)}$, *&c.* may be extracted, if it hath such a Root, in manner following, *viz.*

First, extract the Root of the difference of the Squares of the parts of the Binomial qualified as aforesaid, *viz.* the cubick Root, when $\sqrt{(3)}$ is sought; but $\sqrt{(5)}$ when $\sqrt{5}$, or $\sqrt{(7)}$ when $\sqrt{(7)}$, *&c.* which Root so extracted is to be reserved for a Dividend.

Secondly,

Secondly, find out a Rational number a little greater than the Root fought, with this caution, that the Rational number found out may not exceed the faid Root above $\frac{1}{2}$, which may eafily be done by Vulgar Arithmetick, and take the faid Rational number for a Divifor.

Thirdly, divide the faid Dividend by the faid Divifor, and if the Rational part of the given Binomial be greater than the other part, add the Quotient to the faid Rational Divifor, and the half of the greateft whole number contained in the fumm fhall be the Rational part of the Root fought; then from the Square of that Rational part fubtract the Root of the difference of the Squares of the parts, (to wit, the Dividend firft found out as above,) fo the Remainder fhall be the Square of the other part, when fuch a Root as was required can be extracted out of the given Binomial; which you may eafily try by multiplying this Root found out into it felf, according to the degree of the Power reprefented by the given Binomial: for the Root found out being multiplied into it felf cubically, if $\sqrt{(3)}$ was fought; or, five times into it felf, if $\sqrt{(5)}$ was fought, ought to produce the given Binomial.

But if the Rational part of the given Binomial be lefs than the other part, then after you have found out the Quotient as above, fubtract it from the Rational Divifor, and the half of the greateft whole number contained in the Remainder fhall be the Rational part of the Root fought; to the Square of which part if there be added the Dividend firft found out as above; the fumm will be the Square of the other part, when the Binomial propofed hath a Root; but by multiplying the Root found out into it felf (as before) you may eafily try whether it be a true Root or not.

Example 1. *To extract the Cubick Root out of* $20 + \sqrt{392}$.

Firft, the difference of the Squares of the parts of the given Binomial, *viz.* the excefs of 400, the Square of 20, above 392, the Square of $\sqrt{392}$ is 8, whofe cubick Root 2 I referve for a Dividend.

Secondly, I feek a Rational number that may be greater than the cubick Root of $20 + \sqrt{392}$, (the given Binomial,) yet fo that the excefs may not be greater than $\frac{1}{2}$; to which end I extract the fquare Root of 392, and find it to be greater than 19, but lefs than 20; then to 20 the Rational part of the given Binomial I add 19 and 20 feverally, and it makes 39 and 40; which are the neareft Rational whole numbers that can exprefs the true value of the given Binomial; whence the cubick Root thereof will be found greater than 3, but lefs than $3\frac{1}{2}$: this $3\frac{1}{2}$, which, according to the Caution before given, exceeds the true cubick Root of the given Binomial by an excefs not greater than $\frac{1}{2}$, I referve for a Divifor.

Thirdly, I divide 2, the Dividend before referved, by the faid Divifor $3\frac{1}{2}$, and the Quotient is $\frac{4}{7}$. Now becaufe 20 the Rational part of the given Binomial is greater than the other part $\sqrt{392}$, I add the faid Quotient $\frac{4}{7}$ to the faid Divifor $3\frac{1}{2}$, and it makes the fumm $4\frac{1}{14}$, wherein the greateft whole number is 4, whofe half is 2 the Rational part of the Root fought; by the help of which Rational part, the other part is eafily difcovered; for if from 4 the Square of the faid 2, you fubtract 2, the cubick Root of the difference of the Squares of the parts of the given Binomial, there will remain 2 the Square of the other part. So that $2 + \sqrt{2}$ is the cubick Root of $20 + \sqrt{392}$ the Binomial propofed, as will appear by the Proof: For $2 + \sqrt{2}$ being multiplied into it felf cubically produceth $20 + \sqrt{392}$; and for the fame reafon, $2 - \sqrt{2}$ is the cubick Root of $20 - \sqrt{392}$.

Example 2. *To extract the Cubick Root out of* $44 + \sqrt{1944}$.

Firft, the cubick Root of the difference of the Squares of the parts is 2 for a Dividend: Secondly, the fquare Root of 1944 is greater than 44, but lefs than 45; thefe added feverally to 44 the Rational part of the given Binomial, make 88 and 89, whofe cubick Roots being extracted, do fhew that the cubick Root of the given Binomial is greater than 4, but lefs than $4\frac{1}{2}$; this Rational number $4\frac{1}{2}$, which according to the Caution before given exceeds the true Root fought by an excefs not greater than $\frac{1}{2}$, I take for a Divifor: Thirdly, I divide the faid Dividend 2 by the faid Divifor $4\frac{1}{2}$, and the Quotient is $\frac{4}{9}$, which I fubtract from the faid $4\frac{1}{2}$; (I fubtract, becaufe 44 the Rational part of the given Binomial is lefs than the other part $\sqrt{1944}$,) and there remains $4\frac{1}{18}$; then the half of 4, the greateft whole number contained in $4\frac{1}{18}$, is 2, which is the Rational part of the Root fought: Laftly, to 4 the Square of the faid 2, I add 2 the cubick Root of the difference of the Squares of the parts, and it makes 6 the Square of the other part. So that $2 + \sqrt{6}$ is the cubick Root fought, as will appear by the Proof: For if it be multiplied into it felf cubically, it

 produceth

produceth $44 + \sqrt{1944}$ the Binomial propoſed; and for the ſame reaſon, $\sqrt{6} - 2$ is the cubick Root of $\sqrt{1944} - 44$.

Example 3. *To extract* $\sqrt{(5)}$ *out of* $176 + \sqrt{32000}$.

Firſt, the difference of the Squares of the parts will be found 1024, whoſe $\sqrt{(5)}$ is 4 for a Dividend: Secondly, the ſumm of the parts will be found greater than 354, but leſs than 355; and conſequently $\sqrt{(5)}$ of the ſumm of the parts is greater than 3, but leſs than $3\frac{1}{2}$: Thirdly, by the ſaid $3\frac{1}{2}$ I divide the ſaid 4, and the Quotient is $1\frac{1}{7}$, which I ſubtract from the ſaid Diviſor $3\frac{1}{2}$ (becauſe the Rational part of the given Binomial is leſs than the other part) and there remains $2\frac{5}{14}$; then the half of 2 (the greateſt whole number contained in $2\frac{5}{14}$) is 1, the Rational part of the Root ſought: Laſtly, the Square of the ſaid 1, to wit, 1, added to 4 (the $\sqrt{(5)}$ of the difference of the Squares of the parts of the given Binomial) makes 5 the Square of the other part. So that $1 + \sqrt{5}$ is the $\sqrt{(5)}$ of the given Binomial $176 + \sqrt{32000}$, at leaſt if any $\sqrt{(5)}$ can be extracted out of the ſame; but $1 + \sqrt{5}$ multiplied into it ſelf five times makes $176 + \sqrt{32000}$: therefore $1 + \sqrt{5}$ is manifeſtly the deſired $\sqrt{(5)}$ of $176 + \sqrt{32000}$.

Example 4. *To extract* $\sqrt{(7)}$ *out of* $2704 + \sqrt{7311488}$.

Firſt, the $\sqrt{(7)}$ of the difference of the Squares of the parts is 2 for a Dividend: Secondly, the value of the given Binomial will be found greater than 5407, but leſs than 5408; whence the $\sqrt{(7)}$ thereof will be diſcovered to be greater than 3, but leſs than $3\frac{1}{2}$: Thirdly, by the ſaid $3\frac{1}{2}$ I divide the Dividend before found 2, and the Quotient is $\frac{4}{7}$, which I add to the Diviſor $3\frac{1}{2}$, (becauſe the Rational part 2704 is greater than the other part) and it makes the ſumm $4\frac{1}{14}$; and therefore 2, the half of the greateſt whole number contained in $4\frac{1}{14}$, is the Rational part of the Root ſought: Laſtly, from 4, the Square of the ſaid 2, I ſubtract 2, to wit, $\sqrt{(7)}$ of the difference of the Squares of the parts of the given Binomial, and there remains 2 the Square of the other part. So that $2 + \sqrt{2}$ is the deſired $\sqrt{(7)}$ of the given Binomial $2704 + \sqrt{7311488}$; for this is the ſeventh Power of $2 + \sqrt{2}$, as will appear by Multiplication.

But here is to be noted, that when the given Binomial hath been multiplied or divided by ſome number, and thereby reduced to another Binomial, and the Root of this latter is found out, we muſt divide or multiply the Root found out by the Root of the number by which the Binomial was multiplied or divided; ſo there will come forth the Root of the given Binomial.

As, for example, becauſe to extract the cubick Root out of $\sqrt{242} + 12\frac{1}{2}$, we firſt multiplied this Binomial by 2 and found $25 + \sqrt{968}$, whoſe cubick Root by the Rule before given will be fouud $1 + \sqrt{8}$; this muſt be divided by $\sqrt{(3)}2$, and the Quotient $\sqrt{(3)}\frac{1}{2} + \sqrt{(6)}128$ ſhall be the cubick Root of $\sqrt{242} + 12\frac{1}{2}$ the Binomial propoſed.

But that the reaſon of the ſaid Diviſion by $\sqrt{(3)}2$ may the more clearly appear, let there be put $d = 1 + \sqrt{8}$, then it follows that $ddd = 25 + \sqrt{968}$, and $\frac{ddd}{2} = \sqrt{242} + 12\frac{1}{2}$ (the Binomial propoſed.) Therefore by extracting the cubick Root out of each part of the laſt Equation, there ariſeth $\sqrt{(3)}\frac{ddd}{2}$, that is, $\frac{d}{\sqrt{(3)}2}$, $= \sqrt{(3)}:\overline{\sqrt{242} + 12\frac{1}{2}}$: But by ſuppoſition $d = 1 + \sqrt{8}$; therefore $1 + \sqrt{8}$ divided by $\sqrt{(3)}2$, that is to ſay, $\sqrt{(3)}\frac{1}{2} + \sqrt{(6)}128$ ſhall be the cubick Root of $\sqrt{242} + 12\frac{1}{2}$: which was to be ſhewn.

Example 2. *To extract* $\sqrt{(3)}$ *out of* $\sqrt{\tfrac{242}{5}} + \sqrt{\tfrac{125}{4}}$.

Firſt, to prepare it for extraction, we multiplied by $\sqrt{5}$, and found $\sqrt{242} + 12\frac{1}{2}$, whoſe $\sqrt{(3)}$ (as appears in the laſt preceding Example) is $\sqrt{(3)}\frac{1}{2} + \sqrt{(6)}128$, which divided by $\sqrt{(6)}5$ gives the Quotient $\sqrt{(6)}\frac{1}{20} + \sqrt{(6)}\frac{128}{5}$ for the deſired cubick Root of $\sqrt{\tfrac{242}{5}} + \sqrt{\tfrac{125}{4}}$. The reaſon of which diviſion by $\sqrt{(6)}5$ may be thus manifeſted, let there be put $d = \sqrt{(3)}\frac{1}{2} + \sqrt{(6)}128$; then it follows that $ddd = \sqrt{242} + 12\frac{1}{2} = \sqrt{\tfrac{242}{5}} + \sqrt{\tfrac{125}{4}}$ into $\sqrt{5}$, whence $\frac{ddd}{\sqrt{5}} = \sqrt{\tfrac{242}{5}} + \sqrt{\tfrac{125}{4}}$; therefore the cubick Root of each part of the laſt Equation being extracted there ariſeth $\sqrt{(3)}\frac{ddd}{\sqrt{5}}$, that is, $\frac{d}{\sqrt{(6)}5}$ (for $\sqrt{(3)}$ of $\sqrt{5}$ is $\sqrt{(6)}5$) $= \sqrt{(3)}:\overline{\sqrt{\tfrac{242}{5}} + \sqrt{\tfrac{125}{4}}}$: But by ſuppoſition,

$d = \sqrt{(3)}\frac{1}{2}$

$d = \sqrt{(3)}\frac{1}{2} + \sqrt{(6)}128$; therefore $\sqrt{(3)}\frac{1}{2} + \sqrt{(6)}128$ divided by $\sqrt{(6)}5$ gives the true cubick Root of $\sqrt{\frac{242}{5}} + \sqrt{\frac{125}{4}}$: which was to be ſhewn.

Example 3. *To extract* $\sqrt{(3)}$ *out of* $\sqrt{242} + \sqrt{243}$.

Firſt, (according to the ſecond Rule of the precedent Preparation) I multiply it by $\sqrt{2}$, and there comes forth $22 + \sqrt{486}$; this multiplied by 2 (according to the fourth preparatory Rule) makes $44 + \sqrt{1944}$, whoſe cubick Root (as before hath been ſhewn) is $2 + \sqrt{6}$, which muſt be divided by $\sqrt{2}$ and there will come forth $\sqrt{2} + \sqrt{3}$ for the cubick Root ſought of $\sqrt{242} + \sqrt{243}$. But to manifeſt the reaſon of dividing $2 + \sqrt{6}$ by $\sqrt{2}$; let there be put $d = 2 + \sqrt{6}$, then it follows that $ddd = 44 + \sqrt{1944} = \overline{22 + \sqrt{486}}$ into 2, whence $\frac{ddd}{2} = 22 + \sqrt{486}$, and this Equation divided by $\sqrt{2}$ (becauſe in the Preparation we multiplied by $\sqrt{2}$) gives $\frac{ddd}{\sqrt{8}} = \sqrt{242} + \sqrt{243}$; therefore $\sqrt{(3)}$ being extracted out of each part of the laſt Equation there ariſeth $\sqrt{(3)}\frac{ddd}{\sqrt{8}}$, that is, $\frac{d}{\sqrt{(6)}8}$, or $\frac{d}{\sqrt{2}}$, $= \sqrt{(3)}: \overline{\sqrt{242} + \sqrt{243}}$: But by ſuppoſition, $d = 2 + \sqrt{6}$; therefore $2 + \sqrt{6}$ divided by $\sqrt{2}$, *viz.* the Quotient $\sqrt{2} + \sqrt{3}$, ſhall be the cubick Root of $\sqrt{242} + \sqrt{243}$: Which was to be ſhewn.

Example 4. *To extract* $\sqrt{(5)}$ *out of* $\sqrt{(3)}3993 + \sqrt{(6)}17578125$.

Firſt, (according to the ſecond preparatory Rule) I divide the given Binomial by $\sqrt{(3)}3$, and then (according to the fourth preparatory Rule) I multiply the Quotient $\sqrt{(3)}1331 + \sqrt{(6)}1953125$ by 16, and there comes forth $176 + \sqrt{32000}$, whoſe $\sqrt{(5)}$ (as hath before been ſhewn) is $1 + \sqrt{5}$. Now this Root $1 + \sqrt{5}$ divided by $\sqrt{(5)}16$, and the Quotient multiplied by $\sqrt{(15)}3$ will diſcover the true $\sqrt{(5)}$ of $\sqrt{(3)}3993 + \sqrt{(6)}17578125$; the reaſon of which Diviſion and Multiplication may be made manifeſt thus; let there be put $d = 1 + \sqrt{5}$, then it follows that $ddddd = 176 + \sqrt{32000}$; and by dividing each part of the laſt Equation by 16, (becauſe in the preparatory work we multiplied by 16) there ariſeth $\frac{ddddd}{16} = \sqrt{(3)}1331 + \sqrt{(6)}1953125$: and by multiplying each part of this Equation by $\sqrt{(3)}3$, there will be produced $\frac{ddddd \times \sqrt{(3)}3}{16} = \sqrt{(3)}3993 + \sqrt{(6)}17578125$: Therefore $\sqrt{(5)}$ being extracted out of each part of the laſt Equation, there will ariſe $\sqrt{(5)}\frac{ddddd \times \sqrt{(3)}3}{16}$, that is, $\frac{d\sqrt{(15)}3}{\sqrt{(5)}16}$ equal to $\sqrt{(5)}$ of $\sqrt{(3)}3993 + \sqrt{(6)}17578125$. But by ſuppoſition, $d = 1 + \sqrt{5}$; therefore $1 + \sqrt{5}$ multiplied into $\sqrt{(15)}3$, and the Product divided by $\sqrt{(5)}16$; or $1 + \sqrt{5}$ divided by $\sqrt{(5)}16$, and the Quotient multiplied by $\sqrt{(15)}3$ produceth the true $\sqrt{(5)}$ of $\sqrt{(3)}3993 + \sqrt{(6)}17578125$: which was to be ſhewn.

The Demonſtration follows.

The certainty of the preceding Rule will be made manifeſt by the three following Propoſitions.

PROP. 1.

If a Binomial whereof one part and the Square of the other are Rational numbers be multiplied into it ſelf cubically, there will be produced another Binomial, the Square of whoſe leſſer part being ſubtracted from the Square of the greater part, leaves a cubick number, to wit, the Cube of the difference of the Squares of the parts of the Root or firſt Binomial.

To make this manifeſt, let there be propoſed the Binomial $b + \sqrt{d}$, this multiplied into it ſelf cubically produceth $bbb + 3bb\sqrt{d} + 3bd + d\sqrt{d}$, to wit, the Cube of $b + \sqrt{d}$. Here you are to note well, that although in that Cube there be four parts or members, yet they are to be eſteemed but as two, one of which, to wit, $bbb + 3bd$ may deſign a Rational number, and the other, $\overline{3bb\sqrt{d} + d\sqrt{d}}$ (or $\overline{3bb + d} \times \sqrt{d}$) an irrational or ſurd number whoſe Square is Rational; whence it is manifeſt, firſt, that the Cube of a Binomial is alſo a Binomial, *viz.* $b + \sqrt{d}$ multiplied into it ſelf cubically produceth this

Binomial

Binomial $bbb + 3bd$ more $3bb\sqrt{d} + d\sqrt{d}$ (or $\overline{3bb + d} \times \sqrt{d}$;) secondly, the Rational part $bbb + 3bd$ is manifestly composed of the Cube of the Rational part of the Root and of the triple Product made by the multiplication of the same Root into the Square of its other part; and lastly, the difference of the Squares of the said parts $bbb + 3bd$ and $3bb\sqrt{d} + d\sqrt{d}$ is equal to the Cube of $bb - d$, or of $d - bb$, *viz.* to the Cube of the difference of the Squares of the parts of the Root $b + \sqrt{d}$: For the Squares of $bbb + 3bd$ and $3bb\sqrt{d} + d\sqrt{d}$ are $bbbbbb + 6bbbbd + 9bbdd$ and $9bbbbd + 6bbdd + ddd$; and if these Squares be subtracted one from the other, the Remainder is either $bbbbbb - 3bbbbd + 3bbdd - ddd$, which is the Cube of $bb - d$; or else the Remainder is $ddd - 3bbdd + 3bbbbd - bbbbbb$, which is the Cube of $d - bb$.

To illustrate this Proposition by Numbers, let there be put $b = 2$ and $\sqrt{d} = 6$; hence the Binomial $2 + \sqrt{6}$ multiplied into it self cubically produceth the Binomial $44 + \sqrt{1944}$, wherein the difference of the Squares of the parts (*viz.* the Remainder when 1936 the Square of 44 is subtracted from 1944 the Square of $\sqrt{1944}$,) is 8, to wit, the Cube of the difference of the Squares of the parts of the binomial Root $2 + \sqrt{6}$.

Likewise this Binomial $2 + \sqrt{2}$ multiplied into it self cubically produceth the Binomial $20 + \sqrt{392}$, wherein the difference of the Squares of the parts, to wit, 8, is the Cube of the difference of the Squares of the parts of the Root $2 + \sqrt{2}$.

The same properties adhere also to a Residual Root, *viz.* the Cube of the Residual Root $b \backsim \sqrt{d}$ is also a Residual, to wit, $\overline{bbb + 3bd} \backsim \overline{3bb\sqrt{d} + d\sqrt{d}}$, (or $\overline{3bb + d} \times \sqrt{d}$;) and the difference of the Squares of the parts of the latter Residual is equal to the Cube of the difference of the Squares of the parts of the Root or first Residual.

PROP. 2.

If a Binomial whereof one part and the Square of the other are Rational numbers, be multiplied by the difference of the Squares of the parts, the Product will be another Binomial, wherein the difference of the Squares of the parts is a Cubick number, to wit, the Cube of the difference of the Squares of the parts of the Root multiplied.

To make this manifest, let there be proposed the Binomial $b + \sqrt{d}$, and suppose b greater than $\sqrt{d}$; then $b + \sqrt{d}$ multiplied by $bb - d$, the difference of the Squares of the parts, will produce this Binomial, to wit, $\overline{bbb - bd}$ more $\overline{bb\sqrt{d} - d\sqrt{d}}$, the Squares of whose parts are $bbbbbb - 2bbbbd + bbdd$ and $bbbbd - 2bbdd + ddd$; then this latter Square subtracted from the former leaves $bbbbbb - 3bbbbd + 3bbdd - ddd$, which is the Cube of $bb - d$ the difference of the Squares of the parts of the first Binomial $b + \sqrt{d}$. The same property would appear if we supposed b less than $\sqrt{d}$.

To illustrate this Proposition by Numbers, suppose $b = 22$, and $\sqrt{d} = 486$; whence the Binomial $22 + \sqrt{486}$ multiplied by 2, the difference of the Squares of the parts, produceth the Binomial $44 + \sqrt{1944}$; wherein the difference of the Squares of the parts is 8, which is the Cube of 2, the difference of the Squares of the parts of the former Binomial $22 + \sqrt{486}$.

PROP. 3.

If the difference of the Squares of any two numbers be divided by a number which doth not exceed the summ of those two numbers above $\frac{1}{2}$; then the Quotient added to the said Divisor will give a number greater than the double of the greater of the said two numbers, but the excess will be less than unity: and if the said Quotient be subtracted from the said Divisor, the Remainder shall be greater than the double of the lesser of the two numbers, but this excess also shall be less than unity.

To manifest this, let a represent the greater of two numbers, and e the lesser; also, let b represent some Fraction not greater than $\frac{1}{2}$: then I say, first, $a + e + b + \frac{aa - ee}{a + e + b}$ is greater than $2a$; but the excess is less than 1, which I prove thus:

It is evident that $aa + ee + bb + 2ae + 2be + 2ba + aa - ee$ is greater than $2aa + 2ae + 2ba$; therefore by dividing each of those two Compound quantities by $a + e + b$, it follows that the first Quotient $a + e + b + \frac{aa - ee}{a + e + b}$ shall be greater than the latter Quotient $2a$; and if this quantity be subtracted from that, the Remainder $\frac{2be + bb}{a + e + b}$ will be less than 1. For by supposition b is not greater than $\frac{1}{2}$; therefore $2be$ is less than

$a + e$,

$a+e$, and bb less than b; and consequently the Numerator $2be+bb$ is less than the Denominator $a+e+b$: wherefore $\frac{2be+bb}{a+e+b}$ is less than 1.

After the same manner it may be proved that $a+e+b-\frac{aa-ee}{a+e+b}$ is greater than $2e$; but this excess also shall be less than 1: which was to be shewn.

Now to apply the preceding three Propositions to the Demonstration of the Rule before given, let it be required to extract the cubick Root out of the Binomial $100+\sqrt{7803}$, whose Rational part 100 is greater than the other part $\sqrt{7803}$. Here we may suppose $bbb+3bd$ to be 100, and $3bb\sqrt{d}+d\sqrt{d}$ (or $\overline{3bb+d}\times\sqrt{d}$) to be $\sqrt{7803}$; so that $bbb+3bd$ more $\overline{3bb+d}\times\sqrt{d}$ may design the given Binomial $100+\sqrt{7803}$; and its Cubick root $b+\sqrt{d}$ the Root sought, whose greater part may be b, and the lesser $\sqrt{d}$: Then, according to the Rule

To extract $\sqrt{(3)}$ *out of* . . $100+\sqrt{7803}$.

First, from the Square of 100, that is, from . . .	10000
Subtract the Square of $\sqrt{7803}$, that is, ...	7803
The Remainder is	2197
The Cubick root of that Remainder is	13 $(=bb-d.)$

Which Root 13 is (by *Prop.* 1.) equal to the difference of the Squares of the parts of the Binomial root sought.

Secondly, find out a Rational number greater than the summ of the parts of the Cubick root sought, with this Caution, that the excess may not be above $\frac{1}{2}$, *viz.*

To the greater part of the given Binomial, that is, to . .	100
Add the nearest value in whole numbers of the other part $\sqrt{7803}$, that is, ...	88 or 89
So the summ shews, that the value in whole numbers of the given Binomial falls between	188 and 189.

Whence the Cubick root of the given Binomial is greater than $5\frac{1}{2}$, but less than 6; so that the excess of 6 above the true Root sought is less than $\frac{1}{2}$.

Thirdly, having found out (as above) 13 the true difference of the Squares of the parts of the Cubick root sought, and 6 a Rational number which exceeds not the true summ of the same parts above $\frac{1}{2}$; we may by the help of *Prop.* 3, and 1. find out the parts severally in this manner, *viz.*

Divide the said	13
By the said	6
And the Quotient is	$2\frac{1}{6}$
Which added to the said Divisor 6, makes the summ	$8\frac{1}{6}$

Which summ $8\frac{1}{6}$ doth (by *Prop.*3.) exceed the double of the greater (to wit, the Rational) part of the Cubick Root sought, but the excess is less than 1; therefore $7\frac{1}{6}$ is less than the said double, but $8\frac{1}{6}$ is greater than the same: and consequently, because the said greater part is supposed to be a Rational whole number, the double thereof must necessarily be 8, (to wit,) the greatest whole number between $7\frac{1}{6}$ and $8\frac{1}{6}$,) and therefore the said part it self is 4: which being found out, it is easie to find the other part. For, (by *Prop.* 1.) if from 16 the Square of the said greater part 4, there be subtracted 13, the Cubick root of the difference of the Squares of the parts of the given Binomial, there will remain 3, the Square of the other part; so that the Cubick root found out is $4+\sqrt{3}$, which will appear by the Proof to be the true Root sought; for $4+\sqrt{3}$ being multiplied into it self cubically produceth the given Binomial $100+\sqrt{7803}$. And for the same reason $4-\sqrt{3}$ is the Cubick root of $100-\sqrt{7803}$.

Or more briefly, the Proof may be made thus.

To the Cube of 4 the Rational part of the Root found out, *viz.* to	64, that is, bbb
Add the Product of thrice that part multiplied into the Square of the Surd part found out, *viz.* the Product . .	36, that is, $3bd$
And it makes the summ	100, that is, $bbb+3bd$.

Which

Which summ is the same with the Rational part of the given Binomial, and therefore it proves that $4+\sqrt{3}$ is the Cubick root sought.

In like manner, to extract $\sqrt{(3)}$ out of $44+\sqrt{1944}$, where the Rational part 44 is less than the other part $\sqrt{1944}$; we may suppose (as before) $\overline{bbb+3bd}$ to be 44, and $\overline{3bb+d}\times\sqrt{d}$ (that is, $3bb\sqrt{d}+d\sqrt{d}$) to be $\sqrt{1944}$; so that $\overline{bbb+3bd}$ more $\overline{3bb+d}\times\sqrt{d}$ may design the given Binomial $44+\sqrt{1944}$, and its Cubick root $b+\sqrt{d}$ the Root sought; whose lesser part may be b, and the greater $\sqrt{d}$. Then, according to the Rule

To extract $\sqrt{(3)}$ *out of* . . $44+\sqrt{1944}$.

First, from the Square of $\sqrt{1944}$, *viz.* from . . .	1944
Subtract the Square of 44, ...	1936
The Remainder is	8
The Cubick root of that Remainder is	2 $(= d - bb.)$

Which Root 2 is (by *Prop.* 1.) equal to the difference of the Squares of the parts of the Binomial root sought.

Secondly, find out a Rational number greater than the summ of the parts of the Cubick root sought, with this Caution, that the excess may not be above $\frac{1}{2}$; which may be done thus, *viz.*

To the lesser part of the given Binomial, *viz.* to . .	44
Add the nearest value in whole numbers of the other part $\sqrt{1944}$, that is, ...	44 or 45
So the summ shews that the value in whole numbers of the given Binomial, falls between	88 and 89.

Whence the Cubick root of the given Binomial is greater than 4, but less than $4\frac{1}{2}$; so that the excess of $4\frac{1}{2}$ above the true Root sought is less than $\frac{1}{2}$.

Thirdly, having found out 2, the true difference of the Squares of the parts of the Cubick root sought; and $4\frac{1}{2}$ a Rational number which doth not exceed the true summ of the same parts above $\frac{1}{2}$; we may by the help of *Prop.* 3, and 1. find out the parts severally in this manner, *viz.*

Divide the said	2
By the said	$4\frac{1}{2}$
And it gives the Quotient	$\frac{4}{9}$
Which subtracted from the said Divisor $4\frac{1}{2}$, there remains . .	$4\frac{1}{18}$

Which Remainder $4\frac{1}{18}$ doth (by *Prop.* 3.) exceed the double of the lesser part (which in this Example is the Rational part) of the Cubick root sought, but the excess is less than 1; Therefore $3\frac{1}{18}$ is less than the said double, but $4\frac{1}{18}$ is greater than the same; and consequently because the said lesser part is a Rational whole number, the double thereof must necessarily be 4, to wit, the greatest whole number between $3\frac{1}{18}$ and $4\frac{1}{18}$, and therefore the said part it self is 2: which being found, it is easie to find the other part; for if to 4 the Square of the said lesser part 2, there be added 2 the Cubick root of the difference of the Squares of the parts of the given Binomial, the summ 6 shall be the Square of the other part. So that the Cubick root found out is $2+\sqrt{6}$, which will appear to be the true Cubick root sought; for $2+\sqrt{6}$ multiplied into it self cubically produceth the given Binomial $44+\sqrt{1944}$. And for the same reason $\sqrt{6}-2$ is the Cubick root of $\sqrt{1944}-44$.

Or more briefly, the Proof may be made thus:

To the Cube of 2, the Rational part of the Root found out, *viz.* to	8, that is, bbb
Add the Product of thrice that part multiplied into the Square of the Surd part found out, *viz.* the Product . .	36, that is, $3bd$
And the summ is	44, that is, $bbb+3bd$.

Which summ is the same with the Rational part of the given Binomial; and therefore it proves that $2+\sqrt{6}$ is the Cubick root sought.

Lastly, what hath here been shewn concerning the Demonstration of the Extraction of the Cubick Root, may easily be applied to the Extraction of the other Roots before mentioned; so that there is no need of farther discourse in this matter.

CHAP.

CHAP. X.

An Explication of Simon Stevin's *General Rule, to extract one Root out of any possible Equation in Numbers, either exactly, or very nearly true.*

I. EQuations falling under any of the Forms in the fourteenth and fifteenth Chapters of the first Book of these Elements, are capable (as hath there been shewn) of perfect Resolutions in Numbers; *viz.* the value of the Root or Roots sought in any of those Equations may be found out and exprest exactly, either by some Rational or Irrational number or numbers; but the perfect Resolution of all manner of Compound Equations in numbers, I have not found in any Author: and since an Exposition of the General Method of *Vieta*, the Rules of *Huddenius* and others to that purpose, would make a large Treatise, and after all leave the curious Analyst dissatisfied, I shall not clogg these Elements with a tedious discourse upon those difficult Rules, which at the best are exceeding tedious in Operation, and some of them uncertain too, but rather pursue my first Design, which was to explain Fundamentals, and such Rules as are certain and most important in this profound Art. However, I shall lead the industrious Learner a few steps farther in order to his understanding the Resolution of all manner of Compound Equations in numbers, and in this Chapter explain *Simon Stevin*'s General Rule, which with the help of the Rules in the following eleventh Chapter, will discover all the Roots of any possible Equation in numbers, either exactly, if they be Rational, or very nearly true if Irrational.

QUEST. 1.

If . . . . $aaa + 26a = 40188$, what is the number a?

RESOLUTION.

This Equation not falling under any of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. cannot be resolved by any of the Canons in that Chapter, and therefore according to *Simon Stevin*'s general Method I search out the number a by tryals, thus, *viz.*

1. I suppose . . . . $a = 1$
   Thence it follows that . . . . $aaa = 1$
   And . . . . $26a = 26$
   Therefore . . . . $aaa + 26a = 27$

Which 27 ought to have been 40188, but it's too little; whereby I find that by supposing a to be 1, I did not hit upon the true number a, and therefore I make another tryal, in like manner as before, *viz.*

2. I suppose . . . . $a = 10$
   Thence it follows that . . . . $aaa = 1000$
   And . . . . $26a = 260$
   Therefore . . . . $aaa + 26a = 1260$

Which 1260 being yet too little, I make a third tryal, *viz.*

3. I suppose . . . . $a = 100$
   Thence it follows, that . . . . $aaa + 26a = 1002600$

Which 1002600 exceeds the just Result or absolute number 40188 in the latter part of the Equation first propos'd, and therefore the true number a is less than 100; but the second tryal shews it to be greater than 10, and therefore the whole number which expresseth the exact, or at least part of the value of a, must necessarily consist of two Characters, and consequently the first (towards the left hand) must be one of these nine, 1, 2, 3, 4, 5, 6, 7, 8, 9; but because by the second Inquiry 10 was found too little, I now make tryal with 2 for the first figure of the Root a, viz.

4. I suppose . . . . $a = 20$
   Thence . . . . $aaa + 26a = 8520$

Which Result 8520 being yet less than the just Result 40188, I make tryal again, *viz.*

5. I suppose . . . . $a = 30$
   Thence . . . . $aaa + 26a = 27780$

 Which

Which is yet too little; therefore;

6. I suppose . . . . $a = 40$

Thence . . . . $aaa + 26a = 65040$

Which 65040 being greater than 40188, it shews me that the true Root or value of a is less than 40; but by the fifth tryal its greater than 30, and consequently the first figure of the Root is 3.

Now the second Character of the Root must necessarily be one of these, *viz.* 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; and because it hath been discovered that the true value of the Root a is greater than 30, the second Character cannot be 0, I therefore make tryal with 1, and suppose $a = 31$, which proving too little, I make tryal with 32, 33, 34, &c. severally, in like manner as before, and at length I find 34 to be the true number a sought, by which the Equation propos'd may be expounded; for if $a = 34$, then consequently $aaa + 26a = 40188$.

II. But if after tryals made (as before) the value of a the Root sought happens to fall between two whole numbers that differ by Unity; then tryals are to be made with the lesser whole number increased with $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, &c. until you have found the value of a in some mixt number consisting of a whole number and some certain tenth parts of an Unit: But if the said value of a happens not to be exprest exactly by the said lesser whole number increased with certain tenth parts, then you are to make tryals with the said lesser whole number increased with a decimal Fraction having for its Numerator a number greater than 10, but less than 100; and for its Denominator 100, as with $\frac{11}{100}$, $\frac{12}{100}$, &c. and by proceeding in that manner you may find the exact value of the Root a when its fractional part is exactly equal to some decimal Fraction, or else approach infinitely near to the said exact value when 'tis irrational or surd, as in this following

QUEST. 2.

If . . . . . . $aaaa + 50a = 184638.6801$; (or, $184638\frac{6801}{10000}$;) what is the number a?

RESOLUTION.

First, I suppose $a = 1$, but this proving too little I put $a = 10$, this also proving too little, I assume $a = 100$, which after tryal I find to be greater than the true number a, and consequently the number a falls between 10 and 100; then making tryal with 20 I find it too little, but making tryal with 30 I find this too great, and therefore the true Root a falls between 20 and 30. Again, making tryal with 21 I find it too great, but 20 was before found too little; therefore the true Root a is between 20 and 21; then I make tryal with 20.1, (that is, $20\frac{1}{10}$,) 20.2; 20.3, &c. and at length find 20.7 to be the true number a sought; for if $a = 20.7$ (that is, $20\frac{7}{10}$,) it will make $aaaa + 50a = 184638.6801$ the Equation proposed.

But if 20.7 had proved too little, and 20.8 too great, then tryals must have been made with 20.71, (that is, $20\frac{71}{100}$;) 20.72; 20.73, &c. In like manner if 20.7 had been too little, but 20.71, (that is, $20\frac{71}{100}$) too great, then tryals must have been made with 20.701, (that is, $20\frac{701}{1000}$;) 20.702; 20.703, &c. This will be partly exercis'd in resolving the Equation in this following

QUEST. 3.

If . . . . $aaa + 20aa = 1954$, what is the number a?

Answ. . . $a = 8.308$, &c. found out by tryals, as before.

III. When the value of (a) the required Root of an Equation happens to be less than Unity, then tryal is to be made with $\frac{1}{10}$; but if this prove too great, then with $\frac{1}{100}$, &c. Now suppose .1 (that is, $\frac{1}{10}$) be too great, but .01 (that is, $\frac{1}{100}$) too little; then tryal must be made with .02 | .03 | .04 | &c. until you have found out the greatest figure that must stand in the second place of the decimal Fraction expressing the Root sought; supposing then such figure to be found 8, *viz.* that .08 (or $\frac{8}{100}$) is less, but .09 (or $\frac{9}{100}$) is greater than the Root, tryal must be made with .081, (that is, $\frac{81}{1000}$,) .082 | .083 | &c. as in this following

QUEST. 4.

If . . . . . $aaa + 3240a = 269$, what is the number a?

Answ. . . $a = .083$, &c. that is, $\frac{83}{1000}$, &c.

IV. The

IV. The preceding Examples may fuffice to fhew the ufe of this General Method when all the Terms of the unknown part of an Equation are Affirmative, (*viz.* when + is prefixt to each Term,) in which cafe there is but one Affirmative Root; in the fearch whereof by tryals (as before) if the numbers affumed feverally for the value of the Root fought do afcend greater and greater, then the Abfolute numbers refulting from thofe affumed values will likewife afcend; and contrarily, if the affumed Roots do defcend from a greater to a lefs, the Refults will likewife grow lefs and lefs: whence by comparing an Abfolute number refulting from an affumed Root with the juft Abfolute number of the Equation propos'd; you may certainly know (if the faid Refult and juft Abfolute be not equal to one another) whether you are to take a number greater or lefs than that laft before affumed.

But when the unknown part of an Equation confifts of affirmative and negative Terms mingled one with another, then the fearch by tryals will be more intricate and doubtful than before; for fometimes it will be hard to difcern whether a following affumed Root muft be taken greater or lefs than that which was taken next before. Moreover, a Compound Equation of this latter kind may happen to be fuch, that it may be expounded by as many feveral affirmative Roots as there be Unities in the Index of the higheft unknown Power, *viz.* a Cubical Equation may be fo conftituted that it fhall have three different affirmative Roots, a Biquadratick Equation four feveral Roots; and fo of higher Equations, as will be fhewn in the following *Chapt.* 11. But in what manner foever any poffible Equation is conftituted in Rational numbers, this general Method will alwayes find out one affirmative Root, either exactly true, or at leaft very near the truth; as will farther appear by the following Queftions.

QUEST. 5.

If . . . . $aaa - 22aa + 157a = 360$, what is the number a?

RESOLUTION.

1. I fuppofe . . . . $a = 1$
 Thence it follows that . . . . . . $aaa - 22aa + 157a = 136$

Which 136 is lefs than the juft abfolute number 360, and therefore I make another tryal; *viz.*

2. I fuppofe . . . . $a = 10$
 Thence it follows that . . . . . . $aaa - 22aa + 157a = 370$

Which 370 exceeds the juft abfolute number 360, and therefore I conclude there is one affirmative value of a, (either rational or irrational) between 1 and 10; which value, after tryals made with 2, 3, 4, 5; I find to be 5; this will conftitute the Equation propofed; for if $a = 5$, then $aaa - 22aa + 157a$ will exactly make 360.

But there are two other Roots or values of a, to wit, 8 and 9; each of which will likewife conftitute the Equation firft propofed; but how they are found out will be fhewn in *Sect.* 9. of the following *Chapt.* 11.

QUEST. 6.

If . . . . $3200a - aaa = 46577$ (juft,) what is the number a?

RESOLUTION.

1. I fuppofe . . . . $a = 1$
 Thence . . . . $3200a - aaa = 3199$ (lefs than juft.)
2. I fuppofe . . . . $a = 10$
 Thence . . . . $3200a - aaa = 31000$ (lefs than juft.)
3. I fuppofe . . . . $a = 100$
 Thence . . . . $3200a - aaa = -680000$ (lefs than juft.)

Now becaufe the fecond Refult (or abfolute number) +31000 is Affirmative, and the laft Refult -680000 is Negative, I make tryals with numbers between 10 and 100 for the value of a; for if the Equation propofed be poffible, before the affirmative Refults fall off to negatives, there will be a Root or value of a producing an affirmative Refult either exactly equal, or very near to the juft Refult 46577; therefore,

4. I fuppofe . . . . $a = 20$
 Thence . . . . $3200a - aaa = 56000$ (greater than juft.)

 Now

Now becauſe by taking 20 for the value of a, the Reſult 56000 exceeds the juſt Reſult 46577; but by taking 10 for a, the Reſult 31000 happened to be leſs than the ſaid 46577; it ſhews there is one affirmative Root or value of a between 10 and 20; which Root, after tryals made with intermediate numbers (as in former Examples) will be found 15.7, &c. Moreover, becauſe by ſuppoſing $a = 20$, the Reſult 56000 happened to exceed the juſt Reſult 46577, but by putting $a = 100$ the Reſult -680000 proved to be leſs than the ſame 46577, it ſhews there is an Affirmative value of a between 20 and 100, which value after tryals made will be found 47: ſo that there are two affirmative Roots or values of a found out, to wit, 15.7, &c. (or $15\frac{7}{10}$, &c.) and 47; the former of which will nearly, and the latter exactly conſtitute the Equation propoſed.

V. *Florimond de Beaune* in the latter of two ſmall Treatiſes printed in 1659, concerning the Nature, Conſtitution and Limits of Equations, ſhews how to find out Limits within which the Roots of all compound Equations not aſcending above the Biquadratick kind are confined; which Limits when they may be diſcovered without much trouble, and are not very wide aſunder, will help to leſſen the tryals in the general Method before delivered: As, in the laſt Example, where

The Equation propoſed was	$3200a - aaa = 46577$
Firſt, becauſe aaa muſt be ſubtracted from $3200a$ and leave a Remainder equal to 46577, it preſuppoſeth	. . . . . $aaa < 3200a$
Therefore by dividing each part by a, ...	. . . . . . $aa < 3200$
And by extracting the ſquare Root out of each part, it follows that	. . . . . . $a < 56.5$, &c.
Again, from the Equation propos'd, by tranſpoſition 'tis evident that	$3200a - 46577 = aaa$
Whence 'tis alſo manifeſt that	. . . $3200a > 46577$
And conſequently by dividing each part by 3200,	. . . . . . $a > 14.5$, &c.

Thus it is found that the value of a the Root ſought is greater than 14.5, &c. but leſs than 56.5, &c. and therefore tryals according to the general Method aforeſaid need not be made with any numbers that are not within thoſe Limits.

From the premiſes 'tis evident, that this general Method finds not a perfect Root of an Equation, unleſs ſuch Root be a whole number, or elſe a Fraction exactly equal to ſome decimal Fraction; or laſtly, a mixt number compos'd of a whole number and a perfect decimal Fraction.

Note. When the Coefficients or known numbers multiplied into any of the unknown Powers under the higheſt, (which muſt have no Coefficient but Unity,) are vulgar (not decimal) Fractions, or mixt numbers whoſe fractional parts are vulgar Fractions; likewiſe, when the Abſolute number that ſolely poſſeſſeth the latter part of the Equation propos'd is a vulgar Fraction, or mixt number whoſe fractional part is a vulgar Fraction; all thoſe vulgar Fractions muſt be reduced to decimal Fractions, or elſe the Equation muſt be reduced to another Equation in Integers (by *Sect.* 7. in the following *Chapt.* 11.) before you enter upon the Reſolution by tryals as aforeſaid.

CHAP. XI.

Extractions out of the Algebraical Treatiſes of Vieta *and* Renates des Cartes, *concerning the Conſtitution and Reſolution of Compound Equations in Numbers; eſpecially thoſe which have many Roots.*

I. THE ſcope of this Chapter is, firſt, to ſhew how to form an Equation that ſhall have as many different Roots or values of the Quantity ſought as ſhall be deſired; then how to free an Equation from Fractions, and to caſt away the ſecond Term; and laſtly, how to find out the Roots of all manner of Compound Equations in numbers, either exactly, if they be Rational, or very near the truth if irrational.

But

But that the Learner may the more easily perceive my meaning, I shall premise a few Definitions in three Sections next following.

II. When the known Absolute number in an Equation solely possesseth one part thereof, let it be transferr'd to the other part by the sign —, and then there will be an Equation which hath o or nothing for one part, and the other part is by *Cartesius* called the Summ of the Equation proposed. As, for example, if this Equation be proposed, *viz.* $aaa - 9aa + 26a = 24$, by transposition of 24 it makes $aaa - 9aa + 26a - 24 = 0$, whose first part is called the Summ of the Equation proposed.

III. In the Equations handled in this Chapter, I put a, e or y to signifie an unknown Quantity; and by the first Term of an Equation is meant the highest unknown Power, to wit, that which hath most Dimensions or Degrees of a; by the second Term that which hath fewer Dimensions by one than the first, and so downwards. As in this Equation, $aaa - 9aa + 26a - 24 = 0$, the first Term is aaa, whose Index is 3; the second Term is $-9aa$, where the Index of aa is 2; the third Term is $+26a$, where the Index of a is 1; and the last Term is -24, the known Absolute number whose Index is o.

IV. The Roots of an Equation are of three kinds, *viz.* either Affirmative, or Negative, or Impossible: an affirmative Root is a quantity greater than nothing, as $+5$ or $+20$: a negative Root (which *Cartesius* calls a false Root) expresseth a quantity whose Denomination is opposite to an affirmative; as -5, or -20; the former of which wants 5, and the latter 20 of being equal to nothing: lastly, impossible Roots are such whose values cannot be conceived or comprehended either Arithmetically or Geometrically: As in this Equation, $a = 2 - \sqrt{-1}$, where $\sqrt{-1}$, that is, the square Root of -1 is no manner of way intelligible; for no number can be imagined, which being multiplied by it self according to any Rule of Multiplication, will produce -1.

V. These things premised, I shall proceed to the forming of Equations which shall have many Roots.

PROP. I.

To form an Equation which shall have two Affirmative Roots.

1. Suppose . . . . $\left\{\begin{array}{ll} a = 2; & \text{that is, } a - 2 = 0 \\ a = 3; & \text{that is, } a - 3 = 0 \end{array}\right.$
2. Then by multiplying the said $a - 2 = 0$ by $a - 3 = 0$, this Equation is produced, *viz.* . . . $aa - 5a + 6 = 0$
3. That is, by transposition, ... $5a - aa = 6$

Which last Equation falls under the last of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. and may be expounded by either of two Roots or values of a, which by the Canon in *Sect.* 10. of the same *Chapt.* will be found 2 and 3, to wit, those from which the said Equation was produced by Multiplication, as above.

Again, if this Equation $aa + 6a - 55 = 0$, (that is, $aa + 6a = 55$,) which hath one affirmative Root, to wit, 5, be multiplied by $a - 6 = 0$, there will be produced $aaa - 91a + 330 = 0$, (that is, $91a - aaa = 330$,) which hath two affirmative Roots or values of a, to wit, 5 and 6; which may be found out by the Rule hereafter delivered in *Sect.* 9. of this *Chapt.*

PROP. II.

To form an Equation which shall have one Affirmative, and one Negative Root.

1. Suppose . . . . $\left\{\begin{array}{ll} a = 3; & \text{that is, } a - 3 = 0 \\ a = -2, & \text{that is, } a + 2 = 0 \end{array}\right.$
2. Then by multiplying the said $a - 3 = 0$ by $a + 2 = 0$, this Equation is produced, *viz.* . . . $aa - a - 6 = 0$
3. That is, ... $aa - a = 6$

Which last Equation falls under the second of the three Forms in *Sect.* 1. *Chap* 15. *Book* 1. and may be expounded by either of two Roots or values of a, whereof one is Affirmative, and the other Negative; which, after the manner of resolving *Quest* 1. in *Sect.* 7. of the same *Chapt.* will be found $+3$ and -2, to wit, those from which the said Equation was produced by Multiplication, as before.

PROP.

PROP. III.

To form an Equation which shall have three Affirmative Roots.

1. Suppose . . . . $\left\{\begin{array}{l} a = 2, \text{ that is, } a - 2 = 0 \\ a = 3; \text{ that is, } a - 3 = 0 \\ a = 4, \text{ that is, } a - 4 = 0 \end{array}\right.$

2. Then by multiplying the three last Equations (in each of which the latter part is 0) one into another, this Equation will be produced, $\Big\}\; aaa - 9aa + 26a - 24 = 0$

3. That is, by transposition of -24, . . $\Big\}\; aaa - 9aa + 26a = 24$

Which Equation may be expounded by every one of these three affirmative Roots, or values of a, to wit, 2, 3 and 4; which may be found out by the Rule in the following *Sect.* 9. of this Chapter.

The same Equation may likewise be formed altogether by Letters, thus, *viz.* Let the said known Roots 2, 3 and 4 be represented by b, c, d; and then

4. Suppose . . . . $\left\{\begin{array}{l} a = b, \text{ that is, } a - b = 0 \\ a = c, \text{ that is, } a - c = 0 \\ a = d, \text{ that is, } a - d = 0 \end{array}\right.$

5. Then by multiplying those three last Equations, in each of which the latter part is nothing, one into another, this Equation will be produced, *viz.*

$$aaa \left.\begin{array}{l} -b \\ -c \\ -d \end{array}\right\} aa \left.\begin{array}{l} +bc \\ +bd \\ +cd \end{array}\right\} a - bcd = 0$$

That is, . . . $aaa - 9aa + 26a - 24 = 0$

PROP. IV.

To form an Equation which shall have three Affirmative Roots; and one Negative Root.

1. Suppose . . . . $\left\{\begin{array}{l} a = 2, \text{ that is, } a - 2 = 0 \\ a = 3, \text{ that is, } a - 3 = 0 \\ a = 4, \text{ that is, } a - 4 = 0 \\ a = -5, \text{ that is, } a + 5 = 0 \end{array}\right.$

2. Then by multiplying the four last Equations (in each of which the latter part is 0,) one into another, this following Equation will be produced, *viz.*

$$aaaa - 4aaa - 19aa + 106a - 120 = 0$$

That is, ... $aaaa - 4aaa - 19aa + 106a = 120$

Which last Equation may be expounded by every one of these three affirmative Roots, or values of a, *viz.* 2, 3 and 4; and by one negative Root -5; every one of which may be found out by the Rule in the following *Sect.* 9. of this Chapter.

The same Equation may likewise be formed altogether by Letters, thus, *viz.* Let the said known Roots, 2, 3, 4 and -5 be represented by b, c, d and $-f$; then

3. Suppose . . . . $\left\{\begin{array}{l} a = b, \text{ that is, } a - b = 0 \\ a = c, \text{ that is, } a - c = 0 \\ a = d, \text{ that is, } a - d = 0 \\ a = -f, \text{ that is, } a + f = 0 \end{array}\right.$

4. Then by multiplying the four last Equations, in each of which the latter part is 0, one into another, this following Equation will be produced, *viz.*

$$aaaa \left.\begin{array}{l} -b \\ -c \\ -d \\ +f \end{array}\right\} aaa \left.\begin{array}{l} +bc \\ +bd \\ +cd \\ -bf \\ -cf \\ -df \end{array}\right\} aa \left.\begin{array}{l} -bcd \\ +bcf \\ +bdf \\ +cdf \end{array}\right\} a - bcdf = 0$$

That is,

$$aaaa - 4aaa - 19aa + 106a - 120 = 0.$$

After the same manner you may form an Equation which shall have as many Roots as you please, either all Affirmative, or some of them Affirmative and some Negative.

VI. *Obser-*

VI. *Observations upon the preceeding four Propositions.*

1. By what hath been said 'tis evident, that sometimes an Equation may have as many Roots as there be unities in the Index of the highest unknown Term; I say sometimes, not alwayes: for although this Equation $aaa - 6aa + 13a - 10 = 0$, as to its number of Terms and Signs be like to the Equation formed in the preceding *Prop.* 3. so that one may think it hath three Roots, yet it hath only one affirmative Root, to wit, 2, and no other Root either Affirmative or Negative can constitute the said Equation, for 'tis produced by the multiplication of this impossible Equation $aa - 4a + 5 = 0$ by $a - 2 = 0$; but that $aa - 4a + 5 = 0$, that is, $4a - aa = 5$, is an impossible Equation, the Determination in *Sect.* 9. *Quest.* 1. *Chap.* 15. *Book* 1. makes manifest.

In like manner, although this Equation $aaaa - 60aaa + 1650aa - 22500a + 115344 = 0$, as to its number of Terms and Signs be like to an Equation that hath four affirmative Roots, yet that Equation can be expounded only by two affirmative Roots, to wit, 12 and 18, and by no other Root either Affirmative or Negative; for 'tis made by the multiplication of $aa - 30a + 216 = 0$, which hath two affirmative Roots, 12 and 18, into this impossible Equation $aa - 30a + 534 = 0$.

2. For as much as Division resolves or undoes that which is compos'd or done by Multiplication, if the summ of any Equation which is produced by the multiplication of two or more Equations one into another, (according to the Method in the preceding four Propositions) be divided by a Binomial compos'd of the unknown quantity (a) less by the value of any one of the affirmative Roots, or more by the value of one of the negative Roots, the Quotient shall be an Equation in which the first Term hath fewer Dimensions by one than the first Term of the Equation so divided: And if the Quotient be divided in like manner, there will come forth an Equation whose first Term hath fewer Dimensions by one than the former Quotient. As, for example, let there be proposed the Equation in the preceding *Prop.* 4. to wit, $aaaa - 4aaa - 19aa + 106a - 120 = 0$, which was made by the continual multiplication of $a-2=0$, $a-3=0$, $a-4=0$, $a+5=0$; I say, if the Equation proposed be divided by any one of those Binomials $a-2$, $a-3$, $a-4$, $a+5$, the Quotient will be an Equation wherein the first Term hath only three Dimensions, which are fewer by one than those in $aaaa$ the first Term of the Equation proposed: So if the said $aaaa - 4aaa - 19aa + 106a - 120 = 0$ be divided by $a-2=0$, there will arise $aaa-2aa-23a+60=0$, as you see by the subsequent Division.

$$
\begin{array}{rllll}
a-2\,) & aaaa - 4aaa - 19aa + 106a - 120 \;(\; aaa - 2aa - 23a + 60 \\
 & aaaa - 2aaa \\
\hline
 & \quad -2aaa - 19aa \\
 & \quad -2aaa + 4aa \\
\hline
 & \qquad -23aa + 106a \\
 & \qquad -23aa + 46a \\
\hline
 & \qquad\quad + 60a - 120 \\
 & \qquad\quad + 60a - 120 \\
\hline
 & \qquad\qquad 0 \quad 0
\end{array}
$$

Likewise if the Quotient, to wit, the Equation $aaa - 2aa - 23a + 60 = 0$, where the first Term aaa hath three Dimensions, be divided by $a - 3 = 0$, there will arise $aa + a - 20$, whose first Term aa hath but two Dimensions: And lastly, if the said latter Quotient $aa + a - 20$ be divided by $a - 4 = 0$, there will come forth a simple Equation, to wit, $a + 5 = 0$, that is, the negative Root $a = -5$.

The like Division may be practised with the literal Equations at the latter end of *Prop.* 3, and 4. in the preceding *Sect.* 5.

3. If a compleat Equation, that is, such in which all the Terms are extant, be produced by the multiplication of possible Equations one into another, you may discover how many affirmative, and how many negative Roots that Equation hath, by this Rule; *viz.* As often as $-$ follows next after $+$, or $+$ next after $-$, so often there is an affirmative Root; and as often as two signs $-$ or two signs $+$ stand next to one another, so often there is a negative Root: As, for example, in this Equation, (before formed in *Prop.* 4.)

to wit,

to wit, $aaaa - 4aaa - 19aa + 106a - 120 = 0$, becauſe next after the firſt Term $+ aaaa$ there follows $- 4aaa$, it ſhews there is one affirmative Root; and becauſe next after $- 4aaa$ there comes $- 19aa$, it ſhews that the Equation hath one negative Root: again, becauſe next after $- 19aa$ there follows $+ 106a$, it hints there is another affirmative Root; and becauſe next after $+ 106a$ there follows $- 120$, it ſhews there is a third affirmative Root; ſo that the ſaid Rule diſcovers the Equation propos'd to have three affirmative Roots, and one negative Root.

4. It is alſo manifeſt from the manner of forming Equations according to the Propoſitions in the preceding *Sect.* 5. that in every Equation which hath as many affirmative Roots as there be Dimenſions in the firſt Term, the Coefficient or known quantity in the ſecond Term is equal to the ſumm of all the affirmative Roots; and the known quantity in the third Term is equal to the ſumm of the Products of every two of the ſaid Roots multiplied one by the other; and the known quantity in the fourth Term is equal to the ſumm of the Products of every three of the ſaid Roots, and ſo forward when there be more Terms; but the laſt Term, to wit, the Abſolute quantity given is equal to the Product of all the Roots multiplied one into another: As in the following Equation (before formed in *Prop.* 3.) *viz.*

$$aaa \left.\begin{matrix} -b \\ -c \\ -d \end{matrix}\right\} aa \left.\begin{matrix} +bc \\ +bd \\ +cd \end{matrix}\right\} a - bcd = 0$$

$$\text{That is,} \quad aaa - 9aa + 26a - 24 = 0$$

Firſt, the ſumm of 2, 3 and 4, (that is, of b, c, d) the three Roots of that Equation is 9, which is the known number of the ſecond Term $- 9aa$; ſecondly, the ſumm of the Products of every two of the ſaid Roots multiplied one by the other is 26, that is, $\overline{+bc+bd+cd}$, which is the known Coefficient of the third Term $+ 26a$, or $\overline{+bc+bd+cd}$ into a; and laſtly, the Product of all the three Roots multiplied one into another is 24, or bcd, to which prefixing $-$ it makes $- 24$, or $- bcd$ the laſt Term of the Equation propoſed.

The like Properties enſue when the ſumm of the numbers of multitude of affirmative and negative Roots is equal to the number of Dimenſions in the firſt Term of an Equation; ſaving that here, in ſumming up all the Roots which compoſe the known quantity in the ſecond Term, and likewiſe the Products which compoſe the known quantities in the following Terms, reſpect muſt be had to the Rules of Addition of $+$ and $-$ in ſuch manner as the Equation propoſed if it be formed altogether by letters will direct; as you may eaſily perceive by the Equations formed in *Prop.* 4. of the preceding *Sect.* 5.

VII. *How to free an Equation from Fractions, when 'tis incumbred therewith in the ſecond, third or any of the following Terms; which work is by* Vieta *called* Iſomœria.

The Rules in *Chap.* 12. *Book* 1. ſhew how to reduce an Equation, ſo, as that the firſt Term may have no Coefficient but unity; but if after any Equation is ſo reduced there happens to be any Fraction in the ſecond, third, or any of the following Terms, ſuch Equation may be reduced to another whoſe Terms ſhall be all Integers, by the Method in the five Examples next following.

Example 1.

1. Let this Equation be propos'd to be reduced to another in Integers, *viz.* . . . . $aaa + \frac{3}{2}a = 225$

Operation.

2. Suppoſe $e = 2a$, ($2a$ becauſe 2 is the Denominator of the Fraction $\frac{1}{2}$,) . . . . $e = 2a$

3. Then divide each part of the laſt Equation by 2 (the Denominator aforeſaid) and there ariſeth . . . . $\frac{e}{2} = a$

4. And by multiplying each part of the Equation in the third ſtep cubically, there comes forth . . . . . $\frac{eee}{8} = aaa$

5. Again, by multiplying each part of the Equation in the third ſtep by $\frac{3}{2}$, (the Fraction in the ſecond Term of the Equation firſt propoſed,) it makes . . . . . $\frac{3e}{4} = \frac{3}{2}a$

6. Then

6. Then add the two last Equations into one; and the summ is . . . . $\frac{eee}{8} + \frac{3e}{4} = aaa + \frac{3}{2}a$
7. But by supposition in the first step . . . . $225 = aaa + \frac{3}{2}a$
8. Therefore from the two last Equations, (by 1. *Axiom.* 1. *Elem.* Euclid.) . . . . $\frac{eee}{8} + \frac{3e}{4} = 225$
9. Which last Equation being reduced to Integers, (by *Sect.* 2. *Chap.* 12. *Book* 1.) gives . . . . $eee + 6e = 1800$

Therefore an Equation is found out which is altogether exprest by Integers; and when the value of e in the last Equation is discovered, the value of a in the Equation propos'd is consequently known; for by the third step $a = \frac{1}{2}e$; therefore if e be 12, then a shall be 6.

Example 2.

Again, if this Equation be propos'd, ... $aaa + \frac{3}{2}a = \frac{265}{2}$
It may be reduced in like manner as before in Example 1. to this, *viz.* . . . . $eee + 6e = 1060$
And if e be 10, then a shall be 5.

Example 3.

So likewise this Equation . . . . $aaa + \frac{3}{2}aa = \frac{325}{2}$
May be reduced to this . . . . $eee + 3ee = 1300$
And if e be 10, then a is 5.

Example 4.

1. Again, let there be proposed . . . . $aaa + \frac{11}{12}a = \frac{12}{4}$

Operation.

2. Suppose $e = 12a$; ($12a$, because 12 is the Denominator of the Fraction $\frac{11}{12}$ in the second Term;) . . . . $e = 12a$
3. Then divide each part of the last Equation by 12 (the Denominator aforesaid,) and there arises . . . . . $\frac{e}{12} = a$
4. And by multiplying cubically the last Equation, it produceth . . . . $\frac{eee}{1728} = aaa$
5. And by multiplying the Equation in the third step by $\frac{11}{12}$, it makes . . . . $\frac{11e}{144} = \frac{11}{12}a$
6. And by adding the two last Equations into one, the summ makes . . . . $\frac{eee}{1728} + \frac{11e}{144} = aaa + \frac{11}{12}a$
7. But by the Equation proposed; . . . . $\frac{12}{4} = aaa + \frac{11}{12}a$
8. Therefore from the two last Equations (by 1. *Axiom* 1. *Elem.* Euclid.) . . . . $\frac{eee}{1728} + \frac{11e}{144} = \frac{12}{4}$

Which Equation reduced to Integers gives . . . . $eee + 132e = 5208.$

Thus an Equation is found out in Integers; and when the value of e is discovered, the value of a in the Equation propos'd is consequently known; for by supposition in the second step, e is to a as 12 to 1: therefore if e be 18, then a shall be $1\frac{1}{2}$.

Example 5.

1. Again, let there be proposed . . $aaaa - 10aaa + 45\frac{5}{6}aa - 104\frac{1}{6}a + 89 = 0.$

Operation.

2. Suppose $e = 6a$, ($6a$, because 6 is the Denominator of the Fraction $\frac{5}{6}$,) . . . . $e = 6a$
3. Then by dividing each part of the last Equation by 6, there arises . . . . $\frac{e}{6} = a$
4. And by squaring the last Equation it makes . . . . $\frac{ee}{36} = aa$
5. Likewise by squaring each part of the last Equation, there will be produced, ... $\frac{eeee}{1296} = aaaa$
6. And by multiplying the Equation in the fourth step by that in the third, the Product is . . . . . . $\frac{eee}{216} = aaa$

7. And by multiplying the last Equation by 10, it gives this, *viz.* . . . . $\frac{10eee}{216} = 10aaa$

8. And by multiplying the Equation in the fourth step by $45\frac{5}{6}$, it produceth . . . . $\frac{275ee}{216} = 45\frac{5}{6}aa$

9. And by multiplying the Equation in the third step by $104\frac{1}{6}$, the Product will be . . . . $\frac{625e}{36} = 104\frac{1}{6}a$

10. Then by connecting the Quantities which stand in the first parts of the Equations in the fifth, seventh, eighth and ninth steps, together with 89, by the same signs which respectively belong to each Term of the Equation proposed, the summ shall be equal to the summ of the same Equation, and consequently equal to nothing; hence this Equation ariseth, *viz.*

$$\frac{eeee}{1296} - \frac{10eee}{216} + \frac{275ee}{216} - \frac{625e}{36} + 89 = 0$$

11. Which Equation being reduced to Integers (by *Sect.* 7. *Chap.* 11. *Book* 1.) gives

$$eeee - 60eee + 165ee - 22500e + 115344 = 0.$$

Thus an Equation is found out whose Terms are all Integers; and the value of the Root e in this Equation is to the value of the Root a in the Equation proposed as 6 to 1; (for, by supposition in the second step, $e = 6a$:) and therefore if e be 12, then a shall be 2; or if e be 18, than a shall be 3.

VIII. *How to take away the second Term of a Compound Equation.*

The Rule is this; Divide the Coefficient, (that is, the known Quantity) multiplied into the second Term of an Equation proposed, by the Index (or number of Dimensions) of the Power which is the first Term: Then if the signs of the first and second Terms be unlike, (*viz.* if one be + and the other —,) subtract the Quotient from the affirmative Root sought; but if the signs be like, (that is, both + or both —,) add the said Quotient to the affirmative Root: Then equate the said Summ or Remainder to some letter to represent an unknown Quantity, and proceed according to the Method in the following Examples; so at length a new Equation will arise, wherein the second Term is wanting.

Example 1.

1. Let there be proposed this Equation . . . . $aa - 6a = 72$
2. That is . . . . $aa - 6a - 72 = 0$
3. Here the number of Dimensions in the first Term aa is 2, and the known number multiplied into a making the second Term $6a$ is 6; this divided by the said 2 gives 3, which subtracted from the Root a, (because the signs of the first and second Terms are unlike,) leaves $a - 3$ which is equal to some unknown number, let it be e; then
4. By supposition . . . . $a - 3 = e$
5. And consequently, by adding 3 to each part of that Equation, there ariseth . . . . $a = e + 3$
6. And by squaring each part of the last Equation, there comes forth . . . . $aa = ee + 6e + 9$
7. And by multiplying each part of the Equation in the fifth step by the Coefficient 6 in the proposed Equation, it makes . . . . $6a = 6e + 18$
8. Then by subtracting the last Equation from that in the sixth step, there remains . . . . $aa - 6a = ee - 9$
9. And lastly, by subtracting 72 (the last Term of the Equation propos'd) from the Equation in the eighth step, there remains . . . . $aa - 6a - 72 = ee - 81 = 0$

Thus you see an Equation is found out, to wit, $ee - 81 = 0$, which is equal to the Equation propos'd, and it wants the second Term; (for there is not any number of e in the Equation found out;) now if the value of e be made known, then the value of a is consequently known: But the Equation found out, to wit, $ee - 81 = 0$, that is, $ee = 81$ gives $e = 9$, and by the fifth step $a = e + 3$, therefore $a = 12$.

Example 2.

Example 2.

1. Again, let there be propofed this Equation, *viz.* . . . . . . $aa + 6a = 216$
2. That is, ... $aa + 6a - 216 = 0$
3. Here (as before) I divide 6, the Coefficient in the fecond Term $6a$, by 2, which denotes the number of Dimenfions in the firft Term aa, and the Quotient 3 I add to the Root a, (becaufe the firft and fecond Terms of the Equation have the fame fign $+$) and the fumm $a + 3$ is equal to fome unknown number, let it be e; then
4. By fuppofition . . . . $a + 3 = e$
5. Therefore by fubtracting 3 from each part of that Equation, there arifeth . . . . $a = e - 3$
6. And by fquaring the laft Equation, there comes forth . . . $aa = ee - 6e + 9$
7. And by multiplying the Equation in the fifth ftep by 6, it produceth . . . . $6a = 6e - 18$
8. Then by adding the two laft Equations into one, the fumm is . . . . $aa + 6a = ee - 9$
9. And by fubtracting 216 (the laft Term of the Equation propos'd) from each part of the Equation in the eighth ftep, there remains . . . . . $aa + 6a - 216 = ee - 225 = 0$

Thus an Equation is found out, to wit, $ee - 225 = 0$, which wants a fecond Term, (for there is no number of e in that Equation,) and when the value of e is made known the value of a in the Equation propos'd is known alfo; but the Equation $ee - 225 = 0$, that is, $ee = 225$ gives $e = 15$, and by the fifth ftep, $a = e - 3$; therefore $a = 12$ that is, $15 - 3$.

Example 3.

1. Again, let this Equation be propos'd, . . . $aaa - 18aa - 7a + 696 = 0$
2. According to the Rule before given, I divide 18 the known number of the fecond Term $- 18aa$, by 3, which denotes the number of Dimenfions in the firft Term aaa, and the Quotient is 6, this I fubtract from the Root a, (becaufe the figns of the firft and fecond Terms are unlike,) and the Remainder is $a - 6$, which is equal to fome unknown number, fuppofe it to be e; then
3. By fuppofition . . . . $a - 6 = e$
4. Therefore by adding 6 to each part of that Equation, there arifeth . . . . $a = e + 6$
5. And by fquaring the laft Equation it makes $aa = ee + 12e + 36$
6. And by multiplying the two laft Equations one by the other, the Product is . . . . $aaa = eee + 18ee + 108e + 216$
7. And by multiplying the Equation in the fifth ftep by 18, (the Coefficient in the fecond Term of the Equation propos'd,) it makes . . . $18aa = 18ee + 216e + 648$
8. Likewife, the Equation in the fourth ftep being multiplied by 7, (the Coefficient in the third Term of the Equation propos'd,) produceth $7a = 7e + 42$
9. Then to the Equation in the fixth ftep adding 696, (to wit, the laft Term of the Equation propos'd,) the fumm is

$$aaa + 696 = eee + 18ee + 108e + 912;$$

10. Likewife by adding the eighth Equation to the feventh, it makes

$$18aa + 7a = 18ee + 223e + 690;$$

11. Laftly, by fubtracting the Equation in the tenth ftep from that in the ninth, this following Equation remains, *viz.*

$$aaa - 18aa - 7a + 696 = eee - 115e + 222 = 0.$$

Thus an Equation is found out, to wit, $eee - 115e + 222 = 0$, which wants the fecond Term, (to wit, the Power ee;) and when the value of the Root e is made known, the value of the Root a fhall be known alfo: For by the fourth ftep, $a = e + 6$; therefore if e be 2, then a fhall be 8; and if e be equal to $\sqrt{112} - 1$, then a fhall be equal to $\sqrt{112} + 5$.

Example 4.

1. Again, let there be proposed . . . $aaaa + 6aaa + 11aa + 6a - 100 = 0$
2. According to the Rule before given, I divide 6 the Coefficient in the second Term $+6aaa$, by 4, which denotes the number of Dimensions in the first Term $aaaa$, and the Quotient is $\frac{3}{2}$, which I add to the Root a, (because the signs of the first and second Terms are like) and the summ is $a + \frac{3}{2}$, which is equal to some unknown number, let it be e; then
3. By supposition . . . . $a + \frac{3}{2} = e$
4. Therefore . . . . $a = e - \frac{3}{2}$
5. The Square of the last Equation is . . $aa = ee - 3e + \frac{9}{4}$
6. And the two last Equations multiplied one by the other, make . . . . . $aaa = eee - \frac{9}{2}ee + \frac{27}{4}e - \frac{27}{8}$
7. And the Equation in the sixth step being multiplied by that in the fourth step, will produce . . . . $aaaa = eeee - 6eee + \frac{27}{2}ee - \frac{27}{2}e + \frac{81}{16}$
8. And the Equation in the sixth step multiplied by 6 produceth . . . . $6aaa = 6eee - \frac{54}{2}ee + \frac{162}{4}e - \frac{162}{8}$
9. And the Equation in the fifth step multiplied by 11 produceth . . . . $11aa = 11ee - 33e + \frac{99}{4}$
10. And the Equation in the fourth step multiplied by 6 gives . . . . . $6a = 6e - 9$
11. Now 'tis manifest, that if from the summ of the first parts of the four last Equations there be subtracted 100, the Remainder will be equal to the summ of the Equation first propos'd equal to 0; therefore also if 100 be subtracted from the summ of the latter parts of the said four Equations the Remainder shall be equal to 0, *viz.*
$$eeee - \tfrac{5}{2}ee - 99\tfrac{7}{16} = 0.$$
12. In which last Equation, the second Term, to wit, the Power eee is wanting, as was desired: And when the value of e is made known, the value of the Root a in the Equation propos'd shall be known also; for by the fourth step $a = e - \frac{3}{2}$, but (by the Canon in *Sect.* 8. *Chap* 15. *Book* 1.) the value of e in the Equation in the eleventh step will be found $\sqrt{:1\frac{1}{4} + \sqrt{101}:}$ and therefore $a = \sqrt{:1\frac{1}{4} + \sqrt{101}:} - \frac{3}{2}$.

IX. *The use of the preceding Rules of this Chapter, in the Resolution of all manner of Compound Equations in Numbers.*

After an adfected or Compound Equation different from any of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. is prepared for Resolution by the Rules of *Chap.* 12. *Book* 1. and reduced (if need be) to Integers, and the summ of all the Terms made equal to 0, (or nothing,) according to *Sect.* 7, and 2. of this *Chapt.* search out (by the Rules of *Chap.* 8. of this *Book*) all the just Divisors to the last Term (that is, the known Absolute number of the Equation so reduced. Then try whether any one of those Divisors connected to the unknown Root a by — or + will divide the total summ of the said reduced Equation without leaving a Remainder; for when such Division succeeds, either the known part of the said Binomial Divisor is the desired value of the Root a, or at least the Quotient gives an Equation whose first Term hath fewer Dimensions by one than the Equation divided; and then the Root of this new Equation, if its first Term be a Square, may be found out by some of the Canons in *Sect.* 6, 8, 10. of *Chap.* 15. *Book* 1. But if the first Term contains three or more Dimensions, let this Equation be examined by Division, (as before,) and if none of those Divisions work off just without a Fraction, then by taking away the second Term, (by the Rule in *Sect.* 8. of this *Chapt.*) another Equation more simple, and such as may be resolved by some of the Canons in *Sect.* 6, 8, 10. *Chap.* 15. *Book* 1. will sometimes arise: But if none of those ways prove effectual, you may by the general Method in the foregoing *Chapt.* 10. find out one Affirmative Root very near a true Root, and then joyning this Root found out to the unknown Root a by the sign —, you may by this Binomial divide the Equation, and proceed to find out the rest of the Roots very near the truth: all which will be made manifest by the following Questions.

QUEST. 1.

If . . . $aaa - 9aa + 26a = 24$
That is, if . . . $aaa - 9aa + 26a - 24 = 0$ } What is the number a?

RESOLUTION.

First, (by the Method in *Sect.* 5. *Chap.* 8. of this *Book*) I search out all the numbers that will severally divide the last Term 24 without a Remainder, and find them to be these, *viz.* 1, 2, 3, 4, 6, 8, 12, 24. Then, by examining in order whether the total summ of the Equation propos'd may be divided by $a - 1$, or $a + 1$; by $a - 2$, or $a + 2$, *&c.* I find it may be exactly divided by $a - 2$ without a Remainder, and the Quotient is $aa - 7a + 12$, as you see by this following Division.

$$
\begin{array}{r|l}
a - 2 \,) & aaa - 9aa + 26a - 24 \,(\, aa - 7a + 12 \\
 & aaa - 2aa \\
\hline
 & \quad - 7aa + 26a \\
 & \quad - 7aa + 14a \\
\hline
 & \qquad + 12a - 24 \\
 & \qquad + 12a - 24 \\
\hline
 & \qquad\quad 0 \quad 0
\end{array}
$$

Therefore 2 the known number in the Divisor $a - 2$ is one Real or Affirmative Root of the Equation proposed; for as well the Divisor as the Dividend was supposed equal to nothing, *viz.* $a - 2 = 0$, whence $a = 2$; the Quotient also is consequently equal to 0, *viz.* $aa - 7a + 12 = 0$, that is, $7a - aa = 12$; hence (by the Canon in *Sect.* 10. *Chapt.* 15. *Book* 1.) two other Affirmative values of the Root a will be discovered, to wit, 4 and 3. So that three Real values of a, to wit, 2, 3 and 4 are found out, by every one of which the Equation propos'd may be expounded, as the Proof will easily shew.

QUEST. 2.

If . . . $aaa - 22aa + 157a = 360$
That is, if . . . $aaa - 22aa + 157a - 360 = 0$ } What is $a = ?$

RESOLUTION.

First, the Divisors of the last Term 360 will be found these, *viz.* 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180 and 360; then by examining in order whether the summ of the Equation propos'd may be divided by $a - 1$, or $a + 1$; by $a - 2$, or $a + 2$; by $a - 3$, or $a + 3$; *&c.* I find that $a - 5$ will precisely divide the said summ without a Fraction, and therefore 5 is one Affirmative Root or value of a; then the Quotient $aa - 17a + 72 = 0$, that is, $17a - aa = 72$ affords two other Affirmative values of a, to wit, 8 and 9. Thus you see three Real values of a, to wit, 5, 8 and 9 are found out, by every one of which the Equation proposed, to wit $aaa - 22aa + 157a = 360$ may be expounded, as will appear by the Proof.

QUEST. 3.

If . . . $91a - aaa = 330$
That is, if . . . $aaa - 91a + 330 = 0$ } What is $a = ?$

RESOLUTION.

First, the Divisors of the last Term 330 will be found 1, 2, 3, 5, 6, 10, 11, 15, 22, 30, 55, 66, 110, 165 and 330; then by examining in order whether the summ of the Equation propos'd, to wit, $aaa - 91a + 330$ may be divided by $a - 1$, or $a + 1$; by $a - 2$, or $a + 2$; *&c.* I find it may be divided by $a - 5$ and leave no Remainder; therefore $a - 5 = 0$ gives $a = 5$, which is one Affirmative Root of the Equation propos'd, and the Quotient $aa + 5a - 66 = 0$, that is, $aa + 5a = 66$ affords another Affirmative value of a, to wit, 6. So that two Real values of a are found out, by each of which the Equation propos'd may be expounded; for if $a = 5$, or $a = 6$, from either supposition it follows that $91a - aaa = 330$.

QUEST. 4.

To find two numbers whose summ shall be 5, and that if the summ of their Squares be multiplied by the summ of their Cubes, the Product may be 455.

RESO-

RESOLUTION.

This Question may be solved by the Canon of *Quest.* 13. *Chap.* 16. *Book* 1. but that Canon being raised from Positions that lye out of the common Road, I shall here solve the Question in the ordinary way, and so it will exercise the preceding Rules of this Chapter. First then,

1. For one of the numbers sought put . . . . . a
2. Therefore the other number is . . . . . . $5 - a$
3. The Square of the first number is . . . . . aa
4. The Square of the second is . . . . $aa - 10a + 25$
5. The summ of those Squares is . . . . . . $2aa - 10a + 25$
6. The Cube of the first number is . . . . . . aaa
7. The Cube of the second is . . . . $-aaa + 15aa - 75a + 125$
8. Therefore the summ of those Cubes is . . . $+15aa - 75a + 125$
9. Which summ being multiplied by the summ of the Squares in the fifth step gives this Product, *viz.* $30aaaa - 300aaa + 1375aa - 3125a + 3125$.
10. But according to the Question, the Product in the last step must be equal to the given Product 455, hence this Equation ariseth,
$$30aaaa - 300aaa + 1375aa - 3125a + 3125 = 455.$$
11. And by subtracting 455 from each part of the last Equation, this ariseth,
$$30aaaa - 300aaa + 1375aa - 3125a + 2670 = 0.$$
12. And by dividing every Term in the last Equation by 30, this ariseth,
$$aaaa - 10aaa + 45\tfrac{5}{6}aa - 104\tfrac{1}{6}a + 89 = 0.$$
13. Then by supposing $e = 6a$, and proceeding according to the *Example* 5. in *Sect.* 7. of this *Chapt.* to free the Equation in the preceding twelfth step from Fractions, this will be produced, *viz.*
$$eeee - 60eee + 1650ee - 22500e + 115344 = 0.$$
14. Now the Divisors of the last Term 115344 will be found 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 27, &c. and after tryals made by Division (like as in the three last preceding Questions,) I find that $e - 12 = 0$ will precisely divide the summ of the Equation in the thirteenth step, and therefore 12 is one true value of e. Again, the Quotient of that Division being $eee - 48ee + 1074e - 9612$, I seek the Divisors of the last Term, 9612, and find them to be 1, 2, 3, 4, 6, 9, 12, 18, 27, 36, &c. Then after tryals made (as before) I find that $e - 18$ will exactly divide the said $eee - 48ee + 1074e - 9612$, and therefore 18 is one other Affirmative value of e; and because the Quotient of the last mentioned Division, to wit, $ee - 30e + 534 = 0$, that is, $30e - ee = 534$, is an impossible Equation, (as is evident by the Determination in *Sect.* 9. *Quest.* 1. *Chap.* 15. *Book* 1.) I conclude that the Equation in the thirteenth step hath no other Root or value of e besides 12 and 18 before found. But because by supposition in the thirteenth step, $e = 6a$, $\frac{1}{6}$ of 12 and likewise of 18, that is, 2 and 3 shall be the true values of a to solve the Question; for their summ is 5; and if 13 the summ of their Squares be multiplied by 35 the summ of their Cubes, the Product is 455, as was desired.

Sometimes the taking away of the second Term of an Equation (by the Rule in *Sect.* 8. of this *Chapt.*) will be an expedient to find out an Equation resolvable by some of the Canons in *Sect.* 6, 8 and 10. *Chap.* 15. *Book* 1. when tryals by Division (as before) will be in vain, as will appear by the following fifth Question, which I find resolved two manner of wayes in Page 319. of *Cartesius* his Geometry, set forth with Comments by *Fran. van Schooten*, and printed at *Amsterdam* in 1659.

QUEST. 5.

To find four numbers in Arithmetical Progression continued, such, that their common difference may be unity, and the Product made by their continual multiplication 100.

RESOLUTION.

1. For the first number put . . . . a
2. Then the second shall be . . . . $a + 1$
3. The third . . . . $a + 2$
4. And the fourth . . . . $a + 3$
5. Therefore the Product of their continual multiplication is . . . . . $aaaa + 6aaa + 11aa + 6a$

6. Which

6. Which Product must be equal to 100, therefore . . . . $\}$ $aaaa + 6aaa + 11aa + 6a = 100$
7. That is . . . . $>$ $aaaa + 6aaa + 11aa + 6a - 100 = 0$
8. Of which Equation the last Term 100 may be divided by 1, 2, 4, 5, 10, 20, 25, 50 and 100, but Division being tryed by $a -$ or $+ 1$, by $a -$ or $+ 2$, by $a -$ or $+ 4$, &c. it proves ineffectual. Then by taking away the second Term, (as in Example 4. *Sect.* 8. of this *Chapt.*) this Equation ariseth, *viz.* $eeee - 2\frac{1}{2}ee - 99\frac{7}{16} = 0$, in which the Root e, (by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1.) will be found equal to $\sqrt{:1\frac{1}{4} + \sqrt{101}:}$ but in taking away the second Term, a was put equal to $e - \frac{3}{2}$, and therefore $a = \sqrt{:1\frac{1}{4} + \sqrt{101}:} - \frac{3}{2}$, and consequently from the first, second, third and fourth steps,

The four numbers sought are these, $\left\{\begin{array}{l} \sqrt{:1\frac{1}{4} + \sqrt{101}:} - \frac{3}{2} \\ \sqrt{:1\frac{1}{4} + \sqrt{101}:} - \frac{1}{2} \\ \sqrt{:1\frac{1}{4} + \sqrt{101}:} + \frac{1}{2} \\ \sqrt{:1\frac{1}{4} + \sqrt{101}:} + \frac{3}{2} \end{array}\right.$

Which numbers exceed one another by Unity; and the Product of their multiplication is 100, as before hath been proved in *Quest.* 3. *Sect.* 17. *Chap.* 9. of this *Book*.

Another way of Resolving Quest 5.

For the first number put $a - 1\frac{1}{2}$, for the second $a - \frac{1}{2}$, for the third $a + \frac{1}{2}$, and for the fourth $a + 1\frac{1}{2}$; then by multiplying these four numbers one into another, and comparing the Product to 100, this Equation ariseth, *viz.* $aaaa - 2\frac{1}{2}aa = 99\frac{7}{16}$; whence the four numbers sought will be found the same as before.

QUEST. 6.

1. . . . . If . . . $8a^3 + 63aa - a^4 - 341a = 1304$,
2. That is, If . . . $a^4 - 8a^3 - 63aa + 341a + 1304 = 0$;
   What is the number a?

RESOLUTION.

3. The Divisors of the last Term 1304 are 1, 2, 4, 8, 163, 326 and 1304; then after tryals made by Division (as in the preceding Questions,) I find that $a - 8 = 0$ will exactly divide the summ of the Equation proposed without any Remainder, and therefore 8 is one Affirmative value of the Root a. Again, because the Divisors of 163 the last Term of this Equation $aaa - 63a - 163 = 0$ (which was the Quotient of the said Division) are only Unity and 163, I try to divide the Equation last mentioned by $a - 1$ and $a + 1$, likewise by $a - 163$ and $a + 163$, but none of these Divisions working off just without a Fraction, and there being no second Term to be taken away, I search out one Affirmative value of a out of the said Equation $aaa - 63a - 163 = 0$, (that is, $aaa - 63a = 163$,) by the general Method in the foregoing *Chap.* 10. and thereby discover $a = 9.0055$, &c. then I divide the said Cubick Equation $aaa - 63a - 163 = 0$, by $a - 9.0055 = 0$, and the Quotient (the Remainder after the Division is ended being neglected) is $aa + 9.0055a + 18.09903025 = 0$; but this Equation cannot possibly have any affirmative Root, and therefore I conclude that the Equation first propos'd to be resolved hath only two affirmative Roots or values of a, to wit, 8 and 9.0055, &c. found out as above.

By the like Operation it will appear that this Equation $a^4 - 17a^3 - 212aa + 4979a - 21131 = 0$ may be expounded by every one of these three Roots or values of a, to wit, 11, 7.1125, &c. and 15.8874, &c. but by no other affirmative Root.

When the Index of the Power of the unknown Quantity in every Term of an Equation is an even number, the Resolution of such Equation will admit of a Contraction; which will be made manifest by this following

QUEST. 7.

1. If . . . . $a^6 - 29a^4 + 244a^2 - 576 = 0$, what is $a = ?$

RESOLUTION.

2. Here because the Indices of the unknown Powers are even numbers to wit, 6, 4 and 2, put . . . . $\}$ $e = a^2$

3. Then

3. Then for . . . . $\left\{\begin{array}{l} + a^6 \\ - 29a^4 \\ + 244a^2 \end{array}\right\}$ write $\left\{\begin{array}{l} + e^3 \\ - 29e^2 \\ + 244e \end{array}\right.$

4. To which Powers of e joyn -576 the laſt Term of the given Equation, and it makes
$$e^3 - 29e^2 + 244e - 576 = 0.$$

5. Which laſt Equation being reſolved by Diviſion, (in like manner as in the preceding Examples of this Section,) there will be found three Affirmative values of the Root e, *viz.* 4, 9 and 16; then becauſe e was put equal to aa, the ſquare Roots of 4, 9 and 16, that is, 2, 3 and 4, ſhall be three Roots or values of a in the Equation firſt propoſed, to wit, $a^6 - 29a^4 + 244a^2 - 576 = 0$, as may eaſily be proved.

I might here ſhew how to reduce a Biquadratick Equation not falling under any of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. to a Cubick Equation, and ſometimes into two Quadratick Equations, but I ſhall ſpare that labour for theſe Reaſons; Firſt, that Reduction being ſubject to many Caſes, is very tedious and troubleſome: Secondly, ſuch a Biquadratick Equation is very ſeldom capable of being reduced into two Quadratick Equations; and when 'tis reduced to a Cubick Equation, this may happen to be ſuch as its Root or Roots in numbers cannot be perfectly found out by any Rules hitherto publiſh'd by any Author: Thirdly, by the Method in this ninth Section, all the Roots of any Cubick, Biquadratick or other Equation of higher degrees may be found out in numbers, either exactly, if they be Rational, or as near the truth, if they be Irrational, as ſhall be needful for any practical uſe: And laſtly, my undertaking (as I have before hinted,) is not to handle all, but the moſt uſeful Rules only in this profound Art.

Note. The Reſolutions of the preceding Queſtions of this ninth Section do clearly ſhew, that there is no ſmall labour in making tryals with the Diviſors of the laſt Term of an Equation to find its Root or Roots; and therefore to leſſen that work, firſt, it will be convenient to make ſome tryals by the general Method in the foregoing *Chapt.* 10. to find out limits within which the Root or Roots of an Equation do fall, or to argue the ſame from ſome things given in a Queſtion producing the ſaid Equation, and then to make tryals only with ſuch Diviſors of the laſt Term as fall within thoſe limits; but when all Contractions are uſed, the work is ſufficiently laborious, ſo that one chief ſcope of an Analyſt in reſolving a knotty Queſtion muſt be to frame his Poſitions with ſuch artifice that the Reſolution may end in as ſimple an Equation as is poſſible: And although one way of Reſolution may produce an Equation compoſed of high Powers, yet often-times by another way you may come to a more ſimple Equation, as may partly appear by the foregoing fourth and fifth Queſtions of this Section; but the skill of finding out the moſt ſimple and facil ways of Reſolution, is not attainable, (as I conceive,) by any certain or conſtant Method, but rather by much uſe and exerciſe in the ſolving of Queſtions.

Sect. X. *Concerning the Reſolution of certain Cubick Equations in numbers, by two Rules, the Invention whereof* Cardanus *attributes to* Scipio Ferreus.

1. All Cubick Equations, after the ſecond Term is taken away, when there happens to be any, (by the Rule in *Sect.* 8. of this *Chapt.*) are reducible to theſe three following Forms, in which a repreſents the Root or Quantity ſought, but p and q known Quantities.

$aaa = -6a + 20$	$aaa = -pa + q$
$aaa = +6a + 40$	$aaa = +pa + q$
$aaa = +91a - 330$	$aaa = +pa - q$

2. Now let it be required to reſolve the firſt of thoſe Equations, *viz.*
If $aaa = -6a + 20$; or, $aaa = -pa + q$;
What is the value of a?

Preparation.

3. Suppoſe . . . . $a = e - y$
4. Suppoſe alſo . . . . $20 = eee - yyy$
5. And . . . . $6 = 3ey$
6. Then by multiplying each part of the Equation in the third ſtep into it ſelf Cubically, this is produced, *viz.* $aaa = eee - 3eey + 3eyy - yyy$

7. And

7. And by multiplying the Equations in the third and fifth steps one into the other, it makes . . . . . $6a = 3eey - 3eyy$

8. And by subtracting the Equation in the seventh step from that in the fourth, there remains . . . . $20 - 6a = eee - 3eey + 3eyy - yyy$

9. Therefore by the sixth and eighth steps 'tis manifest that . . . . $aaa = eee - 3eey + 3eyy - yyy = 20 - 6a$

10. From the premisses it's evident, that if in the Equation propos'd to be resolv'd, to wit, $aaa = -6a + 20$, or $aaa = -pa + q$, we suppose the Root a sought to be equal to the difference of two unknown numbers e and y; also the Absolute number 20 (or q) to be equal to the difference of the Cubes of the same two numbers, and the Coefficient 6 (or p) to be equal to the triple Product of their multiplication; then as well aaa, as $20 - 6a$ (that is, $q - pa$) shall be equal to the Cube of the difference of those two numbers, *viz.* to the Cube of $e - y$; and therefore when two such numbers are found out, their difference shall be the Root or number a sought: But to find out the said two numbers (e and y) there is given the Product of their multiplication, to wit, 2, (or $\frac{1}{3}p$,) that is, one third part of the Coefficient, as also 20 (or q) the difference of the Cubes of the same two numbers: and therefore the numbers themselves shall be given severally by the Canon of *Quest.* 15. *Chap* 15. *Book* 1. and consequently the Root a sought shall be given also, as will be made manifest by this following

Operation.

11. To the Square of half the given Absolute number 20 (or q) *viz.* to	100	$\frac{1}{4}qq$
12. Add the Cube of 2 (or $\frac{1}{3}p$) *viz.* the Cube of $\frac{1}{3}$ of the Coefficient 6 (or p,) which Cube is	8	$\frac{1}{27}ppp$
13. The summ is . . .	108	$\frac{1}{4}qq + \frac{1}{27}ppp$
14. The square Root of that summ is . . .	$\sqrt{108}$	$\sqrt{}:\overline{\frac{1}{4}qq + \frac{1}{27}ppp}:$
15. To that square Root add half the Absolute number 20 (or q,) and the summ is . . .	$10 + \sqrt{108}$	$\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}$
16. The Cubick Root of that summ is the greater number e sought, *viz.*	$\sqrt{(3)}:\overline{10 + \sqrt{108}}:$	$\sqrt{(3)}:\overline{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}}:$
17. Again, from the square Root in the fourteenth step, subtract half the Absolute number 20 (or q,) and the Remainder is .	$-10 + \sqrt{108}$	$-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}$
18. Then the Cubick Root of that Remainder shall be the lesser number y sought, *viz.* . . .	$\sqrt{(3)}:\overline{-10 + \sqrt{108}}:$	$\sqrt{(3)}:\overline{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}}:$

19. And then the difference of the two Cubick Roots found out in the sixteenth and eighteenth steps shall be the value of the Root a in the Equation proposed, *viz.*

$$a = \sqrt{(3)}:\overline{10 + \sqrt{108}}: - \sqrt{(3)}:\overline{-10 + \sqrt{108}}: \quad \text{that is,}$$

$$a = \sqrt{(3)}:\overline{\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}qq + \tfrac{1}{27}ppp}}: - \sqrt{(3)}:\overline{-\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}qq + \tfrac{1}{27}ppp}}:$$

20. It remains to make tryal whether the Binomial $10 + \sqrt{108}$ hath a perfect Cubick Root or not; so by the Rule in *Sect.* 18. *Chapt.* 9. of this Second *Book*, it will appear that $1 + \sqrt{3}$ is the Cubick Root of $10 + \sqrt{108}$, and $\sqrt{3} - 1$ is the Cubick Root of $\sqrt{108} - 10$, and consequently the value of the Root a before found out in the nineteenth step is expressible by a Rational number; for if $\sqrt{3} - 1$ be subtracted

from $1 + \sqrt{3}$, the Remainder 2 is the deſired value of a in the Equation propoſed; for if $a = 2$, then $aaa = 20 - 6a$, or $aaa + 6a = 20$.

21. In like manner, by the Canon in the foregoing nineteenth ſtep the value of a in this Equation $aaa + 27a = 64$, will be found this that follows, *viz.*

$$a = \sqrt{(3)}:\overline{32 + \sqrt{1753}}: - \sqrt{(3)}:\overline{-32 + \sqrt{1753}}:$$

But this value of a cannot be expreſt by any Rational number, becauſe the Binomial $32 + \sqrt{1753}$ hath not a perfect Cubick root, and therefore the ſaid value muſt either reſt in that ſurd Form, or elſe be expreſt by ſome Rational number near the true value, which will be found 2.05, &c. that is, $2\frac{5}{100}$, &c.

22. In the next place let it be required to reſolve a Cubick Equation of the ſecond of the three Forms before mentioned, *viz.*

If . . . . . $aaa = 6a + 40$, or, $aaa = pa + q$;
What is the value of a?

Preparation.

23. Suppoſe	$a = e + y$
24. Suppoſe alſo, ...	$40 = eee + yyy$
25. And	$6 = 3ey$
26. Then by multiplying each part of the Equation in the twenty third ſtep into it ſelf cubically, this is produced,	$aaa = eee + 3eey + 3eyy + yyy$
27. And the Equations in the twenty third and twenty fifth ſteps being mutually multiplied one by the other will produce	$6a = 3eey + 3eyy$
28. And the ſumm of the Equations in the twenty fourth and twenty ſeventh ſteps makes	$6a + 40 = eee + 3eey + 3eyy + yyy$
29. Therefore by the twenty ſixth and twenty eighth ſteps 'tis evident that	$aaa = eee + 3eey + 3eyy + yyy = 6a + 40$

30. By the eight laſt preceding ſteps 'tis manifeſt, That if in the Equation propos'd to be reſolved, to wit, $aaa = 6a + 40$, or $aaa = pa + q$, we ſuppoſe the Root a ſought to be equal to the ſumm of two unknown numbers, e and y, alſo the Abſolute number 40 (or q) to be equal to the ſumm of the Cubes of the ſame two numbers, and the Coefficient 6 (or p) to be equal to the triple Product of their multiplication, then as well aaa, as $6a + 40$ (that is, $pa + q$) ſhall be equal to the Cube of $e + y$; and therefore when two ſuch numbers are found out, their ſumm ſhall be the Root or number a ſought. But to find out the ſaid two numbers (e and y) there is given the Product of their multiplication, to wit, 2 (or $\frac{1}{3}p$,) that is, $\frac{1}{3}$ part of the Coefficient, as alſo 40 (or q) the ſumm of the Cubes of the ſame two numbers, and therefore the numbers ſhall be given ſeverally by the Canon of *Queſt.* 14. *Chap.* 16. *Book* 1. and conſequently the Root a ſought ſhall be given alſo: all which will made manifeſt by this following

Operation.

31. From the Square of half the given Abſolute number 40 (or q,) *viz.* from	400	$\frac{1}{4}qq$
32. Subtract the Cube of 2 (or $\frac{1}{3}p$,) *viz.* the Cube of $\frac{1}{3}$ of the Coefficient, which Cube is	8	$\frac{1}{27}ppp$
33. The Remainder is	392	$\frac{1}{4}qq - \frac{1}{27}ppp$
34. The ſquare Root of that Remainder is	$\sqrt{392}$	$\sqrt{:\frac{1}{4}qq - \frac{1}{27}ppp:}$
35. Which ſquare Root added to half the Abſolute number 40 (or q,) makes the ſumm	$20 + \sqrt{392}$	$\frac{1}{2}q + \sqrt{:\frac{1}{4}qq - 2\frac{1}{27}ppp:}$
36. The Cubick Root of the ſumm in the laſt ſtep is the value of e, ...	$\sqrt{(3)}:\overline{20 + \sqrt{392}}:$	$\sqrt{(3)}:\overline{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}ppp}}:$
37. The ſquare Root in the thirty fourth ſtep being ſubtracted from half the Abſolute number 40 (or q,) leaves . .	$20 - \sqrt{392}$	$\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}ppp}$
38. The Cubick Root of that Remainder is the value of y, ...	$\sqrt{(3)}:\overline{20 - \sqrt{392}}:$	$\sqrt{(3)}:\overline{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}ppp}}:$

36. Then

39. Then the summ of the two Cubick Roots found out in the thirty sixth and thirty eighth steps shall be the value of the Root a in the Equation propos'd to be resolved, *viz.*

$$a = \sqrt{(3)}:\overline{20 + \sqrt{392}}: + \sqrt{(3)}:\overline{20 - \sqrt{392}}: \text{ that is,}$$

$$a = \sqrt{(3)}:\overline{\tfrac{1}{2}q + \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}ppp}}: + \sqrt{(3)}:\overline{\tfrac{1}{2}q - \sqrt{\tfrac{1}{4}qq - \tfrac{1}{27}ppp}}:$$

40. It remains to make tryal whether the Binomial $20 + \sqrt{392}$ hath a perfect Cubick Root or not; so by the Rule in *Sect.* 18. *Chap.* 9. of this Second *Book* you will find $2 + \sqrt{2}$ to be the Cubick Root of $20 + \sqrt{392}$, and $2 - \sqrt{2}$ the Cubick Root of $20 - \sqrt{392}$, and consequently the value of the Root a before found out in the thirty ninth step is expressible by a Rational number; for if $2 - \sqrt{2}$ be added to $2 + \sqrt{2}$, the summ 4 is the desired value of a in the Equation propos'd to be resolved: for if $a = 4$, then $aaa = 6a + 40$.

41. Another Example of resolving a Cubick Equation of the second Form may be this; *viz.* Let it be required to find the value of a in this Equation, $aaa = 12a + 18$; that is, $aaa = pa + q$, then the Canon exprest by the literal Equation in the thirty ninth step, will give

$$a = \sqrt{(3)}:\overline{9 + \sqrt{17}}: + \sqrt{(3)}:\overline{9 - \sqrt{17}}:$$

But this value of a is inexpressible by any Rational number, because the Binomial $9 + \sqrt{17}$ hath not a perfect Cubick Root, and therefore the said value must either rest in that surd Form, or else be exprest by some Rational number near the true value, which will be found 4.05, *&c.* that is, $4\frac{5}{100}$, *&c.*

The premisses do clearly shew the rise of two Rules delivered by *Cardanus* in his Algebraical Treatise entituled *Ars magna*, which Rules are mentioned in divers Authors, and the substance of them is contained in the two literal Equations in the foregoing nineteenth and thirty ninth steps; the former of which Equations is a Canon to find out the Root of any Cubick Equation in numbers which falls under the first of the three Forms before mentioned, and to express the same perfectly either by some Rational or Irrational number; and the latter of those literal Equations finds out the like exact Root of any Cubick Equation of the second Form, except in one Case only, *viz.* when the Square of half the Absolute number (q) which is the last Term of the Equation is less than the Cube of one third part of the known Coefficient (p.) But no Author that I have met with, gives a certain Rule, either to find out the Root in that case if it be an Irrational number; or the two affirmative Roots of a Cubick Equation of the third Form, if each of these also be Irrational. *Huddenius* indeed saith, (in page 503. of *Cartesius*'s Geometry before mentioned) he had a Rule (which he intended to publish) by which all Irrational Roots as well of Numeral as of Literal Equations may be found out, but that much desired Rule is not yet come to light. But when a Cubick Equation of what kind soever hath one Root expressible by a Rational number, both that and the rest of the Roots, when the Equation is capable of more than one, may be exactly found out by the help of the Divisors of the last Term, according to *Sect.* 9. of this *Chapter*.

CHAP. XIII.

Of the method of resolving Questions wherein many Quantities are sought, by assuming different Letters to represent the said Quantities severally.

1. Hitherto in the Algebraical Resolution of a Question wherein two or more Quantities have been sought, I have assumed only one letter, as a, or e to represent some one of the unknown Quantities, and formed the Positions for the rest by the help of that letter and the Quantities given in the Question: But many Questions may be more easily resolved by assuming a peculiar letter to represent every one of the Quantities sought; as a for one unknown Quantity, e for a second, y for a third, *&c.* By this Method also those intricate and obscure ways of resolving Questions by second Roots, or (as *Simon Stevin* calls them) postposed Quantities, will be avoided.

In handling the following Method I shall give three principal Rules; and explain them by Examples; but to prescribe Rules for all Cases, is (as I conceive) an impossible work.

RULE I.

When many Quantities are sought by a Question, first let them be severally represented by different letters; then after you have well considered the conditions in the Question, abstract it from words, and express the tenor thereof by Equations; that done, by the help of Transposition find what the first, that is, any single letter representing a number or quantity sought in the first Equation is equal to; then wheresoever that first letter is found in the other Equations, take instead of it those Quantities to which the said first letter was found equal; so such first letter will quite vanish out of those other Equations. Again, by Transposition set a second letter alone in one of those Equations out of which the first letter was expell'd, and proceed as before; so at length one of the numbers sought will be made known, by the help whereof the rest will easily be discovered. This work will be better understood by Examples than many words, and therefore I shall proceed to Questions.

QUEST. 1.

A Factor exchanged 6 *French* Crowns and two Dollars, for 45 Shillings of *English* Money; also at another time he exchanged 9 *French* Crowns and 5 Dollars (each of these being of the same value with the former) for 76 Shillings: I demand the value of a *French* Crown, and also of a Dollar, in *English* Money?

Let a represent the desired value of a Crown, and e the value of a Dollar, then the Question being abstracted from words may be stated thus;

1. If . . . . $6a + 2e = 45$
2. And . . . . $9a + 5e = 76$

What are the numbers a and e? ||

RESOLUTION.

3. By transposition of $2e$ in the first Equation this ariseth, $6a = 45 - 2e$
4. And by dividing each part of the third Equation by 6, it gives . . . . $a = \frac{45 - 2e}{6}$
5. The fourth Equation multiplied by 9, produceth . . $9a = \frac{405 - 18e}{6}$
6. Then if instead of $9a$ in the second Equation, you take the latter part of the fifth, this will arise, ... $\frac{405 - 18e}{6} + 5e = 76$
7. The sixth Equation, after due Reduction, discovers the value of a Dollar, *viz.* . . . . $e = 4\frac{1}{4}$
8. The seventh Equation multiplied by 2 gives . . . $2e = 8\frac{1}{2}$
9. And by setting the latter part of the eighth Equation in the place of $2e$ in the first, this Equation ariseth, . . . $6a + 8\frac{1}{2} = 45$
10. From which last Equation, after due Reduction, the value of a, or one *French* Crown is discovered, *viz.* $a = 6\frac{1}{12}$

Thus by the seventh and tenth Equations it is found that a Dollar was valued at 4 *s.* 3 *d.* and a *French* Crown at 6 *s.* 1 *d.* which numbers will satisfie the conditions in the Question, as may easily be proved.

QUEST. 2.

Three men had every one of them a certain number of Pounds in his Purse; the summ of the first and second man's money was 5 (or b) Pounds, the summ of the second and third man's money was 12 (or c) Pounds, and the summ of the third and first man's money was 11 (or d) Pounds: How many Pounds had every one in his Purse?

Let the three numbers of Pounds sought be represented by a, e and y; then respect being had to the numbers given, the Question may be stated thus, *viz.*

1. If . . . . $a + e = b \ (= 5)$
2. And . . . . $e + y = c \ (= 12)$
3. And . . . . $y + a = d \ (= 11)$

What are the numbers a, e and y? ||

RESO-

RESOLUTION.

4. By transposition of a in the first Equation, there will arise $\quad e = b - a$
5. Then by taking the latter part of the fourth Equation instead of e in the second, this Equation ariseth, $\quad b - a + y = c$
6. And by transposition of $b - a$ in the 5th Equation it gives $\quad y = c - b + a$
7. And by taking the latter part of the sixth Equation instead of y in the third, this ariseth, $\quad c - b + a + a = d$
8. From which seventh Equation, after due Reduction, the number a will be made known, *viz.* $\quad a = \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c$
9. Again, if instead of a in the first Equation we take the latter part of the eighth, this ariseth, $\quad \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c + e = b$
10. Then from the ninth, after due Reduction, the number e will be made known, *viz.* $\quad e = \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d$
11. Again, if instead of a in the third Equation we take the latter part of the eighth, this ariseth, $\quad y + \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c = d$
12. Lastly, from the eleventh Equation, after due Reduction, the number y will be made known, *viz.* $\quad y = \frac{1}{2}d + \frac{1}{2}c - \frac{1}{2}b$

The eighth, tenth and twelfth Equation gives this

CANON.

From the summ of every two of the three numbers given, subtract the remaining number, then the halves of the three Remainders shall be the numbers sought. Whence the numbers sought, to wit, a e and y will be found 2, 3 and 9: for $2 + 3 = 5$; also $3 + 9 = 12$; and $9 + 2 = 11$; as was required.

The foregoing Resolution of this *Quest.* 2. is formed according to *Rule* 1: but the same Canon may be more expeditiously discovered by this following Resolution, *viz.*

The summ of the first, second and third Equations which state the Question is $\quad 2a + 2e + 2y = b + c + d$

The half of that summ is $\quad a + e + y = \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}d$

Then from that half summ subtract the first Equation, and the Remainder will be $\quad y = \frac{1}{2}c + \frac{1}{2}d - \frac{1}{2}b$

Again, from the said half summ subtract the second Equation, and the Remainder is $\quad a = \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c$

Lastly, from the said half summ subtract the third Equation, and the Remainder gives $\quad e = \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d$

Which three last Equations do manifestly give the same values of a, e and y, as were found out by the former Resolution.

QUEST. 3.

Three Men discourse of their moneys in this manner; the first saith to the other two, if 100 *l.* were added to his money, the summ would be equal to both their moneys; the second saith to the other two, if 100 *l.* were added to his money, the summ would be equal to the double of both their moneys; the third saith to the other two, if 100 *l.* were added to his money, the summ would be equal to the triple of both their moneys: The Question is, to find how many Pounds each Man had?

Let the three numbers of Pounds sought be represented by a, e and y; then the Question may be stated thus, *viz.*

1. If $\quad a + 100 = e + y$
2. And $\quad e + 100 = 2a + 2y$
3. And $\quad y + 100 = 3a + 3e$

What are the numbers a, e and y?

RESOLUTION.

4. From the first Equation by transposition of y, this ariseth, $\quad a + 100 - y = e$
5. Then if instead of e in the second Equation, there be taken that which is equal to e, to wit, the first part of the fourth, this will arise, $\quad a + 100 - y + 100 = 2a + 2y$
6. That is, after due Reduction, $\quad 200 = a + 3y$

7. Again,

7. Again, if instead of $3e$ in the third Equation, there be taken the triple of the first part of the fourth Equation, this will arise, to wit, . .	$y + 100 = 3a + 3a + 300 - 3y$
8. Which last Equation, after due Reduction, gives	$y = \frac{3}{2}a + 50$
9. Then if instead of $3y$ in the sixth Equation, there be set the triple of the latter part of the eighth, this will come forth, *viz.* . . .	$200 = a + \frac{9}{2}a + 150$
10. From the ninth Equation, after due Reduction, the number a will be discovered, *viz.*	$a = 9\frac{1}{11}$
11. Again, if instead of a in the sixth Equation, there be taken $9\frac{1}{11}$, to wit, the value of a found out in the tenth, it will give . . .	$200 = 9\frac{1}{11} + 3y$
12. The eleventh Equation duely reduced discovers the number y, viz.	$y = 63\frac{7}{11}$
13. From the fourth, tenth and twelfth Equations by exchange of equal Quantities, this Equation ariseth, *viz.*	$9\frac{1}{11} + 100 - 63\frac{7}{11} = e$
14. The thirteenth reduced gives	$e = 45\frac{5}{11}$

From the 10*th*, 14*th* and 12*th* Equations, the three numbers sought, a, e and y are discovered, *viz.* The first Man had $9\frac{1}{11}$ *l.* the second $45\frac{5}{11}$ *l.* and the third $63\frac{7}{11}$ *l.* which numbers will satisfie the Question, as may easily be proved.

If 121 be given instead of 100 in this third Question, then the three numbers sought will be whole numbers, to wit, 11, 55, 77.

RULE II.

When the same Quantity, suppose a, is found in two several Equations, and equal numbers are prefixed to those Quantities, then if their signs be both +, or both —, subtract the lesser Equation from the greater; but if one of the signs be + and the other —, add those two Equations together; so the said Quantity a will quite vanish, as will appear by the Resolution of the following Question.

QUEST. 4.

The summ of two numbers being given 12 (or b,) and their difference 8 (or c;) to find the numbers.

Let a be put for the greater number, and e for the lesser, and the Question may be stated thus;

1. If	$a + e = b\ (= 12)$
2. And	$a - e = c\ (= 8)$

What are the numbers a and e?

RESOLUTION.

3. For as much as a or $+1a$ is found in each of the Equations proposed, therefore (according to *Rule* 2.) I subtract the lesser Equation from the greater; whence the letter a quite vanisheth, and there remains	$2e = b - c\ (= 4)$
4. Then by dividing each part of the third Equation by 2, the number e is made known, *viz.*	$e = \frac{1}{2}b - \frac{1}{2}c\ (= 2)$
5. And by taking the latter part of the fourth Equation instead of e in the first, there remains . . .	$a + \frac{1}{2}b - \frac{1}{2}c = b\ (= 12)$
6. Lastly, the fifth Equation duely reduced discovers the number a, viz.	$a = \frac{1}{2}b + \frac{1}{2}c\ (= 10)$

The 6*th* and 4*th* Equations discover a Canon to find out the numbers sought, which in this Example are 10 and 2, and the Canon is the same with that before found in *Quest.* 1. *Chap.* 14. *Book* 1.

Otherwise thus;

7. For as much as $+e$ is found in the first Equation, and $-e$ in the second, therefore by adding those two Equations together, (according to *Rule* 2.) the letter e vanisheth, and the summ is	$2a = b + c\ (= 20)$

8. There-

8. Therefore by dividing each part of the seventh Equation by 2, there ariseth the same value of a which was before found in the sixth Equation, *viz.*	$a = \frac{1}{2}b + \frac{1}{2}c\ (= 10)$
9. And by setting the latter part of the eighth Equation in the place of a in the first, this ariseth, . . .	$\frac{1}{2}b + \frac{1}{2}c + e = b\ (= 12)$
10. Which last Equation reduced discovers the same value of e which was before found in the fourth Equation, *viz.*	$e = \frac{1}{2}b - \frac{1}{2}c\ (= 2)$

RULE III.

When the same Quantity, suppose a, is found in two several Equations, but the numbers prefixt to those equal Quantities are unequal, those two Equations may be reduced into two others which shall have equal numbers prefixt to the said Quantity a, by this Rule, *viz.* Multiply all the Quantities in the first Equation by the number which is prefixt to the said Quantity a in the second; multiply likewise all the Quantities in the second Equation by the number which is prefixt before the same Quantity a in the first; so by such alternate multiplication two new Equations will be produced, wherein the numbers prefixt to the said Quantity a will be equal to one another: and then by adding or subtracting, according to the import of *Rule* 2. of this *Chapt.* that Quantity a will quite vanish. That done, renew the like work to expell the same Quantity out of the rest of the Equations; and proceed in like manner with a second Quantity, until at length the value of some one Quantity be made known. This I shall make plain by the Resolutions of Five *Questions* next following.

QUEST. 5.

To find two numbers, that if the quadruple of the greater be increased with the triple of the less, it may make 36; but if the triple of the greater be lessened by the double of the less, the Remainder may be 10.

Put a for the greater number, and e for the lesser, then the Question may be stated thus, *viz.*

1. If	$4a + 3e = 36$
2. And	$3a - 2e = 10$
What are the numbers a and e? ||	

RESOLUTION.

3. The first Equation multiplied by 3, which is prefixt to a in the second, produceth	$12a + 9e = 108$
4. The second Equation multiplied by 4, which is prefixt to a in the first, makes	$12a - 8e = 40$
5. Now for as much as the Quantity $12a$ is found both in the fourth and third Equations, and is Affirmative in each, therefore according to *Rule* 2. I subtract the lesser Equation from the greater, so the Quantity $12a$ vanisheth, and this Equation remains,	$9e + 8e = 68$
6. The fifth Equation, after due Reduction, discovers the number e, viz.	. . . . $e = 4$
7. Then I set 12 (which by the sixth Equation is the value of $3e$) in the place of $3e$ in the first, and this Equation ariseth, . .	$4a + 12 = 36$
8. Lastly, the seventh Equation duely reduced discovers the number a, viz.	. . . $a = 6$

From the 8*th* and 6*th* Equations the two numbers sought are found 6 and 4, which will solve the Question: For four times 6 with thrice 4 makes 36; and thrice 6, to wit, 18, lessened by twice 4 gives 10, as was required.

QUEST. 6.

1. If	$2a + 3e - 2y = 50$
2. And	$5a - 2e + 5y = 240$
3. And	$-a + 5e - 3y = 10$
What are the numbers a, e and y? ||	

RESO-

RESOLUTION.

4. The first Equation multiplied by 5, which is prefixt to a in the second, produceth . . . . $10a + 15e - 10y = 250$

5. Likewise the second Equation multiplied by 2, which is prefixt to a in the first, makes . . . . $10a - 4e + 10y = 480$

6. Then (according to *Rule* 2.) by substracting the fourth Equation from the fifth, the Quantity $10a$ vanisheth, and this Equation ariseth, ... $-19e + 20y = 230$

7. Again, the third Equation multiplied by 5 which is prefixt to a in the second, produceth . . . . $-5a + 25e - 15y = 50$

8. And the second Equation multiplied by 1, which is supposed to be prefixt to a in the third, gives the same second Equation without alteration, *viz.* . . . . $+5a - 2e + 5y = 240$

9. Then because $+5a$ and $-5a$ by addition will destroy one another, therefore (according to *Rule* 2.) I add the seventh and eighth Equations together; so the letter a vanisheth, and this Equation ariseth, ... $+23e - 10y = 290$

10. Again, I proceed with the sixth and ninth Equations according to *Rule* 3. *viz.* I multiply the sixth Equation by 23, (which is prefixt to e in the ninth,) and it makes $-437e + 460y = 5290$

11. Also the ninth Equation multiplied by 19 (which is prefixt to e in the sixth, produceth . . . . . . $+437e - 190y = 5510$

12. Then (according to *Rule* 2.) by adding the tenth and eleventh Equations together, the letter e vanisheth, and this Equation ariseth, *viz.* . . . . $+270y = 10800$

13. And by dividing each part of the twelfth Equation by 270, the number y is discovered, *viz.* . . . . $y = 40$

14. Then instead of $10y$ in the ninth Equation taking ten times 40, that is, 400, (which by the thirteenth Equation is equal to $10y$) the ninth will be reduced to this, . . $+23e - 400 = 290$

15. And from the fourteenth Equation, after due Reduction, the number e will be discovered, *viz.* . . . . . . $e = 30$

16. Then instead of $3e - 2y$ in the first Equation, I take $90 - 80$, (which by the fifteenth and thirteenth Equations will be found equal to $3e - 2y$,) so the first Equation will be converted into this, *viz.* . . . . $2a + 90 - 80 = 50$

17. Lastly, the sixteenth Equation duely reduced discovers the number a, viz. . . . . $a = 20$

From the 17*th*, 15*th* and 13*th* Equations the three desired numbers a, e, y, are 20, 30 and 40, which will constitute the three Equations first proposed, as may easily be proved.

QUEST. 7.

Three Men discourse of their moneys in this manner; the first saith to the other two, if you give me 100 Pounds, my money will be made equal to both your remaining moneys: the second saith to the other two, if ye give me 100 Pounds, my money will be made equal to the double of both your remaining moneys: lastly, the third saith to the other two, if ye give me 100 Pounds, my money will be equal to the triple of both your remaining moneys: I demand how many Pounds each Man had?

Let a letter be assumed to represent each Man's money; as a for the first, e for the second, and y for the third; then the Question may be stated thus, *viz.*

1. If . . . . $a + 100 = e + y - 100$
2. And . . . . $e + 100 = 2a + 2y - 200$
3. And . . . . $y + 100 = 3a + 3e - 300$

What are the numbers a, e, y?

RESOLUTION.

4. The first Equation by transposition will be reduced to this, ... $-a + e + y = 200$

5. Likewise

5. Likewiſe the ſecond Equation by tranſpoſition gives . . . . $+2a - e + 2y = 300$
6. And the 3*d* Equation by tranſpoſition produceth $+3a + 3e - y = 400$
7. Then I proceed with the fourth and fifth Equations according to *Rule* 3. *viz.* I multiply the fourth Equation by 2, (which is prefixt to *a* in the fifth,) and it produceth . . . . . . $-2a + 2e + 2y = 400$
8. The ſumm of the fifth and ſeventh Equations gives . . . $e + 4y = 700$
9. Again, I proceed with the fifth and ſixth Equations according to *Rule* 3. *viz.* multiplying the fifth Equation by 3, (which is prefixt to *a* in the ſixth,) it gives . . . . $6a - 3e + 6y = 900$
10. Alſo the ſixth Equation multiplied by 2, (which is prefixt to *a* in the fifth) produceth . . . . $6a + 6e - 2y = 800$
11. Then by ſubtracting the tenth Equation from the ninth, the Remainder is . . . . $-9e + 8y = 100$
12. Again, I proceed with the eighth and eleventh Equations according to *Rule* 3. *viz.* multiplying the eighth Equation by 9, (which is prefixt to *e* in the eleventh,) it makes . . . . $+9e + 36y = 6300$
13. Then (according to *Rule* 2.) the eleventh and twelfth Equations added together make . . . . $44y = 6400$
14. And by dividing the thirteenth Equation by 44, the number *y* is made known, *viz.* . . . . $y = 145\frac{5}{11}$
15. From the eighth and fourteenth, by exchange of equal Quantities, this ariſeth, *viz.* . . . . $e + 581\frac{9}{11} = 700$
16. And from the fifteenth, by ſubtraction of $581\frac{9}{11}$ from each part, the number *e* is diſcovered, *viz.* . . . . . $e = 118\frac{2}{11}$
17. From the firſt, fourteenth and ſixteenth Equations, by exchange of equal Quantities, this Equation ariſeth, *viz.* . . . . $a + 100 = 118\frac{2}{11} + 145\frac{5}{11} - 100$
18. Laſtly, the ſeventeenth Equation, after due Reduction, diſcovers the number *a*, viz. . . . . $a = 63\frac{7}{11}$

Thus, by the 18*th*, 16*th* and 14*th* Equations it is found that the firſt Man had $63\frac{7}{11}$ *l.* the ſecond $118\frac{2}{11}$ *l.* and the third $145\frac{5}{11}$ *l.* which three numbers will ſatisfie the Queſtion, as may eaſily be proved.

QVEST. 8.

1. If . . . . $a + \frac{2}{3}e + \frac{2}{3}y + \frac{2}{3}u = 112$
2. And . . . . $e + \frac{3}{4}a + \frac{3}{4}y + \frac{3}{4}u = 114$
3. And . . . . $y + \frac{4}{5}a + \frac{4}{5}e + \frac{4}{5}u = 125\frac{2}{5}$
4. And . . . . $u + \frac{5}{6}a + \frac{5}{6}e + \frac{5}{6}y = 133\frac{1}{3}$

What are the numbers *a*, *e*, *y* and *u* ?

RESOLUTION.

5. The firſt Equation multiplied by 3, (the Denominator of the Fraction $\frac{2}{3}$) produceth this Equation in Integers, to wit, ... $3a + 2e + 2y + 2u = 336$
6. Likewiſe the ſecond Equation multiplied by 4, produceth . . . . $3a + 4e + 3y + 3u = 456$
7. And the third Equation multiplied by 5 gives . $4a + 4e + 5y + 4u = 628$
8. Alſo the fourth Equation multiplied by 6 produceth . . . . $5a + 5e + 5y + 6u = 800$
9. For as much as 3*a* is found in the fifth, and alſo in the ſixth Equation, I ſubtract the leſſer from the greater, ſo 3*a* quite vaniſheth, and this Equation ariſeth, ... $2e + y + u = 120$

O o 10. Then

10. Then I proceed with the fifth and seventh Equations according to *Rule* 3. *viz.* I multiply the fifth Equation by 4, (which is prefixt to a in the seventh,) and there comes forth . . . . $12a + 8e + 8y + 8u = 1344$

11. Also I multiply the seventh Equation by 3, (which is prefixt to a in the fifth,) and it produceth $12a + 12e + 15y + 12u = 1884$

12. Then by subtracting the tenth Equation from the eleventh, the quantity $12a$ quite vanisheth, and this Equation ariseth, to wit, ... $4e + 7y + 4u = 540$

13. The ninth Equation multiplied by 2, produceth . . $4e + 2y + 2u = 240$

14. Then by subtracting the thirteenth Equation from the twelfth, this ariseth, to wit, ... $5y + 2u = 300$

15. Again, I proceed with the fifth and eighth Equations according to *Rule* 3. *viz.* I multiply the fifth Equation by 5, (which is prefixt to a in the eighth,) and it produceth . . . . $15a + 10e + 10y + 10u = 1680$

16. Likewise the eighth Equation multiplied by 3, (which is prefixt to a in the fifth,) produceth . . $15a + 15e + 15y + 18u = 2400$

17. Then by subtracting the fifteenth Equation from the sixteenth, this ariseth, *viz.* . . . . . . $5e + 5y + 8u = 720$

18. Again, I proceed with the ninth and seventeenth Equations according to *Rule* 3. *viz.* I multiply the ninth Equation by 5, (which is prefixt to e in the seventeenth,) and it produceth . . . . $10e + 5y + 5u = 600$

19. And the seventeenth Equation multiplied by 2, (which is prefixt to e in the ninth,) produceth . $10e + 10y + 16u = 1440$

20. Then by subtracting the eighteenth Equation from the nineteenth, there remains . . . . . $5y + 11u = 840$

21. And by subtracting the 14*th* Equation from the 20*th*, (for since $5y$ is found in each of those Equations, they need no Reduction according to *Rule* 3.) there remains . . . . $9u = 540$

22. Which twenty first Equation divided by 9 discovers the number u, viz. . . . . $u = 60$

23. From the 20*th* and 22*d* Equations, by setting eleven times 60, to wit, 660 in the place of $11u$ in the 20*th*, there ariseth . . . . $5y + 660 = 840$

24. Therefore from the twenty third Equation, after due Reduction, the number y is discovered, *viz.* $y = 36$

25. And from the 9, 24 and 22 Equations, this ariseth, $2e + 36 + 60 = 120$

26. The 25*th* duly reduced discovers the number e, viz. $e = 12$

27. From the 5*th*, 26*th*, 24*th*, and 22*d* Equations, by exchange of equal Quantities, this Equation ariseth, $3a + 24 + 72 + 120 = 336$

28. Lastly, from the 27*th*, after due Reduction, the number a is discovered, *viz.* . . . . $a = 40$

Thus by the 28*th*, 26*th*, 24*th* and 22*d* Equations the four numbers sought, (to wit, a, e, y, u) are found 40, 12, 36 and 60, which will constitute the four Equations in *Quest.* 8.

QUEST. 9.

A Maid being at the Market is offer'd 10 Apples for a penny, and 25 Pears for two pence; now if at those rates she would lay out $9\frac{1}{2}$ pence to buy 100 Apples and Pears together, how many Apples, and how many Pears ought she to have?

1. For the number of Apples sought put . . . . a
2. And for the number of Pears sought put . . . . e
3. Then search out the cost of the number of Apples in the first step, and say, If 10 . 1 :: a . $\left(\frac{a}{10}\right.$: so the cost of the number of Apples sought is . . . . $\frac{a}{10}$

4. Search

4. Search out also the cost of the number of Pears in the second step, and say, If 25 . 2 :: e . ($\frac{2e}{25}$. so the cost of the number of Pears sought is found	$\frac{2e}{25}$
5. Then (according to the Question) the money laid out for all the Apples and Pears sought must be equal to $9\frac{1}{2}$ Pence; hence this Equation, ...	$\frac{a}{10} + \frac{2e}{25} = 9\frac{1}{2}$
6. But the number of Apples, together with the number of Pears bought must make 100, therefore	$a + e = 100$
7. Then the Equation in the fifth step, after due Reduction, will give this Equation in Integers, to wit, ...	$50a + 40e = 4750$
8. And the Equation in the sixth step being multiplied by 50 produceth	$50a + 50e = 5000$
9. Then by subtracting the Equation in the seventh step from that in the eighth, there ariseth	$10e = 250$
10. And the Equation in the ninth step divided by 10, discovers the number e, viz.	$e = 25$
11. Lastly, from the sixth and tenth steps, the number a is also made known, *viz.*	$a = 75$

By the first, second, eleventh and tenth steps it appears that there might be bought 75 Apples, and 25 Pears; which numbers will solve the Question, as may easily be proved.

QUEST. 10.

To divide 90 into four such numbers, that if the first be increased with 2; the second lessened by 2; the third multiplied by 2; and the fourth divided by 2; the Summ, Remainder, Product and Quotient may be equal between themselves.

Let b and d be put for the two given numbers, 90 and 2; also a, e, y and u for the four numbers sought, then the Question may be stated thus;

1. If . . . . $a+e+y+u = b$
2. And . . . . $a+d = e-d$
3. And . . . . $a+d = dy$
4. And . . . . $a+d = \frac{u}{d}$

What are the numbers a, e, y and u?

RESOLUTION.

5. The first number sought is equal to it self, *viz.* . . . $a = a$
6. From the second Equation, by transposition of $-d$, this ariseth, ... $a + 2d = e$
7. And by dividing each part of the third Equation by d, this ariseth, ... $\frac{a+d}{d} = y$
8. And the fourth Equation multiplied by d produceth $da+dd = u$
9. The summ of the four last Equations gives

$$2a + 2d + \frac{a+d}{d} + da + dd = a+e+y+u = b$$

10. Which last Equation, after due Reduction, gives $a = \frac{bd-ddd-2dd-d}{dd+2d+1}$
11. Then from the tenth and sixth Equations, by exchange of equal Quantities, ... $e = \frac{bd+ddd+2dd+d}{dd+2d+1}$
12. And from the tenth and seventh Equations, . $y = \frac{b}{dd+2d+1}$
13. And from the tenth and eighth Equations, . $u = \frac{bdd}{dd+2d+1}$

The four last Equations give a Canon to find out the four numbers sought, which are 18, 22, 10 and 40, which will solve the Question. For, first, their summ is 90; then if the first number 18 be increased with the given number 2, it makes 20: and if the second

number 22 be lessened by 2, the Remainder is also 20: moreover, if the third number 10 be multiplied by 2, it likewise produceth 20: lastly, if the fourth number 40 be divided by 2, the Quotient is also 20. Therefore the conditions in the Question are satisfied.

But the Numerator of the Fraction in the latter part of the tenth Equation shews, That the numbers b and d must not be given at random, but so, that $ddd + 2dd + d$ may be subtracted from bd and leave a Remainder greater than nothing; therefore bd must be greater than $ddd + 2dd + d$, and consequently b must be greater than $dd + 2d + 1$. Therefore, to the end the Question may be possible, the numbers given must be subject to this

Determination.

The number given to be divided (b) must be greater than the Square of ($d + 1$) the summ of the other number given and Unity.

QUEST. 11.

There are two numbers whose summ is equal to the difference of their Squares; and if the summ of the Squares of those two numbers be subtracted from the Square of their summ, the Remainder will be 60: what are the two numbers?

Put b for the given number 60, also a for the greater number sought, and e for the lesser; then the Question may be stated thus, *viz.*

1. If . . . . $aa - ee = a + e$
2. And . . . . . . $aa + ee + 2ae - aa - ee = b$

What are the numbers a and e?

RESOLUTION.

3. The second Equation after its first part is duely contracted is . . . . $2ae = b$
4. And the third Equation divided by 2 gives . . $ae = \frac{1}{2}b$
5. And if each part of the first Equation be divided by $a + e$ it will give . . . . $a - e = \dfrac{a+e}{a+e} = 1$
6. From the fifth Equation, by transposition of e, there ariseth . . . . $a = e + 1$
7. The sixth Equation multiplied by e produceth . $ae = ee + e$
8. From the fourth and seventh Equations, by exchanging equal Quantities, ... $ee + e = \frac{1}{2}b$
9. Then the eighth Equation being resolved by the Canon in *Sect.* 6. *Chapt.* 15. *Book* 1. the lesser number sought will be made known, *viz.* . . $e = \sqrt{:\tfrac{1}{4} + \tfrac{1}{2}b:} - \tfrac{1}{2} = 5$
10. And from the ninth and sixth Equations the greater number sought will also be made known, *viz.* . . . . $a = \sqrt{:\tfrac{1}{4} + \tfrac{1}{2}b:} + \tfrac{1}{2} = 6$

The two last Equations give a Canon to find out the two numbers sought, which are 6 and 5; as may easily be proved.

QUEST. 12.

There are two numbers, such, that if their summ be subtracted from the summ of their Squares, the Remainder is 42; but if the summ of the said two numbers be added to the Product of their multiplication, it makes 34: what are the numbers?

Let a and e represent the two numbers sought, then the Question may be stated thus, *viz.*

1. If . . . . $aa + ee - a - e = 42$
2. And . . . . $ae + a + e = 34$

What are the numbers a and e?

RESOLUTION.

3. By adding the first and second Equations together, the summ will be . . . . $aa + ee + ae = 76$
4. And by adding the second Equation to the third, the summ will be . . . . $aa + ee + 2ae + a + e = 110$
5. Suppose . . . . $y = a + e$

6. Then

6. Then by squaring each part of the fifth Equation, this ariseth, ... $yy = aa + ee + 2ae$
7. The summ of the two last Equations makes . $yy + y = aa + ee + 2ae + a + e$
8. And from the seventh and fourth Equations, by exchange of equal quantities, this Equation ariseth, $yy + y = 110$
9. Which eighth Equation being resolved by the Canon in *Sect.6. Chap.15. Book* 1. the number y, to wit, $a + e$ will be made known, *viz.* . . $y (= a + e) = 10$
10. Then by setting 10 (the value of $a + e$) in the place of $a + e$ in the second Equation, there ariseth $ae + 10 = 34$
11. And by subtracting 10 from each part of the tenth Equation, there remains . . . . . . $ae = 24$
12. And from the ninth Equation, by transposition of a, there ariseth . . . . $e = 10 - a$
13. And if a in the eleventh be multiplied by $10 - a$ instead of e, the said eleventh Equation will be reduced to this, ... $10a - aa = 24$
14. Wherefore the last Equation being resolved by the Canon in *Sect.* 10. *Chapt.* 15. *Book* 1. the two numbers sought will be discovered, *viz.* . $\begin{cases} a = 6 \\ e = 4 \end{cases}$

Thus 6 and 4 are found out, which will solve the Question proposed, as will be evident by the Proof.

QUEST. 13.

There are two numbers, such, that the summ of their Squares makes 100, and if the summ of the two numbers be added to the Product of their multiplication, it makes 62; what are the numbers?

Let a and e be put for the two numbers sought, then the Question may be stated thus, *viz.*

1. If . . . . $aa + ee = 100$
2. And . . . . $ae + a + e = 62$

What are the numbers a and e? ||

RESOLUTION.

3. The second Equation multiplied by 2 produceth . $2ae + 2a + 2e = 124$
4. The summ of the first and third Equations gives $aa + ee + 2ae + 2a + 2e = 224$
5. Suppose . . . . $y = a + e$
6. Then by squaring each part of the fifth Equation this is produced, *viz.* . . . . $yy = aa + ee + 2ae$
7. And by adding the double of the fifth Equation to the sixth, it gives . . . . $yy + 2y = aa + ee + 2ae + 2a + 2e$
8. And from the seventh and fourth Equations, by exchange of equal quantities, this Equation ariseth $yy + 2y = 224$
9. Which last Equation being resolved by the Canon in *Sect. 6. Chap.* 15. *Book* 1. the number y, to wit $a + e$, will be made known, *viz.* . . $y = a + e = 14$
10. Then from the ninth and second Equations, by taking 14 instead of $a + e$, the second Equation will be reduced to this, *viz.* . . . . . . $ae + 14 = 62$
11. Which last Equation, by equal subtraction of 14, gives . . . . $ae = 48$
12. The ninth Equation by transposition of a gives $e = 14 - a$
13. Then by multiplying a in the eleventh Equation by $14 - a$ instead of e, this Equation is produced, to wit, ... $14a - aa = 48$
14. Wherefore the last Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two numbers sought will be discovered, *viz.* $\begin{cases} a = 8 \\ e = 6 \end{cases}$

So the numbers sought are found 8 and 6, which will solve the Question, as will appear by the Proof.

QUEST. 14.

There are two numbers, such, that their summ is equal to the Product of their multiplication; and if the Product or summ of the said numbers be added to the summ of their Squares, it makes $15\frac{3}{4}$: what are the numbers?

Let a and e be put for the two numbers sought, then the Question may be stated thus, *viz.*

1. If . . . . $ae = a + e$
2. And . . . . $aa + ee + ae = 15\frac{3}{4}$

What are the numbers a and e?

RESOLUTION.

3. The summ of the first and second Equations is . $aa + ee + 2ae = a + e + 15\frac{3}{4}$
4. And from the third Equation, by transposition of $a + e$, there ariseth . . . . $aa + ee + 2ae - a - e = 15\frac{3}{4}$
5. Suppose . . . . $y = a + e$
6. Then by squaring each part of the fifth Equation, . . $yy = aa + ee + 2ae$
7. And by subtracting the fifth Equation from the sixth, there remains . . . . $yy - y = aa + ee + 2ae - a - e$
8. And from the fourth and seventh Equations, by exchange of equal Quantities, there will arise . $yy - y = 15\frac{3}{4}$
9. Which last Equation being resolved by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1, the number y, to wit, $a + e$ will be made known, *viz.* . . . . . $y = a + e = 4\frac{1}{2}$
10. Therefore from the first and ninth Equations, . $a + e = ae = 4\frac{1}{2}$
11. From the ninth Equation, by transposition of a, ... $e = 4\frac{1}{2} - a$
12. The eleventh Equation multiplied by a, produceth . . . . $ae = 4\frac{1}{2}a - aa$
13. And from the tenth and twelfth Equations, by exchange of equal Quantities, ... $4\frac{1}{2}a - aa = 4\frac{1}{2}$
14. Wherefore the last Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two numbers sought will be discovered, *viz.* . . . $\left\{\begin{array}{l} a = 3 \\ e = 1\frac{1}{2} \end{array}\right.$

So the numbers sought are found 3 and $1\frac{1}{2}$, which will solve the Question; for their summ is equal to the Product of their multiplication, and if their summ $4\frac{1}{2}$ be added to $11\frac{1}{4}$ the summ of their Squares, it makes $15\frac{3}{4}$, as the Question requires.

QUEST. 15.

There are two numbers, such, that the Square of their difference is equal to the Product of their multiplication; and the summ of their Squares makes 20 : what are the numbers?

Let a and e be put for the two numbers sought, and let a be the greater; then the Question may be stated thus, *viz.*

1. If . . . . $aa - 2ae + ee = ae$
2. And . . . . $aa + ee = 20$

What are the numbers a and e?

RESOLUTION.

3. From the first Equation by transposition of $-2ae$, this ariseth, ... $aa + ee = 3ae$
4. Therefore from the second and third Equations . . . $3ae = 20$
5. And the third Equation divided by 3, gives . . . . $ae = \frac{20}{3}$
6. And by adding the double of the fifth Equation to the second, it makes . . . . $aa + ee + 2ae = \frac{100}{3}$
7. Therefore by extracting the square Root of each part of the sixth Equation, the summ of the two numbers sought will be made known, *viz.* . . . . $a + e = \sqrt{\frac{100}{3}}$
8. From the seventh Equation, by transposition of a, this ariseth, ... $e = \sqrt{\frac{100}{3}} - a$
9. The eighth Equation multiplied by a, produceth . . $ae = \sqrt{\frac{100}{3}} \times a, - aa$

10. And

10. And from the fifth and ninth Equations this ariseth, . . . $\sqrt{\frac{100}{3}} \times a, - aa = \frac{20}{3}$

11. Wherefore the last Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two numbers sought will be discovered, *viz.* . . . . $\left\{ \begin{array}{l} a = \sqrt{8\frac{1}{3}} + \sqrt{1\frac{2}{3}} \\ e = \sqrt{8\frac{1}{3}} - \sqrt{1\frac{2}{3}} \end{array} \right.$

The Proof.

The difference of the two numbers in the eleventh step is . . . $\sqrt{1\frac{2}{3}} + \sqrt{1\frac{2}{3}} = \sqrt{\frac{20}{3}}$

The Square of the said difference is . . . . $\frac{20}{3}$

And (by the last of the three Rules in *Sect.* 10. *Chap.* 9. of this *Book*) the Product of the multiplication of the same two numbers is also . . . . $\frac{20}{3}$

Lastly, (by the first and second of the said three Rules) the summ of the Squares of the said two numbers is 20

QUEST. 16.

There are two numbers, such, that if their summ be multiplied by their difference, the Product is 21; but if the summ of the Squares of those two numbers be multiplied by the difference of their Squares, the Product is 609: what are the numbers?

Let a and e be put for the two numbers sought, and let a represent the greater; then the Question may be stated thus, *viz.*

1. If . . . . $\overline{a+e} \times \overline{a-e}$, that is, $aa - ee, = 21$
2. And . . . . $aa+ee \times aa-ee$, that is, $aaaa - eeee, = 609$

What are the numbers a and e? ||

RESOLUTION.

3. By supposition in the first Equation, ... $aa - ee = 21$
4. Therefore (by transposition of $-ee$) . . . . $aa = ee + 21$
5. And by squaring each part of the fourth Equation this ariseth, ... $aaaa = eeee + 42ee + 441$
6. And by taking the latter part of the fifth Equation instead of $aaaa$ in the second, the said second Equation will be reduced to this, ... $eeee + 42ee + 441 - eeee = 609$
7. The sixth Equation, after due Reduction, gives . . . . $ee = 4$
8. Therefore by extracting the square Root out of each part of the seventh Equation, the lesser number sought is discovered, *viz.* . . . . $e = 2$
9. Then from the fourth and seventh Equations this ariseth, ... $aa = 4 + 21 = 25$
10. Therefore by extracting the square Root out of each part of the last Equation, the greater number sought is also made known, *viz.* . . . . . . $a = 5$

So the numbers sought are found 5 and 2, which will solve the Question, as will be evident by the Proof.

QUEST. 17.

There are two numbers, such, that if their summ be multiplied by the summ of their Squares, the Product is 272; but if the difference of the same two numbers be multiplied by the difference of their Squares the Product is 32: what are the numbers?

Put a for the greater number sought, and e for the lesser; then the Question may be stated thus, *viz.*

1. If . . . . $\overline{a+e} \times \overline{aa+ee} = 272$
2. And . . . . $a-e \times aa-ee = 32$

What are the numbers a and e? ||

RESOLUTION.

3. By multiplying $a+e$ into $aa+ee$, the first Equation will be reduced to this, ... $aaa + aae + aee + eee = 272$

4. Likewise,

4. Likewise by multiplying $a-e$ into $aa-ee$, the second Equation will be reduced to this, . . . $aaa-aae-aee+eee=32$
5. The summ of the third and fourth Equations gives $2aaa \quad +2eee=304$
6. The half of the fifth Equation is . . . . $aaa \quad +eee=152$
7. The fourth Equation subtracted from the third, leaves . . . . $2aae+2aee=240$
8. The half of the seventh Equation is . . . . $aae+aee=120$
9. The summ of the seventh and eighth Equations is $3aae+3aee=360$
10. The summ of the sixth and ninth Equations is . $aaa+3aae+3aee+eee=512$
11. The cubick Root of the tenth being extracted, there ariseth . . . . $a+e=8$
12. By dividing each part of the first Equation by the respective part of the eleventh, there will arise $aa+ee=34$

By the two last Equations, the summ of the two numbers sought is found 8, and the summ of their Squares 34; therefore by the Canon of *Quest.* 7. *Chapt.* 16. *Book* 1. the numbers themselves will be found 5 and 3, which will solve the Question, as may easily be proved.

QUEST. 18.

To divide a given number 14 (or b) into three continual Proportionals, such, that if the said given number be divided severally by every one of the said three Proportionals, the summ of the three Quotients may be equal to $12\frac{1}{4}$ (or d) a number given.

RESOLUTION.

1. For the first (or least) of the three Proportionals sought put . . . . e
2. For the second (or mean) Proportional put . . . a
3. Then the Square of the mean Proportional being divided by the first gives the third, to wit, . . . $\frac{aa}{e}$
4. Therefore the summ of the three Proportionals is . $e+a+\frac{aa}{e}$
5. Which summ must be equal to the given number 14, (or b,) whence this Equation ariseth, *viz.* . . . $e+a+\frac{aa}{e}=b$
6. Then by reducing that Equation to Integers, this ariseth $ee+ae+aa=be$
7. Again, (according to the Question) let the given number b be divided by every one of the three Proportionals in the fourth step, so the three Quotients added together, will give . . . . $\frac{b}{e}+\frac{b}{a}+\frac{be}{aa}$
8. But the summ of the three Quotients in the seventh step must be equal to the given summ $12\frac{1}{4}$, (or d,) hence this Equation ariseth, ... $\frac{b}{e}+\frac{b}{a}+\frac{be}{aa}=d$
9. Which last Equation reduced to Integers will produce $baaa+baae+baee=daaae$
10. And by dividing every Term of the Equation in the ninth step by a, this ariseth, ... $baa+bae+bee=daae$
11. The sixth Equation multiplied by b, produceth . . $baa+bae+bee=bbe$
12. And from the tenth and eleventh Equations, (where each of two Quantities is found equal to a common third) this ariseth, *viz.* . . . . $daae=bbe$
13. The twelfth Equation divided by e gives . . . $daa=bb$
14. And the thirteenth Equation divided by d gives . . . $aa=\frac{bb}{d}$
15. Therefore by extracting the square Root out of each part of the fourteenth Equation, the mean Proportional sought will be made known, *viz.* . . $a=\sqrt{\frac{bb}{d}}=4$
16. And because a is now known, to wit, 4; and $b=14$; therefore the Equation in the sixth step may be reduced into this, *viz.* . . . . . . $ee+4e+16=14e$

17. Which

17. Which last Equation, after due Reduction, will give . . $10e - ee = 16$

18. Lastly, the Equation in the seventeenth step being resolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the first and third Proportionals will be discovered, *viz.* . . . $e = \begin{cases} 2 \\ 8 \end{cases}$

Thus the three Proportionals sought are found 2, 4, 8, which will satisfie the conditions in the Question: For first, 2, 4 and 8 are manifestly in continual proportion; secondly, their summ is 14; thirdly, if 14 be divided by 2, 4 and 8 severally, the summ of the Quotients 7, $3\frac{1}{2}$ and $1\frac{3}{4}$ is $12\frac{1}{4}$; as was prescribed in the Question.

It may also be observed, that those three Quotients are continual Proportionals, as will be manifest from the seventh step of the Resolution, where they are represented by $\frac{b}{e}$, $\frac{b}{a}$ and $\frac{be}{aa}$; for the Product made by the multiplication of the two extremes, to wit, the Product $\frac{bbe}{aae}$, that is, $\frac{bb}{aa}$, is equal to the Square of the mean Proportional $\frac{b}{a}$.

QUEST. 19.

To find three numbers in Arithmetical Progression, such, that if the first be multiplied by 1, the second by 2, the third by 3, the summ of the Products may be 62; and that the summ of the Squares of the three numbers may make 275.

Let the three numbers sought be represented by a, e, y, and suppose a to be the smallest and first Term, then the Question may be stated thus, *viz.*

1. If . . . . $e - a = y - e$
2. And . . . . $a + 2e + 3y = 62$
3. And . . . . $aa + ee + yy = 275$

What are the numbers a, e, y?

RESOLUTION.

4. By supposition in the first step . . . . $e - a = y - e$
5. Therefore by transposition of $-a$ and $-e$, there ariseth . . . . $a + y = 2e$
6. And by dividing each part of the last Equation by 2, it gives . . . . $\frac{1}{2}a + \frac{1}{2}y = e$
7. And by squaring the Equation in the sixth step, there comes forth . . . . $\frac{1}{4}aa + \frac{1}{2}ay + \frac{1}{4}yy = ee$
8. Then if instead of $2e$ in the second Equation, there be taken the first part of the fifth, the second will be converted into this, *viz.* . . $a + a + y + 3y = 62$
9. That is, ... $2a + 4y = 62$
10. The half of the last Equation is . . . . $a + 2y = 31$
11. And by transposition of Quantities in the tenth Equation this ariseth, *viz.* . . . . . $31 - 2y = a$
12. And by squaring the eleventh Equation, there comes forth . . . . $961 - 124y + 4yy = aa$
13. From the seventh, eleventh and twelfth Equations this ariseth, ... $\frac{961}{4} - \frac{31}{2}y + \frac{1}{4}yy = ee$
14. It is evident that . . . . $yy = yy$
15. And by adding the twelfth, thirteenth and fourteenth Equations into one summ, it makes $\frac{21}{4}yy - \frac{279}{2}y + \frac{4805}{4} = aa + ee + yy$
16. But by supposition in the third step, ... $275 = aa + ee + yy$
17. Therefore from the fifteenth and sixteenth Equations, by exchange of equal Quantities, $\frac{21}{4}yy - \frac{279}{2}y + \frac{4805}{4} = 275$
18. And after due Reduction the Equation in the seventeenth step gives . . . . $\frac{186}{7}y - yy = \frac{1235}{7}$
19. Therefore by resolving the Equation in the 18 step, (according to the Canon in *Sect.* 10. *Ch.* 15. *Book* 1.) two values of y will be discovered, *viz.* . . . $y = 13$, or $13\frac{4}{7}$
20. And from the 19*th* and 11*th* Equations . . . . . $a = 5$, or $3\frac{6}{7}$
21. Lastly, from the 20*th*, 19*th* and 6*th* Equations . . . . . $e = 9$, or $8\frac{5}{7}$

From

From the three last Equations 'tis evident, that the three desired numbers a, e, y may be either 5, 9, 13, or $3\frac{6}{7}$, $8\frac{5}{7}$, and $13\frac{4}{7}$: For first, 5, 9, 13 are in Arithmetical Progression; and if 5 be multiplied by 1, 9 by 2, and 13 by 3, the summ of the three Products is 62; moreover, the summ of the Squares of 5, 9, 13 makes 275, as was required. The like may be proved by $3\frac{6}{7}$, $8\frac{5}{7}$ and $13\frac{4}{7}$.

QUEST. 20.

To find three such numbers, that the Square of the first being added to the Product of the first multiplied into the second may make the summ 48; also, that the Square of the first being subtracted from the Product of the first multiplied into the third, the Remainder may be 32; and that the summ of the Squares of the first and third may have the same proportion to the Square of the second as 5 to 2.

Let the three numbers sought be represented by a, e, y, and then the Question may be stated thus, *viz.*

1. If	$aa + ae = 48$
2. And	$ay - aa = 32$
3. And	$aa + yy \,.\, ee :: 5 \,.\, 2$
What are the numbers a, e, y?	

RESOLUTION.

4. From the first Equation by transposition of aa, this ariseth, *viz.*	$ae = 48 - aa$
5. And by dividing each part of the last Equation by a, it gives	$e = \frac{48 - aa}{a}$
6. And by transposition of $-aa$ in the second Equation, it makes	$ay = aa + 32$
7. And by dividing the sixth Equation by a, there ariseth	$y = \frac{aa + 32}{a}$
8. From the Analogy in the third step, by comparing the Product of the extremes to the Product of the means, this Equation ariseth,	$5ee = 2aa + 2yy$
9. The Square of the seventh Equation is	$yy = \frac{1024 + 64aa + a^4}{aa}$
10. The double of the ninth Equation is	$2yy = \frac{2048 + 128aa + 2a^4}{aa}$
11. If instead of $2yy$ in the latter part of the eighth Equation there be taken the latter part of the tenth, the eighth will be converted into this, *viz.*	$5ee = \frac{2048 + 128aa + 4a^4}{aa}$
12. The Square of the fifth Equation is	$ee = \frac{2304 - 96aa + a^4}{aa}$
13. The twelfth Equation multiplied by 5 gives	$5ee = \frac{11520 - 480aa + 5a^4}{aa}$
14. From the eleventh and thirteenth Equations, by comparing their latter parts one to the other, and reducing the Equation thereby resulting, this Equation ariseth, *viz.*	$608aa - a^4 = 9472$
15. Which Equation in the 14*th* step being resolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. will discover two values of a, viz.	$a = \sqrt{592}$ or 4
16. But the lesser of those two values of a, to wit, 4, is the first number sought by the Question, for the Square of the greater value $\sqrt{592}$ exceeds 48, but according to the supposition in the first step it ought to be less than 48; supposing then $a = 4$, it follows from the fifth step, that	$e = 8$
17. Lastly, from the 15*th* and 7*th* Equations,	$y = 12$

So three numbers are found out, to wit, 4, 8 and 12; which will satisfie the Question, as may easily be proved.

QUEST. 18.

QVEST. 21.

To find three such numbers, that the Square of the first, together with the Product of the first multiplied by the second may make 10; also, that the Square of the second with the Product of the second into the third may make 21; and lastly, that the Square of the third, with the Product of the third into the first may make 24.

Let the three numbers sought be represented by a, e, y, and then the Question may be stated thus;

1. If . . . . $aa + ae = 10$
2. And . . . . $ee + ey = 21$
3. And . . . . $yy + ya = 24$

What are the numbers a, e, y?

RESOLUTION.

4. By transposition of aa in the first Equation this ariseth, ... $ae = 10 - aa$
5. And by dividing each part of the fourth Equation by a, it gives . . $e = \frac{10 - aa}{a}$
6. And by squaring the fifth Equation it makes . . . . $ee = \frac{a^4 - 20aa + 100}{aa}$
7. And from the second, fifth and sixth Equations this ariseth, ... $\frac{a^4 - 20aa + 100}{aa} + \frac{10 - aa}{a}y = 21$
8. And by subtracting $\frac{a^4 - 20aa + 100}{aa}$ from each part of the seventh Equation, this remains, ... $\frac{10 - aa}{a}y = \frac{41aa - 100 - a^4}{aa}$
9. And by dividing each part of the 8*th* Equation by $\frac{10 - aa}{a}$, this ariseth, $y = \frac{41aa - 100 - a^4}{10a - aaa}$
10. And by squaring the ninth Equation it makes . . . . $yy = \frac{a^8 - 82a^6 + 1881a^4 - 8200aa + 10000}{100aa - 20a^4 + a^6}$
11. And by multiplying the ninth Equation by a, it produceth . . . . $ya = \frac{41aaa - 100a - a^5}{10a - aaa}$
12. And by adding the eleventh Equation to the tenth, the summ makes
$$yy + ya = \frac{2a^8 - 133a^6 + 2391a^4 - 9200aa + 10000}{100aa - 20a^4 + a^6}$$
13. Therefore from the third and twelfth Equations this ariseth,
$$\frac{2a^8 - 133a^6 + 2391a^4 - 9200aa + 10000}{100aa - 20a^4 + a^6} = 24.$$
14. Which last preceding Equation, after due Reduction, gives this that follows, *viz.*
$$-a^8 + 78\tfrac{1}{2}a^6 - 1435\tfrac{1}{2}a^4 + 5800aa = 5000.$$
15. That is, after transposition of 5000,
$$-a^8 + 78\tfrac{1}{2}a^6 - 1435\tfrac{1}{2}a^4 + 5800aa - 5000 = 0.$$
16. Then by supposing $u = 2a$, and proceeding according to the Rule in *Sect.* 7. *Chap.* 11. of this Second *Book*, the Equation last above written will be reduced to this following Equation in Integers, *viz.*
$$-u^8 + 314u^6 - 22968u^4 + 371200uu - 1280000 = 0.$$
17. And by supposing $x = uu$ we may instead of $-u^8$ in the last preceding Equation write $-x^4$, and instead of $+314u^6$ we may set $314x^3$, also $-22968xx$ in the place of $-22968u^4$, and $+371200x$ instead of $+371200uu$, and last of all the Absolute number -1280000: whence this following Equation ariseth; and then after x is made known, its square Root shall be the number u; (for by supposition $x = uu$,)
$$-x^4 + 314x^3 - 22968xx + 371200x - 1280000 = 0.$$
18. Now because the last Term -1280000 in the Equation last above written hath many Divisors which will be useless in the finding of the value of x, it will be convenient before they be found out, to search out limits, within which such a value of the Root x doth fall as will produce a value of a capable of solving the Question proposed; to which end I proceed thus, *viz.*

19. By the latter part of the fourth Equation it's manifest that — $a \sqsupset \sqrt{10}$
20. And by the second Equation, after transposition of *ee*, it will likewise appear that . . . . — $e \sqsupset \sqrt{21}$
21. Now suppose . . . . — $e = \sqrt{21}$
22. Then by multiplying $\sqrt{21}$ instead of *e* by *a* in the first Equation, it will be reduced to this, *viz.* . . . . . . — $aa + \sqrt{21} \times a = 10$
23. Which last Equation being resolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. gives . . . . — $a = 1\frac{61}{100}$, &c.
24. And because when *e* is supposed to be equal to $\sqrt{21}$, the Equation in the twenty second step gives $a = 1\frac{61}{100}$, it may easily be conceived that when *e* is less than $\sqrt{21}$, (as it ought to be) then the first Equation, to wit, $aa + ea = 10$ will necessarily give . . . . — $a \sqsubset 1\frac{61}{100}$, &c.
25. Therefore by doubling each part of the nineteenth and twenty fourth steps, it is manifest that . . . . — $2a \begin{cases} \sqsupset \sqrt{40} \\ \sqsubset 3\frac{22}{100}, \text{ \&c.} \end{cases}$
26. And by squaring each part in the twenty fifth step, it follows that . . . . — $4aa \begin{cases} \sqsupset 40\frac{36}{100}, \text{ \&c.} \\ \sqsubset 10 \end{cases}$
27. But by supposition in the sixteenth step $u = 2a$, and consequently . . . . — $uu = 4aa$
28. Therefore from the two last precedent steps it's evident that — $uu \begin{cases} \sqsupset 40 \\ \sqsubset 10\frac{36}{100}, \text{ \&c.} \end{cases}$
29. And because by supposition in the seventeenth step, . . — $x = uu$
30. Therefore from the twenty eighth and twenty ninth steps it follows that . . . . — $x \begin{cases} \sqsupset 40 \\ \sqsubset 10\frac{36}{100}, \text{ \&c.} \end{cases}$
31. Having found that such a value of *x* in the Equation in the seventeenth step as is capable of producing a true value of the desired first number *a*, must be less than 40, but greater than $10\frac{36}{100}$; it is manifest that among the Divisors of 1280000, the last Term of that Equation, these three only, to wit, 16, 20, 32, are necessary to make tryals in finding out the said value of *x*, and consequently of *a*; and therefore (according to the *Rule* in *Sect.* 9. *Chap.* 11. of this *Book*) I first divide the said Equation in the seventeenth step, to wit, $-x^4 + 314x^3 - 22968xx + 371200x - 1280000 = 0$ by $a - 16$, and the Quotient is exactly $-x^3 + 298xx - 18200x + 80000$, wherefore 16 shall be a true value of *x* in that Equation: And because by supposition $x = uu = 4aa$, it follows that $\sqrt{16}$ (that is, $\sqrt{x}$) $= u = 2a$, and consequently $2 = a$ the first number sought.
32. Now since 2 is found equal to *a*, the first Equation, to wit, $aa + ae = 10$ will be reduced to this, *viz.* . . . . — $4 + 2e = 10$
33. Whence the second number *e* is discovered, *viz.* . . . — $e = 3$
34. And consequently the second Equation will be reduced to this, ... — $9 + 3y = 21$
35. Whence the third number *y* is discovered, *viz.* . . . — $y = 4$

Thus the three numbers sought (to wit, *a*, *e*, *y*,) are found 2, 3, 4, which will solve the Question: For the Square of the first with the Product of the first and second makes 10; also the Square of the second with the Product of the second and third makes 21; and the Square of the third with the Product of the third and first makes 24, as was required.

Note, That the Quotient found out in the thirty first step, to wit, the Equation $-x^3 + 298xx - 18200x + 80000 = 0$ hath three Affirmative Roots, whose values (by the *Rule* in *Sect.* 9. *Chap.* 11. of this Second *Book*) will be found very near equal to $4\frac{76}{100}$, $78\frac{12}{100}$, and $215\frac{12}{100}$; but these are without the limits of *x* discovered in the thirtieth step, and therefore although the Equation in the fifteenth step may be expounded by four Affirmative values of *a*, yet only one of them, to wit, 2, is capable of solving the Question proposed.

Note also, That if none of those Divisors which were discovered to be within the limits for the finding of a due value of *x* had produced an exact Quotient without a Remainder, and consequently in such case the number *a* had been Irrational, yet a Rational number near the true value of *x*, and consequently of *a*, might be found out by the help of the General Method in *Chap.* 10. of this Second *Book*.

CHAP.

CHAP. XIII.

Concerning the Resolution of such Arithmetical Questions as are capable of innumerable Answers.

I. AFter a Question is stated by Equations in such manner as hath been shewn in the foregoing twelfth *Chapter*, if those Equations be equal in multitude to the Quantities sought, then the Question hath a certain determinable number of Answers; but whensoever a Question affords not as many given Equations, not mutually depending upon one another, as there be Quantities required, it is capable of innumerable Answers. Questions of this latter kind are very pleasant and delightful, but oftentimes exceeding hard to be resolved, especially when all the Answers in whole numbers that a Question is capable of are desired; and therefore I suppose it will not be unacceptable to the Learner, if in this *Chapter* I give him a taste of that vast skill, by expounding three Propositions found out by Monsieur *Bachet*; the two first of which contain the substance of the eighteenth and twenty-first in his ingenious little Book, entituled *Problemes plaisans & delectables, qui se font par les Nombres*, (printed at *Lyons* in 1624;) but his Method of solving and demonstrating the same being very tedious and obscure, I shall wave it; and deliver two wayes of my own finding out, which are both intelligible and demonstrative. The third Proposition (which is handled by the same Author in his Comment upon the 41. *Prop.* of the fourth Book of *Diophantus*,) I shall also explain at large by various Questions.

PROP. I.

Two whole numbers prime between themselves being given, to find out two others, suppose a and b; that if a be multiplied by the greater of the two given numbers, and to the Product there be added a given whole number, the summ shall be equal to the Product of b multiplied by the lesser of the two numbers first given. Moreover, to find out all the whole numbers a and b that are capable of producing the same effect.

Explication.

1. Numbers prime between themselves are such as have only Unity for their common Divisor; (*per Defin.* 12. *Elem.* 7. Euclid.) so 12 and 5 are said to be Prime between themselves, because they have no common Divisor but 1, to divide them severally, so as to leave no Remainder; the like may be said of 20 and 21, 7 and 3, *&c.*
2. I call a number the *Multiple* of another when it exactly contains that other twice, thrice, or more times, without any Remainder: As, 6 is a Multiple of 3, because it contains 3 exactly twice; likewise 18 is a Multiple of 6, because it contains 6 just thrice without any Remainder. Moreover I take the liberty to call a number the Multiple of it self, because it contains it self just once. These things premised, I shall proceed to shew two ways of solving the preceding *Prop.* 1. and explain the same by Questions.

Sect. II. *The first Method of solving the foregoing* Prop. 1.

QUEST. 1.

To find out all the values of a and b in whole numbers that may make $9a + 6 = 7b$, *viz.* that nine times the whole number a with 6 added may make seven times the whole number b.

The Equation proposed, . . . $9a + 6 = 7b$;

The Resolution, ...

1	15	14	2
2	24	21	3
3	33	28	4
4	42	35	5
5	51	42	6
6	60		
7	69		

Expli-

Explication.

1. To the number 9 prefixt to a I add 6, (to wit, $+6$ which follows $9a$) and it makes 15, to this I add again 9 and the summ is 24, to which I add again 9, and it gives 33; and in like manner I continue the addition of 9 to every next preceding summ until I have found out these seven numbers, 15, 24, 33, 42, 51, 60, 69, which stand (as you see in the Example) under $9a$, and on the left hand of those numbers I set 1, 2, 3, 4, 5, 6, 7. These two Colmels of numbers do shew that if 1 be taken for the value of a, then $9a+6$ makes 15; but if $2=a$, then $9a+6=24$; if $3=a$, then $9a+6=33$; and so of the rest. The addition aforesaid is in this Example continued only to the seventh summ inclusive, because (as hereafter will appear) the smallest whole number that can express the value of a, never exceeds the number prefixt to b in the Equation propos'd.
2. Then under $7b$ I set the Multiples of 7 orderly one under another, *viz.* 14, (to wit, twice 7,) 21, 28, &c. until I have found out a number equal to one of the seven numbers 15, 24, 33, &c. so at length among the Multiples of 7, I find 42, that is, six times 7, to be equal to 42 that stands among the numbers in the second Colmel, which latter 42 (by the construction aforesaid) is compos'd of 6 and four times 9. Whence 'tis manifest that if 4 be taken for the value of a, and 6 for the value of b, then $9a+6=7b$ $(=42;)$ *viz.* nine times 4 together with 6 is equal to seven times 6, and therefore one Answer to the Question is discovered.

Note 1. When the given whole number prefixt to b in the Equation propos'd is a single figure, or some small number of two places, then this first Method will readily discover the smallest values of a and b in whole numbers; for the smallest whole number a never exceeds the given number prefixt to b, as hereafter will be made manifest: But if the number prefixt to b be large, then the work by this first Method will be intollerably tedious, especially in the solving of *Prop.* 2.

Note 2. If the two given whole numbers which are prefixt to a and b in the Equation propos'd be not prime between themselves, then it will sometimes be impossible to find out any whole numbers for the values of a and b to solve the Proposition: as, if two whole numbers a and b be desired that may make $6a+3=2b$, it may easily be shewn that 'tis impossible to find out two such whole numbers; For the whole number a must be either even or odd, but whether it be even or odd, if it be multiplied by the even number 6 the Product shall be even; (by *Prop.* 21, & 28. *Elem.* 9. Euclid.) to which adding 3 the summ will be odd, (for odd added to even makes odd,) which summ must be equal to $2b$, and consequently the half of that summ is the number b; but the half of an odd number cannot be a whole number, and therefore b in the Equation propos'd cannot be a whole number: But if the given whole numbers which are prefixt to a and b be Prime to one another, then whatever whole number be given to be added to the desired Multiple of a, innumerable whole numbers may be found out for the values of a and b, as hereafter will be shewn.

3. After the two smallest whole numbers are found out for the values of a and b to constitute the Equation proposed, all other pairs of whole numbers that are capable of producing the same effect, may be orderly enumerated in two Arithmetical Progressions thus formed; *viz.* Having found 4 for the smallest whole number a, and 6 for the smallest whole number b to constitute the Equation before proposed, to wit, $9a+6=7b$, let the said 4 be made the first Term, and 7, which is prefixt to b, the common difference of the Terms of the first Progression; then let 6, the smallest whole number b, be the first Term, and 9 which is prefixt to a in the said Equation, the common difference of the Terms of the latter Progression, so the Terms of those Progressions will be these, *viz.*

Values of a;	4,	11,	18,	25,	32,	39,	46,	53,	&c.
Values of b;	6,	15,	24,	33,	42,	51,	60,	69,	&c.

4. Now out of the first of those Progressions you may take any Term for the value of a, as 11, (the second Term,) and then the correspondent Term in the latter Progression, to wit, 15, shall be the value of b; by which two numbers 11 and 15 the Equation $9a+6=7b$ may be expounded, *viz.* nine times 11 with 6 added is equal to seven times 15. Likewise 18 and 24, also 25 and 33, and every pair of correspondent Terms in those two Progressions will cause the same effect, as I shall now demonstrate.

Prepa-

Preparation.

5. Let c and n repreſent two whole numbers Prime between themſelves, and a, b, d three other whole numbers, ſuch, that all five will make this Equation, *viz.* . . . . . $ca + d = nb$

6. Let an Arithmetical Progreſſion be ſo formed that a may be the firſt and leaſt Term, and n the common difference of the Terms, as, ... $a\,,\ a+n\,,\ a+2n\,,$ &c.

7. Let another Arithmetical Progreſſion be formed from b the firſt and leaſt Term, and c the common difference of the Terms, as, ... $b\,,\ b+c\,,\ b+2c\,,$ &c.

8. I ſay, if you multiply c by $a+n$, (the ſecond Term of the firſt Progreſſion,) inſtead of a in the Equation in the fifth ſtep, and to the Product add d, the ſumm ſhall be equal to a Multiple of n, to wit, the Product of n multiplied into $b+c$, (the ſecond Term of the latter Progreſſion;) and the like may be affirmed of every following Term in each Progreſſion.

Demonſtration.

9. By ſuppoſition in the fifth ſtep, ... $ca + d = nb$

10. And by adding cn to each part of that Equation, this ariſeth, ... $ca + cn + d = nb + cn$

11. Therefore from the laſt Equation, ... $c \times \overline{a+n}, + d = n \times \overline{b+c}$
Which was to be ſhewn.

12. Again, if to each part of the Equation firſt granted in the ninth ſtep you add $2cn$, it makes . . . $ca + 2cn + d = nb + 2cn$

13. That is, ... $c \times \overline{a+2n}, + d = n \times \overline{b+2c}$

14. After the ſame manner it may be ſhewn that . . $c \times \overline{a+3n}, + d = n \times \overline{b+3c}$
And ſo forwards. Which was to be proved.

15. Now ſuppoſing a and b to expreſs the ſmalleſt whole numbers that are capable of conſtituting the Equation in the fifth ſtep, to wit, $ca + d = nb$, I ſhall demonſtrate that no other whole numbers beſides the Terms which follow a and b in the two Progreſſions formed in the ſixth and ſeventh ſteps, can be taken inſtead of a and b to produce the ſame effect: If it be poſſible, let $a+$ ſome whole number f, viz. $a+f$ be taken inſtead of a; and let $b+$ ſome whole number g, viz. $b+g$ be taken inſtead of b; then c multiplied by $a+f$ makes $ca+cf$, to which adding d, the ſumm is $ca+cf+d$, which muſt be equal to the Product of n multiplied by $b+g$, to wit, $nb+ng$, whence . . . . $ca + cf + d = nb + ng$

16. And by ſuppoſition in the fifth ſtep, ... $ca + d = nb$

17. Therefore by ſubtracting the laſt Equation from the laſt but one, this remains, ... $cf = ng$

18. And by reſolving the laſt Equation into Proportionals, this Analogy ariſeth, *viz.* . . . . . $n\ .\ c :: f\ .\ g$

19. Whence it is manifeſt that the whole numbers f and g are in the ſame Reaſon (or Proportion) as the whole numbers n and c; and conſequently, ſince n and c are by ſuppoſition whole numbers Prime between themſelves, f muſt neceſſarily be equal either to n, or $2n$, or $3n$, &c. and g muſt be equal to c, or $2c$, or $3c$, &c. Wherefore $a+n$, $a+2n$, $a+3n$, &c. *viz.* the Terms which follow a in the Progreſſion in the ſixth ſtep, and $b+c$, $b+2c$, $b+3c$, &c. *viz.* the Terms which follow b in the Progreſſion in the ſeventh ſtep, are the only whole numbers that can be taken inſtead of a and b, the leaſt whole numbers to conſtitute the Equation propoſed, to wit, $ca+d=nb$. Which was to be ſhewn.

20. If there be two whole numbers a and b, given or found out, which will conſtitute the Equation before propoſed, or ſuch like, and thoſe two numbers be not the ſmalleſt values of a and b, you may by the help of thoſe given find out the ſmalleſt, by this Rule; *viz.* Divide the given whole number a, by the given number which is prefixt to b in the Equation propoſed, then after the diviſion is finiſh'd there will remain either a number or nothing; if a number remain, it ſhall be the ſmalleſt value of a, but if 0 remain, then the number prefixt to b is the ſmalleſt value of a, and conſequently the correſpondent value of b is eaſily diſcovered by the Equation. The Reaſon of this *Rule* is manifeſt by *Sect.* 9. *Chap.* 17. *Book* 1. For if any Term greater than the leaſt of an Arithmetical Progreſſion

Progression be given, as also the common Difference, the least Term shall be given also, either by a continual subtraction of the common Difference, or by the *Rule* above exprest.

As, for example, If in the former of the two Arithmetical Progressions in the third step, which express values of a and b to constitute the Equation $9a + 6 = 7b$, there be given 32 for the value of a, I divide 32 by 7 which is prefixt to b, and find 7 contain'd four times in 32, and there remains 4; now this Remainder 4 is the smallest value of a, whence the correspondent whole number b is easily discovered; for if $a = 4$, then $9a + 6 = 42 = 7b$; Therefore 42 divided by 7 gives 6 for the whole number b.

Again, If $a = 20$, and $b = 26$, then this will be a true Equation, *viz.* $5a + 4 = 4b$; now if you desire the smallest whole numbers a and b to constitute that Equation, divide 20 the given value of a by 4 which is prefixt to b, and there remains 0, therefore (according to the *Rule* before given) the said 4 shall be the smallest value of a; whence $5a + 4 = 24 = 4b$, and consequently $6 = b$.

Lastly, from what hath been said in the third step, all the values of a and b in whole numbers that are capable of constituting the said Equation $5a + 4 = 4b$ are the Terms of these two Arithmetical Progressions, *viz.*

Values of a; 4, 8, 12, 16, 20, 24, 28, 32, &c.
Values of b; 6, 11, 16, 21, 26, 31, 36, 41, &c.

Sect. III. *Another way of solving the foregoing* Prop. 1.

In this latter Method there are four principal Cases, which I shall first explain by Questions, and then shew how the Resolution of the Proposition will alwayes run into one of those four Cases.

QUEST. 2.

To find all the whole numbers a and b that are capable of constituting this Equation, *viz.* $8a + 97 = 5b$.

The Equation proposed, . . .	1	$8a + 97 = 5b$
The Resolution, ...	2	$8 + 97 = 105$
	3	$\frac{105}{5} = 21 = b$
	4	$1 = a$

Explication.

First I add 97 (to wit, $+97$ in the Equation proposed) to 8, which is prefixt to a, and it makes 105, this I divide by 5 the number prefixt to b; and because the Quotient 21 happens to be exactly a whole number without any Remainder, it shall be the smallest whole number b sought, and the whole number a in this case is always 1. The Reason is evident, for if $a = 1$, then $8a + 97 = 8 + 97$; and if this summ happens to be a Multiple of the given number prefixt to b, then b is necessarily a whole number. This is the first of the four Cases above mentioned.

Then after 1 and 21, the smallest whole numbers a and b to constitute the Equation propos'd, are found out, all the other values of a and b in whole numbers will be found in these two following Arithmetical Progressions formed according to the *Rule* in the third step of the foregoing *Sect.* 2. *viz.*

Values of a; 1, 6, 11, 16, 21, 26, &c.
Values of b; 21, 29, 37, 45, 53, 61, &c.

I say, every two correspondent numbers in those Progressions may be taken for values of a and b in this Equation, $8a + 97 = 5b$; as, for example, if 11 be taken for a, and 37 for b, then eight times 11, with 97 added shall be equal to five times 37, *viz.* $185 = 185$. And so of the rest.

QUEST. 3.

To find all the whole numbers a and b that are capable of constituting this Equation, *viz.* $49a + 6 = 13b$.

The

The Equation propoſed,	1	$49a + 6 = 13b$
The Reſolution,	2	$55 = 65 - 10$
	3	$49 = 39 + 10$
	4	$104 = 104$
	5	$\frac{104}{13} = 8 = b$
	6	$\frac{104 - 6}{49} = 2 = a$

Explication.

Firſt, I add 6 (to wit, $+6$ in the Equation propoſed) to 49 which is prefixt to a, and it makes 55; now if this 55 were exactly diviſible by 13 which is prefixt to b, the Quotient would be the whole number b ſought, and 1 the number a, (as in *Queſt.* 2.) But 55 not being a Multiple of 13, I proceed thus, *viz.* I ſeek the Multiple of 13 which is next greater than 55, by dividing 55 by 13, ſo I find that four times 13 is leſs than 55, but five times 13, that is, 65, exceeds 55 by 10; and therefore 55 is equal to 65 wanting 10, *viz.* $55 = 65 - 10$. This is the ſecond Equation in the Example.

2. Then I divide 49 which is prefixt to a, by 13 which is prefixt to b, ſo I find that three times 13, that is, 39, is the greateſt Multiple of 13 contained in 49, and there remains 10; therefore $49 = 39 + 10$: which is the third Equation.

3. Now becauſe $+ 10$ is found in the third Equation, and $- 10$ in the ſecond, I add thoſe Equations together, ſo the ſaid 10 vaniſheth, and there ariſeth $104 = 104$; which is the fourth Equation.

4. Then I divide 104, that is, either part of the fourth Equation, by 13 which is prefixt to b in the Equation propos'd, and the Quotient 8 is the whole number b ſought.

5. Then from the ſaid 104 in the fourth Equation, I ſubtract 6, (to wit, $+6$ in the Equation propos'd) and divide the Remainder 98 by 49 which is prefixt to a, ſo the Quotient gives 2 for the whole number a ſought.

I ſay $2 = a$ and $8 = b$ will make $49a + 6 = 13b$, as was required in *Queſt.* 3. and all the values of a and b in whole numbers that are capable of producing the ſame effect, are the Terms of theſe two following Arithmetical Progreſſions whoſe conſtruction hath been ſhewn before.

Values of a; 2, 15, 28, 41, 54, 67, &c.
Values of b; 8, 57, 106, 155, 204, 253, &c.

Note, That the manner of forming the ſecond and third Equations in the foregoing Reſolution of *Queſt.* 3. muſt be diligently obſerved, becauſe the like work is conſtantly uſed in the following fourth, fifth, ſixth, ſeventh, eighth and ninth Queſtions: But it's by accident, that the ſame number 10 follows the ſigns $-$ and $+$ in the ſaid ſecond and third Equations, and therefore the adding them together to produce the fourth Equation, is an Operation peculiar only to this and the like accident, which I call the ſecond of the four Caſes before mentioned.

But that in this ſecond Caſe, the Reſolution infallibly produceth whole numbers for the values of a and b, I prove thus; Firſt by Conſtruction, $65 - 10$ (the latter part of the ſecond Equation) wants 10 of a Multiple of 13, and $39 + 10$ (the latter part of the third Equation) exceeds a Multiple of 13 by 10; therefore the ſumm of the ſaid $65 - 10$ and $39 + 10$, to wit, 104 (the latter part of the fourth Equation) ſhall be a Multiple of 13; and conſequently 104 divided by 13 will exactly give a whole number, to wit, 8, for the value of b. Secondly, becauſe 104 (the firſt part of the fourth Equation) is by conſtruction compos'd of a Multiple of 49 together with 6; by ſubtracting 6 from 104, the Remainder 98 ſhall be a Multiple of 49, and conſequently 98 divided by 49 will give the Quotient an exact whole number, to wit, 2, for the value of a. Whence it is manifeſt, that if after the ſecond and third Equations are formed out of the firſt, (to wit, the Equation propoſed) according to the preceding directions for ſolving *Queſt.* 3. it happens that the number following $+$ in the latter part of the third Equation, is the ſame with the number following $-$ in the latter part of the ſecond, there will certainly ariſe two whole numbers for the values of a and b.

QUEST. 4.

To find all the whole numbers a and b that may make $82a + 66 = 13b$.

The Equation propos'd, . . .	1	$82a + 66 = 13b$
The Resolution, ...	2	$148 = 156 - 8$
	3	$82 = 78 + 4$
	4	$164 = 156 + 8$
	5	$312 = 312$
	6	$\frac{312}{13} = 24 = b$
	7	$\frac{312 - 66}{82} = 3 = a$

Explication.

1. The second and third Equations are formed out of the first in such manner as before hath been explain'd in the Resolution of *Quest.* 3.

2. Because the number 4 which follows the sign + in the latter part of the third Equation, happens to be an Aliquot part, to wit, $\frac{1}{2}$ of 8 which follows the sign — in the latter part of the second Equation, I multiply each part of the third Equation by 2 (the Denominator of the said Aliquot part,) to the end there may be + 8 in the Equation made by that Multiplication; so there is produced $164 = 156 + 8$, which is the fourth Equation.

3. Now since + 8 is found in the fourth Equation, and — 8 in the second, I add those Equations together, so the said 8 vanisheth, and there ariseth $312 = 312$; which is the fifth Equation.

4. Then I divide 312, (to wit, either part of the fifth Equation) by 13 which is prefixt to b in the Equation proposed, and the Quotient 24 is the whole number b sought.

5. Lastly, from the said 312 (in the fifth Equation) I subtract 66, to wit, + 6 in the Equation propos'd, and divide the Remainder 246 by the given number 82, (which is prefixt to a;) so the Quotient 3 is the whole number a sought.

I say, $3 = a$ and $24 = b$ will make $82a + 66 = 13b$, as was required in *Quest.* 4. and all the values of a and b in whole numbers that are capable of producing that Equation, are the Terms of these two Arithmetical Progressions, (whose Construction hath been shewn before in the third step of *Sect.* 2.) *viz.*

Values of a; 3, 16, 29, 42, 55, 68, &c.
Values of b; 24, 106, 188, 270, 352, 434, &c.

Note, That it was by meer chance that the number following the sign + in the third Equation happened to be an Aliquot part of the number following the sign — in the second, and therefore the multiplying of the third Equation by the Denominator of the Aliquot part, is an Operation peculiar only to that and the like accident, which is the third of the four Cases before-mentioned. The reason of the Operation in this fourth Question (or third Case,) may be easily discerned by the Demonstration before given in *Quest.* 3. but for further illustration I shall add another Example of *Case* 3.

QUEST. 5.

To find all the whole numbers that may be values of a and b in this Equation, *viz.* $601a + 9 = 200b$.

The Equation proposed, . .	1	$601a + 9 = 200b$
The Resolution, ...	2	$610 = 800 - 190$
	3	$601 = 600 + 1$
	4	$114190 = 114000 + 190$
	5	$114800 = 114800$
	6	$\frac{114800}{200} = 574 = b$
	7	$\frac{114800 - 9}{601} = 191 = a$

Explica-

Explication.

The Refolution of this Queftion is like that in the foregoing *Queft.* 4. for fince $+ 1$ in the latter part of the third Equation happens to be an Aliquot part of 190 which followeth — in the fecond Equation, I multiply each part of the third by 190, to the end that $+ 190$ may be found in the Product, as you fee in the fourth Equation; then by adding the fourth Equation to the fecond, the fumm makes the fifth, which is free from the figns $+$ and $-$; laftly, from the fifth Equation the whole numbers 574 and 191 expreffing the values of b and a are difcovered, in like manner as in the preceding third and fourth Queftions; which numbers will conftitute the Equation propofed: For 601 times 191 together with 9 is equal to 200 times 574, that is, 114800; and all the reft of the values of a and b in whole numbers to make that Equation will be found in thefe two following Arithmetical Progreffions formed by the Rule before given in the third ftep of *Sect.* 2.

Values of a; 191, 391, 591, 791, 991, *&c.*
Values of b; 574, 1175, 1776, 2377, 2978, *&c.*

QUEST. 6.

If	1	$121a + 5 = 93b$,	What are a and b in whole numbers?
Out of 1.	2	$126 = 186 - 60$	
	3	$121 = 93 + 28$	
Suppofe	4	$93c + 60 = 28d$	$c = ?$ $d = ?$
Out of 4.	5	$153 = 168 - 15$	
	6	$93 = 84 + 9$	
Suppofe	7	$28e + 15 = 9f$	$e = ?$ $f = ?$
Out of 7.	8	$43 = 45 - 2$	
	9	$28 = 27 + 1$	
Eq. 9×2.	10	$56 = 54 + 2$	
Eq. $8 + 10$.	11	$99 = 99$	
Out of 11 and 7.	12	$\frac{99}{9} = 11 = f$	Here the Regreffive work begins.
12, 6 and 5.	13	$11 \times 93 + 153 = 1176$	
13 and 4.	14	$\frac{1176}{28} = 42 = d$	
14, 3 and 2.	15	$42 \times 121 + 126 = 5208$	
15 and 1.	16	$\frac{5208}{93} = 56 = b$	
15 and 1.	17	$\frac{5208-5}{121} = 43 = a$	

Explication.

1. The fecond and third Equations are formed out of the firft in like manner as before in the Explication of *Queft.* 3.

2. But becaufe 28 which follows $+$ in the third Equation, is not equal to, nor an Aliquot part of 60 which follows — in the fecond, the procefs cannot be made like that in the third, fourth and fifth Queftions; fo that now a fourth Cafe takes rife, and the fcope of a new fearch is to find out a number d, fuch, that if it multiply the faid $+ 28$, the Product may exceed a Multiple of 93 (which is prefixt to b) by 60; for then it will be evident, that if the third Equation be multiplied by that number d, an Equation will be produced whofe firft part fhall be a Multiple of 121, and the latter part fhall exceed a Multiple of 93 by 60, and then the reft of the work will be like that in Cafe 2. in *Queft.* 3. In the fearch therefore of the number d, the fourth Equation is affumed, to wit, $93c + 60 = 28d$.

3. The fifth and fixth Equations are formed out of the fourth, in like manner as the fecond and third out of the firft.

4. Becaufe 9 which follows $+$ in the fixth Equation, is neither equal to, nor an Aliquot part of 15 which follows the fign — in the fifth, the next fcope (for the like reafon before

given concerning the number d) is to find out a number f, such, that if it multiply the said $+ 9$, the Product may exceed a Multiple of 28 which is prefixt to d, by the said 15; to which end the seventh Equation is assumed, to wit, $28e + 15 = 9f$.

5. The eighth and ninth Equations are formed out of the seventh, in like manner as the second and third out of the first.

6. Because 1 which follows $+$ in the ninth Equation, is an Aliquot part of 2 which stands next after $-$ in the eighth, the ninth is multiplied by 2 the Denominator of the said part; (according to the Rule in Case 3. *Quest.* 3.) whence the tenth Equation is produced, to wit, $56 = 54 + 2$.

7. The eleventh Equation, to wit, $99 = 99$ is the summ of the eighth and tenth; and since the said eleventh is free from the signs $+$ and $-$, a Regressive work now begins, to find out the whole numbers f, d, b and a; in this manner, *viz.*

8. By dividing either part of the eleventh Equation, to wit, 99, by 9 which is prefixt to f in the seventh; there ariseth $11 = f$, as in the twelfth Equation.

9. Then multiplying the number f, to wit, 11, by 93, that is, either part of the sixth Equation, and to the Product adding 153, that is, either part of the fifth Equation, the summ makes 1176, (as you see in the thirteenth Equation,) which 1176 is a Multiple of 28, to wit, that which is represented by $28d$ in the fourth Equation; Therefore,

10. By dividing the said 1176 by 28, the Quotient 42 is the number d, as in the fourteenth Equation.

11. Then multiplying the number d, to wit, 42, by 121, that is, either part of the third Equation, and to the Product adding 126, that is, either part of the second Equation, the summ makes 5208, as you see in the fifteenth Equation, which 5208 is a Multiple of 93, to wit, that which is represented by $93b$ in the first Equation; Therefore,

12. By dividing either part of the fifteenth Equation, to wit, 5208, by 93, the Quotient 56 is the number b sought.

13. Then from the said 5208 subtracting 5, to wit, $+ 5$ in the first Equation, and dividing the Remainder 5203 by 121 which is prefixt to a in the first Equation, the Quotient gives 43 for the number a sought, as in the seventeenth and last Equation. Therefore, if 43 be taken for a, and 56 for b, then $121a + 5 = 93b$, which is the Equation proposed in *Quest.* 6. and all the values of a and b in whole numbers that are capable of constituting that Equation are the Terms of these two following Arithmetical Progressions, whose Construction hath been shewn before in the third step of *Sect.* 2.

Values of a;	43,	136,	229,	322,	415,	508,	&c.
Values of b;	56,	177,	298,	419,	540,	661,	&c.

14. After the numbers f and d in the foregoing Resolution of *Quest.* 6. are known, the numbers e and c in the seventh and fourth Equations may easily be discovered; but there is no need of their help in the finding out of the desired numbers a and b.

15. But me-thinks I hear the Reader make this Objection, *viz.* How doth it appear, that from every three whole numbers given in such sort as before is declared in *Prop.* 1. there may infallibly be found out two whole numbers a and b to solve the said Proposition, by the Operation before explained in the four Cases before mentioned: For Answer to this Objection, I shall here shew how far the Process need be continued at the farthest, to find out an Equation having $+ 1$ in its latter part; for when such Equation ariseth, 'tis manifest by the Operation in the third Case explain'd in *Quest.* 4, and 5. that two whole numbers a and b will infallibly be discovered to satisfie the Proposition, and consequently innumerable other pairs of whole numbers to produce the same effect. First then in the foregoing *Quest.* 6. the given number 121 which is prefixt to a, being divided by the given number 93 which is prefixt to b, after the Division is finish'd there remains 28, to wit, $+ 28$ in the latter part of the third Equation: Secondly, the said Divisor 93 being divided by the said Remainder 28, after the Division is ended there remains 9, to wit, $+ 9$ in the latter part of the sixth Equation: Again, the last Divisor 28 being divided by the last Remainder 9, after this Division is ended there remains 1, that is, $+ 1$ in the latter part of the ninth Equation, which Remainder 1 you will alwayes infallibly come unto by a continued Division in that manner, because the two given numbers prefixt to a and b are (as the Proposition requires) Prime between themselves; and that continued Division is nothing else but the Method of finding out the greatest common Divisor unto two numbers; so

ſo that you may at firſt (if you pleaſe) diſcover unto what letter at the fartheſt, the proceſs need be continued before you return backward according to the Operation explain'd in *Queſt.* 6. But oftentimes before you come to the ſaid Remainder 1, the Reſolution will run into one of the three Caſes explain'd in *Queſt.* 2, 3, 4, and 5. as will appear by the following ſeventh, eighth, and ninth Queſtions.

QUEST. 7.

If	1	$97a + 1 = 26b$,	What are a and b in whole numbers?
Out of 1.	2	$98 = 104 - 6$	
	3	$97 = 78 + 19$	
Suppoſe	4	$26c + 6 = 19d$	$c = ?$ $d = ?$
Out of 4.	5	$32 = 38 - 6$	
	6	$26 = 19 + 7$	
Suppoſe	7	$19e + 6 = 7f$	$e = ?$ $f = ?$
Out of 7.	8	$25 = 28 - 3$	
	9	$19 = 14 + 5$	
Suppoſe	10	$7g + 3 = 5h$	$g = ?$ $h = ?$
Out of 10.	11	$7 + 3 = 10$	
Out of 10, & 11.	12	$\frac{10}{5} = 2 = h$	Here the Regreſſive work begins.
Out of 12, 9, 8.	13	$2 \times 19, + 25 = 63$	
13, and 7.	14	$\frac{63}{7} = 9 = f$	
14, 6, and 5.	15	$9 \times 26, + 32 = 266$	
15, and 4.	16	$\frac{266}{19} = 14 = d$	
16, 3, and 2.	17	$14 \times 97, + 98 = 1456$	
17, and 1.	18	$\frac{1456}{26} = 56 = b$	
17, and 1.	19	$\frac{1456 - 1}{97} = 15 = a$	

Explication.

In this ſeventh Queſtion the proceſs is formed like that in the foregoing ſixth, and the laſt letter in the work is h, whoſe value is diſcovered in the twelfth Equation by the help of the tenth and eleventh, according to the Operation in *Queſt.* 2. and then by the help of the number h, the work returns backward to find out the numbers f, d, b and a, in like manner as in *Queſt.* 6. But in this ſeventh Queſtion the laſt letter in the proceſs, to wit, h, is made known before an Equation ariſeth which hath $+1$ in its latter part; and the like effect happens in the following eighth and ninth Queſtions.

Now in anſwer to this ſeventh Queſtion, all the values of a and b in whole numbers that are capable of conſtituting the Equation propoſed, to wit, $97a + 1 = 26b$, are the Terms of the two following Arithmetical Progreſſions, which are deduced from the two ſmalleſt values of a and b, (to wit, 15 and 56 found out as above,) according to the Rule in the third ſtep of *Sect.* 2.

Values of a; 15, 41, 67, 93, 119, 145, &c.

Values of b; 56, 153, 250, 347, 444, 541, &c.

QUEST. 8.

QUEST. 8.

If	1	$119a + 6 = 57b,$	What are the whole numbers a and b?
Out of 1.	2	$125 = 171 - 46$	
	3	$119 = 114 + 5$	
Suppose	4	$57c + 46 = 5d$	$c = ?$ $d = ?$
Out of 4.	5	$103 = 105 - 2$	
	6	$57 = 55 + 2$	
5 + 6.	7	$160 = 160$	
7, 4.	8	$\frac{160}{5} = 32 = d$	Regress.
8, 3, 2.	9	$32 \times 119, + 125 = 3933$	
9, 1.	10	$\frac{3933}{57} = 69 = b$	
9, 1.	11	$\frac{3933 - 6}{119} = 33 = a$	

Values of a; 33, 90, 147, 204, 261, 318, &c.
Values of b; 69, 188, 307, 426, 545, 664, &c.

In which Progressions, every two correspondent Terms may be taken for values of a and b to constitute the Equation in *Quest.* 8.

QUEST. 9.

If	1	$173a + 1 = 71b,$	What are the whole numbers a and b?
Out of 1.	2	$174 = 213 - 39$	
	3	$173 = 142 + 31$	
Suppose	4	$71c + 39 = 31d$	$c = ?$ $d = ?$
Out of 4.	5	$110 = 124 - 14$	
	6	$71 = 62 + 9$	
Suppose	7	$31e + 14 = 9f$	$e = ?$ $f = ?$
Out of 7.	8	$31 + 14 = 45$	
8, and 7.	9	$\frac{45}{9} = 5 = f$	Regress.
9, 6, 5.	10	$5 \times 71, + 110 = 465$	
10, 4.	11	$\frac{465}{31} = 15 = d$	
11, 3, 2.	12	$15 \times 173, + 174 = 2769$	
12, 1.	13	$\frac{2769}{71} = 39 = b$	
12, 1.	14	$\frac{2769 - 1}{173} = 16 = a$	

Values of a; 16, 87, 158, 229, 300, 371, &c.
Values of b; 39, 212, 385, 558, 731, 904, &c.

Sect. 4. *PROP.* II.

Two whole numbers Prime between themselves being given, to find out two others, suppose a and b, that if a be multiplied by the lesser of those two numbers given, and to the Product there be added a whole number given, the summ shall be equal to the Product of b multiplied by the greater of the two numbers first given. Moreover, to discover all the whole numbers a and b that are capable of producing the same effect.

When each of the two given numbers which are Prime between themselves is a single figure, or some small number consisting of two Characters, then the first of the two ways of solving the foregoing *Prop* 1. will readily solve this second; but waving that Method, I shall shew two other ways by the help of the latter of those two Methods.

The

The firſt Method of ſolving Prop. 2.

QVEST. 10.

If	1	$71a + 3 = 173b$,	What are a and b in whole numbers?
Out of 1.	2	$145 = 173 - 28$	
By *Prop.* 1.	3	$2769 = 2768 + 1$	
Eq. 3 × 28.	4	$77532 = 77504 + 28$	
2 + 4.	5	$77677 = 77677$	
Out of 5, 1.	6	$\frac{77677}{173} = 449 = b$	true Values.
5, 1.	7	$\frac{77677 - 3}{71} = 1094 = a$	
By the Rule in	8	$56 = a$	the leaſt Values.
Sect. 2. *num.* 20.	9	$23 = b$	

Explication.

1. I multiply 71 which is prefixt to a in the Equation propoſed, by ſuch a number, that when 3, to wit, $+3$ in the ſame Equation is added to the Product, the ſumm may be either equal to, or leſs than ſome Multiple of 173; ſo multiplying 71 by 2, the Product 142 increaſed with 3 makes 145, which is equal to 173 wanting 28, *viz.* $145 = 173 - 28$, which is the ſecond Equation.

2. Then by *Prop.* 1. of this *Chapt.* I ſeek two ſuch numbers a and b, that if a be multiplied by 173, and the Product increaſed with $+1$, the ſumm may be equal to the Product of b multiplied by 71; *viz.* Suppoſing $173a + 1 = 71b$, and proceeding according to the foregoing *Queſt.* 9. I find 16 for the value of a, and 39 for b; therefore $173 \times 16, + 1 = 71 \times 39$; or $71 \times 39 = 173 \times 16, + 1$; that is, $2769 = 2768 + 1$, which is the third Equation.

3. Becauſe $+1$ in the latter part of the third Equation is an Aliquot part of 28 in the ſecond, I multiply the third Equation by 28 the Denominator of the ſaid part, and it makes the fourth Equation, to wit, $77532 = 77504 + 28$.

4. Then by adding the fourth Equation to the ſecond the ſumm gives the fifth, which is free from the ſigns $+$ and $-$; and from the fifth Equation the whole numbers 449 and 1094 are diſcovered for values of b and a, in like manner as in *Queſt.* 4, and 5. and by the help of thoſe the ſmalleſt values of a and b, to wit, 56 and 23 are found out by the Rule in the twentyeth ſtep of *Sect.* 2.

5. Laſtly, by the help of the two ſmalleſt values of a and b, and the Rule in the third ſtep of *Sect.* 2. all that are capable of ſolving *Queſt.* 10. will be found in the two following Arithmetical Progreſſions, which may be continued as far as you pleaſe.

Values of a; 56, 229, 402, 575, 748, 921, 1094; *&c.*
Values of b; 23, 94, 165, 236, 307, 378, 449, *&c.*

QVEST. 11.

If	1	$22a + 5000 = 65b$,	What are a and b in whole numbers?
Out of 1.	2	$5022 = 5070 - 48$	
By *Prop.* 1.	3	$66 = 65 + 1$	
Eq. 3 × 48.	4	$3168 = 3120 + 48$	
2 + 4.	5	$8190 = 8190$	
Out of 5, and 1.	6	$\frac{8190}{65} = 126 = b$	true Values.
5, 1.	7	$\frac{8190 - 5000}{22} = 145 = a$	
By the Rule in	8	$15 = a$	the leaſt Values.
Sect. 2. *num.* 20.	9	$82 = b$	

Expli-

Explication.

1. I add 22 to 5000 and it makes 5022, which is not exactly divisible by 65, for 77 times 65 is less than 5022, but 78 times 65, that is, 5070, exceeds 5022 by 48; therefore $5022 = 5070 - 48$, which is the second Equation.

2. Then by *Prop.* 1. of this *Chapt.* I seek two such whole numbers a and b, that if a be multiplied by 65, and to the Product there be added 1, the summ may be equal to the Product of b multiplied by 22; *viz.* Supposing $65a + 1 = 22b$, and proceeding according to the latter Method of resolving the foregoing *Prop.* 1. I find 1 and 3 to be values of a and b; therefore, $\overline{65 \times 1}, + 1 = 22 \times 3$; or $22 \times 3 = \overline{65 \times 1}, + 1$; that is, $66 = 65 + 1$, which is the third Equation.

3. By prosecuting the work as before in the Explication of *Quest.* 10. all the desired values of a and b in whole numbers that are capable of constituting the Equation first proposed in this eleventh Question will be found to be the Terms of these two following Arithmetical Progressions, *viz.*

Values of a; 15, 80, 145, 210, 275, 340, *&c.*
Values of b; 82, 104, 126, 148, 170, 192, *&c.*

Another way of solving Prop. 2.

QUEST. 12.

If	1	$71a + 3$	$= 173b$,	What are a and b in whole numbers?
Out of 1.	2	145	$= 173 - 28$	
	3	213	$= 173 + 40$	
Suppose	4	$173c + 28$	$= 40d$	$c = ?$ $d = ?$
Out of 4.	5	201	$= 240 - 39$	
	6	173	$= 160 + 13$	
6×3.	7	519	$= 480 + 39$	
$5 + 7$.	8	720	$= 720$	
8, 4.	9	$\frac{720}{40}$	$= 18 = d$	Regress.
9, 3, 2.	10	$\overline{18 \times 213}, + 145$	$= 3979$	
10, 1.	11	$\frac{3979}{173} = 23$	$= b$	
10, 1.	12	$\frac{3979 - 3}{71} = 56$	$= a$	

Explication.

1. In this Question, which is the same with the foregoing tenth, the second Equation is formed as is there directed.

2. The third Equation is thus formed: For as much as the given number 71 is less than 173 which is prefixt to b, I multiply 71 by such a number that the Product may exceed 173, and be also Prime to it; so multiplying 71 by 3, the Product 213 exceeds 173, also 213 and 173 are Prime to one another; then I divide the said 213 by 173, and find that 213 contains 173 once, and 40 over and above; therefore $213 = 173 + 40$, which is the third Equation.

3. The fourth, fifth, and sixth Equations here, are formed like the fourth, fifth and sixth Equations in the foregoing *Quest.* 6.

4. Then because 13 which follows + in the sixth Equation is an Aliquot part of 39 which follows — in the fifth, I multiply the sixth Equation by 3 the Denominator of the said part, (for 13 is $\frac{1}{3}$ of 39,) and it produceth the seventh Equation, to wit, $519 = 480 + 39$.

5. The eighth Equation is the summ of the fifth and seventh, (according to the Operation in Case 2.) and then in the ninth Equation the Regressive work begins, to find out the values of d, b and a in such manner as hath been shewn in divers preceding Questions of this Chapter: So at length all the values of a and b in whole numbers to solve this twelfth Question will by this latter Method be found the same as before in *Quest.* 10.

Sect. 5. *PROP.* III.

* To divide a given number into three or more numbers, ſuch, that if every one of them be multiplied by a different number given, the ſumm of the Products may be equal to a given number. But the ſumm of thoſe Products muſt fall between the two Products made by multiplying the given Dividend into the greateſt and leaſt of the given Multiplicators.

The Solution of this Problem is explain'd by the following Queſtions of this Chapter, and oftentimes requires the help of the two preceding Propoſitions, as will partly appear by the fifteenth Queſtion.

QUEST. 13.

To divide 24 into three ſuch whole numbers, that if the firſt be multiplied by 36, the ſecond by 24, and the third by 8, the ſumm of the three Products may make 516.

Let the numbers ſought be repreſented by a, e and y, then the Queſtion may be ſtated thus;

1. If . . . . $a + e + y = 24$
2. And . . . . $36a + 24e + 8y = 516$

What are the whole numbers a, e and y?

RESOLUTION.

3. The firſt Equation multiplied by 36, which is prefixt to a in the ſecond, produceth . . . . $36a + 36e + 36y = 864$
4. The ſecond Equation ſubtracted from the third, leaves . . . $12e + 28y = 348$
5. The fourth Equation by tranſpoſition of $+28y$, gives . . . $12e = 348 - 28y$
6. The fifth Equation divided by 12 gives . . . . . $e = 29 - \frac{7y}{3}$
7. If inſtead of e in the firſt Equation there be taken the latter part of the ſixth, this ariſeth, ... $a + 29 - \frac{7y}{3} + y = 24$
8. That is, ... $a + 29 - \frac{4y}{3} = 24$
9. From the eighth Equation by tranſpoſition of $29 - \frac{4y}{3}$, this ariſeth, ... $a = 24 - 29 + \frac{4y}{3}$
10. That is, ... $a = \frac{4y}{3} - 5$
11. By the latter part of the tenth Equation 'tis evident that . . . . $\frac{4y}{3} \sqsubset 5$
12. Therefore by multiplying each part in the eleventh ſtep by 3, it follows that . . . . $4y \sqsubset 15$
13. And by dividing each part in the twelfth ſtep by 4, $y \sqsubset 3\frac{3}{4}$
14. And from the latter part of the ſixth Equation, by arguing in like manner as in the eleventh, twelfth and thirteenth ſteps, it will be manifeſt that . . . . $y \sqsupset 12\frac{3}{7}$
15. Now if Fractions or mixt numbers were admitted to be the values of a, e and y, then by the thirteenth, fourteenth, tenth and ſixth ſteps 'tis evident that

$$y = \text{any number between } 3\tfrac{3}{4} \text{ and } 12\tfrac{3}{7};$$

$$a = \frac{4y}{3} - 5;$$

$$e = 29 - \frac{7y}{3}.$$

16. But to find out whole numbers to ſolve the Queſtion, the limits in the thirteenth and fourteenth ſteps do ſhew that y muſt be ſome whole number greater than 3, but not greater than 12; yet every whole number within thoſe limits will not ſerve the turn, for the values of a and e before diſcovered will not be whole numbers unleſs $\frac{4y}{3}$ and $\frac{7y}{3}$ be whole numbers; but $\frac{4y}{3}$ and $\frac{7y}{3}$ cannot be whole numbers unleſs y be 3, or ſome Multiple of 3; and becauſe 3 is without the limits, y may be 6, or 9, or 12, and conſequently

a	e	y
3	15	6
7	8	9
11	1	12

 from

from the fifteenth step a shall be 3, or 7, or 11; and e, 15, or 8, or 1. Now in answer to the Question, 3, 15 and 6, (to wit, a, e and y) are three such whole numbers, that their summ is 24; and if the first be multiplied by 36, the second by 24, and the third by 8, the summ of the three Products makes 516, as was required. The like may be said of each of the two other Answers. But if Fractions or mixt numbers were admitted, innumerable Answers might be given to the Question, as before hath been shewn in the fifteenth step.

Note. When one part of an Equation consists of an Affirmative letter and some Negative Absolute number, a limit may thence be inferr'd, above which the number signified by that letter ought to be taken. But if one part of an Equation consists of a Negative letter and of an Affirmative Absolute number, it will give a limit beneath which the number represented by that letter must be chosen. Sometimes also two limits will be discovered, (as in this thirteenth Question for the choice of the number y;) and sometimes but one, (as in divers of the following Questions.)

QUEST. 14.

To find three such whole numbers that their summ may make 100; and that if the first be multiplied by 4, the second by 3, and the third by $1\frac{2}{5}$, the summ of the three Products may make 300.

For the three numbers sought put a, e and y, then the Question may be stated thus;

1. If . . . . $a + e + y = 100$
2. And . . . . $4a + 3e + 1\frac{2}{5}y = 300$

What are the whole numbers a, e and y?

RESOLUTION.

3. The first Equation multiplied by 4, (which is prefixt to a in the second Equation,) produceth . . . $4a + 4e + 4y = 400$
4. The second Equation subtracted from the third, leaves . . $e + \frac{11y}{5} = 100$
5. The fourth Equation by transposition of $+\frac{11y}{5}$ gives . . $e = 100 - \frac{11y}{5}$
6. If instead of e in the first Equation there be taken the latter part of the fifth, this will arise, ... $a + 100 - \frac{11y}{5} + y = 100$
7. That is, after due Reduction, ... $a = \frac{6y}{5}$
8. From the latter part of the fifth Equation it's manifest that . . . . $\frac{11y}{5} \sqsupset 100$
9. And consequently by multiplying each part in the eighth step by 5, ... $11y \sqsupset 500$
10. And by dividing each part in the ninth step by 11, it follows that . . . . $y \sqsupset 45\frac{5}{11}$

Whence 'tis manifest, that if the three numbers sought were not restrained to whole numbers, any number less than $45\frac{5}{11}$ might be taken for the number y, and then the numbers a and e would be discovered from the seventh and fifth steps. But to have the Question solved by whole numbers, the number y must be some whole number not greater than 45, and such as may cause $\frac{11y}{5}$ and $\frac{6y}{5}$ to be whole numbers, for otherwise the values of e and a in the fifth and seventh steps will not be expressible by whole numbers; but $\frac{11y}{5}$ and $\frac{6y}{5}$ cannot be whole numbers unless y be 5, or some Multiple of 5, and therefore y may be 5, or 10, or 15, or any of the rest of the numbers in the third Columel of this Table; and consequently, from the fifth and seventh steps of the Resolution, the whole numbers e and a will be such as stand under e and a.

a	e	y
6	89	5
12	78	10
18	67	15
24	56	20
30	45	25
36	34	30
42	23	35
48	12	40
54	1	45

Thus you see that the Question receives nine Answers in whole numbers, which are all that it's capable of: So that if you take 6 for a; 89 for e; and 5 for y, their summ is 100; and if 6 be multiplied by 4;

by 4; 89 by 3; and 5 by $1\frac{4}{5}$, the ſumm of the three Products makes 300, as the Queſtion requires. The like may be proved of every one of the other eight Anſwers.

Note. When three numbers are ſought by a Queſtion of this nature that is capable of many Anſwers in whole numbers, all the values of every one of the letters in whole numbers are in Arithmetical Progreſſion, and therefore when two of thoſe Anſwers are found out, all the reſt within the limits diſcovered by the Reſolution are conſequently given by Addition or Subtraction of the common Difference in each Rank, as may eaſily be perceived by the values of a, e, y in the Table above-written. But when four numbers are ſought, the values of a letter are oftentimes found in ſeveral Arithmetical Progreſſions, as in the following *Queſt.* 20.

QUEST. 15.

To divide 1533 into three whole numbers, ſuch, that $\frac{1}{8}$ of the firſt, together with $\frac{3}{8}$ of the ſecond and $\frac{2}{113}$ of the third may make 167.

For the three whole numbers ſought put a, e and y, then the Queſtion may be ſtated thus;

1. If . . . . $a + e + y = 1533$
2. And . . . . $\frac{1}{8}a + \frac{3}{8}e + \frac{2}{113}y = 167$

What are the whole numbers a, e and y?

RESOLUTION.

3. The firſt Equation multiplied by $\frac{1}{8}$ produceth . . . $\frac{1}{8}a + \frac{1}{8}e + \frac{1}{8}y = 191\frac{5}{8}$
4. The ſecond Equation ſubtracted from the third, leaves . . . . $-\frac{1}{4}e + \frac{97}{904}y = 24\frac{5}{8}$
5. The fourth Equation by tranſpoſition of $-\frac{1}{4}e$ gives . . . . $\frac{97}{904}y = 24\frac{5}{8} + \frac{1}{4}e$
6. The fifth Equation divided by $\frac{97}{904}$ gives . . $y = \frac{22261}{97} + \frac{226e}{97}$
7. If inſtead of y in the firſt Equation there be taken the latter part of the ſixth, this ariſeth, $a + e + \frac{22261}{97} + \frac{226e}{97} = 1533$
8. The ſeventh Equation, after due Reduction, gives . . . . $a = \frac{126440}{97} - \frac{323e}{97}$
9. By the eighth Equation it's manifeſt that . . $323e \sqsupset 126440$
10. And conſequently by dividing each part of the laſt ſtep by 323, ... $e \sqsupset 391\frac{147}{323}$
11. Now to find out the values of a, e and y in whole numbers, (if there be a poſſibility,) I multiply the ſixth Equation by the Denominator 97, and it makes . . . . . $97y = 22261 + 226e$
12. That is, ... $226e + 22261 = 97y$
13. Then by the foregoing *Prop.* 1. of this Chapter, I ſearch out all ſuch whole numbers as may be values of e and y to conſtitute the laſt Equation, that is, $226e + 22261 = 97y$; but with this condition, *viz.* That the greateſt whole number among thoſe that are found out for the values of e may not exceed 391, as the preceding tenth ſtep requires; ſo I find four values of e, to wit, 47, 144, 241, 338; and four values of y, to wit, 339, 565, 791 and 1017: Then the ſumm of every two correſpondent values of e and y being ſubtracted from 1533 the number firſt given to be divided, the Remainders ſhall be the deſired values of a, to wit, 1147, 824, 501 and 178; ſo there are only four Anſwers to the Queſtion in whole numbers, to wit, thoſe inſerted in the Table in the Margin.

a	e	y
1147	47	339
824	144	565
501	241	791
178	338	1017

The Proof of the firſt Anſwer.

The ſumm of 1147, 47 and 339 is 1533,
$\frac{1}{8}$ of 1147 is . . . . $143\frac{3}{8}$,
$\frac{3}{8}$ of 47 is . . . . $17\frac{5}{8}$,
$\frac{2}{113}$ of 339 is 6,
Laſtly, the ſumm of thoſe three Products is . . . . . 167.

Therefore

Therefore all the conditions in the Question are satisfied, and the like may be proved by every one of the other three Answers in whole numbers; but if Fractions were admitted, innumerable Answers might be given by the tenth eighth, and sixth steps of the Resolution.

QUEST. 16.

To find three numbers, that their summ may make 300; and that if the first be multiplied by 6, the second by 5, and the third by $2\frac{7}{300}$, the summ of the three Products may make 1496.

Let a, e, y be put for the three numbers sought; then by forming the Resolution in like manner as in the preceding thirteenth, fourteenth and fifteenth Questions, it will appear that

$$y = \text{any number between } 1\tfrac{307}{893} \text{ and } 76\tfrac{532}{1193};$$

$$e = 304 - \frac{1193y}{300};$$

$$a = \frac{893y}{300} - 4.$$

Whence 'tis evident that there cannot be three whole numbers found out to solve this Question, for 300 is the smallest whole number that can be taken for y to cause $\frac{1193y}{300}$ and $\frac{893y}{300}$ to be whole numbers; but 300 exceeds the greater of the two limits above discovered for chusing of the number y.

QUEST. 17.

If one would lay out 98 pence to buy 40 Birds, suppose Partridges, Larks and Quails; how many of each kind may be bought when Partridges are at 3 pence a piece, Larks at an half-penny a piece, and Quails at 4 pence a piece?

Let a represent the number of Partridges, e the number of Larks, and y the number of Quails; then according to the Question, $a + e + y = 40$; and because the number of all the Partridges multiplied by the price of one of them produceth the full cost of all, it's manifest that $3a$ is the full cost of all the Partridges; and for the like reason $\frac{1}{2}e$ signifies the full cost of all the Larks; likewise $4y$ the full cost of all the Quails: But those three particular summs of money must be equal to 98 pence, therefore $3a + \frac{1}{2}e + 4y = 98$; so that the Question may be stated thus;

1. If . . . . $a + e + y = 40$
2. And . . . . $3a + \frac{1}{2}e + 4y = 98$

What are the whole numbers a, e and y?

RESOLUTION.

3. The first Equation multiplied by 3 (which is prefixt to a in the second,) produceth . . . . $3a + 3e + 3y = 120$
4. The second Equation subtracted from the third, leaves . . $\frac{5e}{2} - y = 22$
5. From the fourth Equation, after due transposition, this ariseth, ... $y = \frac{5e}{2} - 22$
6. Then instead of y in the first Equation, if there be set the latter part of the fifth, the first will be reduced to this, $a + e + \frac{5e}{2} - 22 = 40$
7. The sixth Equation, after due Reduction, gives . . . $a = 62 - \frac{7e}{2}$
8. By the latter part of the fifth Equation it's evident that $\frac{5e}{2} \sqsubset 22$
9. And consequently by multiplying each part in the eighth step by 2, ... $5e \sqsubset 44$
10. Whence by dividing each part by 5, it follows that $e \sqsubset 8\frac{4}{5}$
11. Again, from the latter part of the seventh Equation, by arguing in like manner as in the eighth, ninth and tenth steps, it will appear that . . . . $e \sqsupset 17\frac{5}{7}$

12. Now

12. Now ſince the nature of this Queſtion requires that the deſired values of a e and y be whole numbers, it's evident from the fifth and ſeventh ſteps that e muſt be an even number, otherwiſe $\frac{5e}{2}$ and $\frac{7e}{2}$ will not be whole numbers; for if e be an odd number, the Dividends $5e$ and $7e$ will be odd, (for odd multiplied by odd produceth odd,) and therefore their halves cannot be whole numbers. Since then e muſt be an even number, it's manifeſt by the tenth and eleventh ſteps, that e may be 10, or 12, or 14, or 16, but no other even number whatever; and conſequently from the fifth ſtep y ſhall be 3, or 8, or 13, or 18; and from the ſeventh ſtep, a ſhall be 27, or 20, or 13, or 6. Thus it appears that the Queſtion may be ſolved by four ſeveral Anſwers (and not more) in whole numbers, *viz.* Firſt, 27 Partridges, 10 Larks, and 3 Quails, which are in multitude 40, may be bought for 98 pence at their reſpective prices given in the Queſtion; or 20 Partridges, 12 Larks, and 8 Quails, which are likewiſe 40 in multitude, and the like may be affirmed of the other two Anſwers inſerted in the Table in the Margin.

Partr. a	Larks. e	Quails. y
27	10	3
20	12	8
13	14	13
6	16	18

But if a Queſtion of the ſame nature be deſired that hath but one Anſwer in whole numbers, the following Epigram (cited by Monſieur *Bachet* in his Comment upon the one and fourtieth Queſtion of the fourth Book of *Diophantus*,) will be ſatisfactory.

QUEST. 18.

Ut tot emantur aves, bis denis utere nummis;
Perdix, Anſer, Anas empta vocetur avis.
Sit ſimplex obolus pretium Perdicis, ematur
Sex obolis Anſer, biſque duobus Anas.
Ut tua procedat in lucem quæſtio, mentem
Conſule, ſic loquitur pectoris arca mihi.
Sint Anates tres atque duæ, ſimplex erit Anſer,
Accipe Perdices quatuor atque decem.

The ſenſe is this: If the price of a Partridge be an half-penny, a Gooſe 3 pence, and a Duck 2 pence; how many of each kind may be bought at thoſe rates, if it be deſired that all the Birds bought may be 20 in number, and coſt 20 pence?

Let a repreſent the number of Partridges, e the number of Geeſe, and y the number of Ducks, then this Queſtion (like the preceding ſeventh,) may be ſtated thus;

1. If . . . . $a + e + y = 20$
2. And . . . . $\frac{1}{2}a + 3e + 2y = 20$

What are the whole numbers a, e and y?

RESOLUTION.

3. The firſt Equation multiplied by $\frac{1}{2}$ produceth . . . $\frac{1}{2}a + \frac{1}{2}e + \frac{1}{2}y = 10$
4. The third Equation ſubtracted from the ſecond, leaves . . $\frac{5e}{2} + \frac{3y}{2} = 10$
5. By tranſpoſition of $\frac{3y}{2}$ in the fourth Equation, this ariſeth, . . $\frac{5e}{2} = 10 - \frac{3y}{2}$
6. The fifth Equation divided by $\frac{5}{2}$ gives . . . . . . $e = 4 - \frac{3y}{5}$
7. By ſetting the latter part of the ſixth Equation in the place of e in the firſt, this ariſeth, ... $a + 4 - \frac{3y}{5} + y = 20$
8. Which laſt Equation, after due Reduction, gives . . . . $a = 16 - \frac{2y}{5}$
9. From the latter part of the ſixth Equation it may be inferr'd, (in like manner as in divers of the preceding Queſtions,) that . . . . $y \sqsupset 6\frac{2}{3}$
10. But the ſixth and eighth ſteps do ſhew, that to the end the values of e and a may be whole numbers, as the nature of this Queſtion requires, it is requiſite that $\frac{3y}{5}$ and $\frac{2y}{5}$

be

be whole numbers; but $\frac{3y}{5}$ and $\frac{2y}{5}$ cannot be whole numbers unless y be 5 or some Multiple of 5; and by the ninth step y must be less than $6\frac{2}{3}$, therefore 5 is the only whole number that can be taken for y, or the number of Ducks; and consequently the sixth step gives 1 for the value of e, that is, 1 Goose; and by the eighth step, the the value of a is 14, that is, 14 Partridges; which three numbers will solve the Question, as may easily be proved.

The Resolutions of the following nineteenth and twentyeth Questions do shew how to find out innumerable Answers to any Question belonging to the Rule of Alligation alternate *in Vulgar Arithmetick, when three or more things are to be mixed together, according to the import of that Rule.*

QUEST. 19.

A Vintner having three sorts of Wines, the prices whereof *per* Gallon are 24 pence, 22 pence, and 18 pence, desires to make a Mixture out of them that may contain 60 Gallons, in such manner, that the total Mixture being sold at some mean price *per* Gallon between 24 pence and 18 pence, suppose at 20 pence, may make the same summ of money, as all the particular quantities of Wine in the Mixture at their own prices. The Question is, to find what Quantity of each sort of Wine may be taken to make that Mixture.

For the desired number of Gallons of the first sort of Wine to make the Mixture, put a; for the number of the second sort e; and of the third y: Then $a+e+y=60$, (the total number of the Gallons in the Mixture;) and because every Gallon of the mix'd quantity must be sold for 20 pence, the 60 Gallons mix'd are worth 1200 pence, and so much also must all the Products of the particular Quantities of each sort of Wine multiplied by their peculiar prices amount unto; therefore, $24a+22e+18y=1200=60\times20$. So that the Question may be stated thus;

1. If	$a+e+y=60$
2. And	$24a+22e+18y=1200\ (=60\times20)$
What are the numbers a, e, y?	

RESOLUTION.

3. The first Equation multiplied by 24, (which is prefixt to a in the second Equation,) produceth	$24a+24e+24y=1440$
4. The second Equation subtracted from the third, leaves	$2e+6y=240$
5. The fourth Equation by transposition of $6y$ gives	$2e=240-6y$
6. The fifth Equation divided by 2, gives	$e=120-3y$
7. By taking the latter part of the sixth Equation instead of e in the first, this ariseth,	$a+120-3y+y=60$
8. The seventh Equation, after due Reduction, discovers the value of a, viz.	$a=2y-60$
9. From the eighth Equation it's evident that	$y \sqsubset 30$
10. And from the sixth Equation, . . .	$y \sqsupset 40$

11. By the 10*th*, 9*th*, 8*th* and 6*th* steps it's manifest that innumerable Answers may be given to the Question proposed; for since Fractions are not here excluded from being Answers, you may esteem . . $y=$ any number between 30 and 40;
$a=2y-60$;
$e=120-3y$.

12. Whence nine Answers in whole numbers are discovered, to wit, those exprest in this Table. But the Rule of Alligation in Vulgar Arithmetick finds out only one Answer to this Question, to wit, the sixth. And because innumerable numbers may be taken between 30 and 40 for values of y, you may find out as many Answers as you please in Fractions, (which are not excluded in Questions of this nature;) so if for y you take $30\frac{1}{2}$, then $a=1$, $(=2y-60,)$ and $e=28\frac{1}{2}$, $(=120-3y.)$

a	e	y
2	27	31
4	24	32
6	21	33
8	18	34
10	15	35
12	12	36
14	9	37
16	6	38
18	3	39

The

The Proof of the firſt Anſwer.

Two Gallons of Wine at 24 pence *per* Gallon, together with 27 Gallons at 22 pence *per* Gallon, and 31 Gallons at 18 pence *per* Gallon, amount to 1200 pence; which is alſo the value of 60 Gallons at 20 pence *per* Gallon.

QVEST. 20.

A Vintner having four ſorts of Wines, whoſe prices *per* Quart are 16 pence, 10 pence, 8 pence, and 6 pence, deſires to make a Mixture out of them that may contain 100 Quarts, ſo as this mixt Quantity being ſold at ſome mean price *per* Quart between 16 pence and 6 pence, ſuppoſe at 12 pence, may produce the ſame ſumm of money, as all the particular quantities of Wine in the Mixture if they were ſold at their own prices. The Queſtion is, to find what quantity of Wine of each ſort may be taken to make that Mixture?

Let a, e, y and u be put for the unknown Quantities of Wine that are ſought to make the Mixture; then $a+e+y+u=100$, (the total number of Quarts in the Mixture,) and by multiplying thoſe Quantities ſeverally into their peculiar prices, the ſumm of the Products is $16a+10e+8y+6u$; which ſumm muſt be equal to the Product of 100 multiplied into 12, that is, 1200 pence: So that the Queſtion may be ſtated thus;

1. If . . . . $a+e+y+u=100$
2. And . . . . $16a+10e+8y+6u=1200$

What are the numbers a, e, y and u?

The given Equations being fewer in multitude than the numbers ſought, it's a ſign that the Queſtion is capable of innumerable Anſwers; now that you may find out as many of them as you pleaſe, the firſt ſcope in the Reſolution muſt be to diſcover limits to direct your choice of ſome one of the numbers ſought, and accordingly, the drift in the eight Equations next following is to ſearch out limits for the firſt number a.

RESOLVTION.

3. From the firſt Equation by tranſpoſition of a, this ariſeth, ... $e+y+u=100-a$
4. And from the ſecond Equation by tranſpoſition of $16a$, this ariſeth, ... $10e+8y+6u=1200-16a$
5. The third Equation multiplied by 6, to wit, the leaſt of the known numbers which are prefixt to the letters in the firſt part of the fourth Equation, produceth . . . . $6e+6y+6u=600-6a$
6. Again, the third Equation multiplied by 10, that is, the greateſt of the known numbers which are prefixt to the letters in the firſt part of the fourth Equation, produceth . . . . $10e+10y+10u=1000-10a$
7. It is manifeſt that the firſt part of the fifth Equation is leſs than the firſt part of the fourth, therefore alſo the latter part of the fifth ſhall be leſs than the latter part of the fourth, *viz.* . . . . $600-6a \sqsupset 1200-16a$
8. Therefore from the ſeventh ſtep, after due Reduction, it follows, that . . . . $a \sqsupset 60$
9. Again, for as much as the firſt part of the ſixth Equation is greater than the firſt part of the fourth, therefore alſo the latter part of the ſixth ſhall be greater than the latter part of the fourth, *viz.* . . . $1000-10a \sqsubset 1200-16a$
10. Therefore from the ninth ſtep, after due Reduction, it follows that . . . . $a \sqsubset 33\frac{1}{3}$

Now ſince it is found by the eighth and tenth ſteps, that a the number of Quarts ſought of the firſt ſort of Wine to make the Mixture muſt be leſs than 60, but greater than $33\frac{1}{3}$, let ſome number within thoſe limits be taken for the value of a, viz.

11. Suppoſe

11. Suppose . . . . $47 = a$

12. Then by setting 47 in the place of a in the first Equation, this ariseth, *viz.* . . . . $47 + e + y + u = 100$

13. Whence by equal subtraction of 47 there remains . . . . $e + y + u = 53$

14. And by multiplying the Equation in the eleventh step by 16, (the number prefixt to a in the second,) it gives . . . . $752 = 16a$

15. Then by setting 752 in the place of $16a$ in second Equation, this ariseth, ... $752 + 10e + 8y + 6u = 1200$

16. And by subtracting 752 from each part of the Equation in the fifteenth step, this remains, *viz.* . . . . $10e + 8y + 6u = 448$

17. The Equation in the thirteenth step multiplied by 10, (which is prefixt to e in the sixteenth,) produceth . . . . $10e + 10y + 10u = 530$

18. Then by subtracting the Equation in the sixteenth step from that in the seventeenth, the letter e vanisheth, and this Equation remains, *viz.* . . . . $2y + 4u = 82$

19. From the eighteenth step, by transposition of $+ 4u$, this Equation ariseth, ... $2y = 82 - 4u$

20. And by dividing each part of the Equation in the nineteenth step by 2, it gives . . . . $y = 41 - 2u$

21. Then by setting the latter part of the Equation in the twentieth step in the place of y in the thirteenth step, it makes . . . . . $e + 41 - 2u + u = 53$

22. Whence, after due Reduction, ... $e = u + 12$

23. By the latter part of the Equation in the twentieth step, it's evident that $2u \sqsupset 41$, therefore . . . . $u \sqsupset 20\frac{1}{2}$

And because the known number 12 which follows $+ u$ in the twenty-second step, (expressing the value of e) is Affirmative, there is not any limit to shew above which the number u ought to be taken; and therefore, according to the three and twentieth step, u may be any number less than $20\frac{1}{2}$: Therefore,

24. Suppose . . . . $u = 20$

25. Then from the twentieth and twenty-fourth steps it follows, that . . . . $y = 1, (= 41 - 2u)$

26. And from the twenty-second and twenty-fourth steps, ... $e = 32, (= u + 12)$

Thus by the eleventh, twenty-sixth, twenty-fifth and twenty-fourth steps; four whole numbers are discovered, to wit, 47, 32, 1 and 20 for the values of a, e, y and u, which numbers will solve the Question. For if 42 quarts of the first sort of Wine, 37 quarts of the second, 1 quart of the third, and 20 of the fourth be mixed together, the summ makes 100 quarts, which at 12 pence *per* Quart yields 1200 pence; and the same number of pence will be produced by selling 47 quarts at 16 pence *per* Quart, 32 quarts at 10 pence, 1 quart at 8 pence, and 20 quarts at 6 pence; which was required.

But because (by the twenty-third step) u may be any whole number less than $20\frac{1}{2}$, nineteen Answers more in whole numbers may be found out by repeating the Process in the twenty-fourth, twenty-fifth and twenty-sixth steps; so that 47 being taken for a, there will be twenty Answers in whole numbers, which are inserted in the following Table. And by putting a equal to every whole number severally between $33\frac{1}{3}$ and 60, which are the limits discovered in the eighth and tenth steps, for the chusing of the number a, after a due repetition of the Process with every one of those whole numbers, in like manner as before with 47 from the eleventh step to the end of the Resolution, two hundred ninety four Answers more in whole numbers will be discovered, which with those twenty in the Table make three hundred and fourteen Answers in whole numbers to this twentieth

Question,

Queſtion, to which the Rule of *Alligation* in Vulgar Arithmetick gives only one Anſwer, which conſiſts partly of Fractions too; but by the Method above deliver'd, innumerable Anſwers may be found out in Fractions. The Table follows.

a	e	y	u
47	32	1	20
47	31	3	19
47	30	5	18
47	29	7	17
47	28	9	16
47	27	11	15
47	26	13	14
47	25	15	13
47	24	17	12
47	23	19	11
47	22	21	10
47	21	23	9
47	20	25	8
47	19	27	7
47	18	29	6
47	17	31	5
47	16	33	4
47	15	35	3
47	14	37	2
47	13	39	1

QUEST. 21.

Fourty-one perſons conſiſting of Men, Women and Children ſpent in the whole at a Feaſt 40 ſhillings; whereof every Man paid 4 ſhillings, every Woman 3 ſhillings, and every Child 4 pence, or $\frac{1}{3}$ of a ſhilling: It's deſired to find the number of Men, likewiſe of the Women and Children.

The nature of this Queſtion not admitting Fractions in the Anſwer, the ſcope of the Reſolution muſt be to divide 41 into three ſuch whole numbers, that if the firſt be multiplied by 4, the ſecond by 3, and the third by $\frac{1}{3}$, the ſumm of the three Products may make 40: To which purpoſe, let a, e and y be put for the deſired numbers of Men, Women and Children, and then the Queſtion may be ſtated thus, *viz.*

1. If . . . . $a + e + y = 41$
2. And . . . . $4a + 3e + \frac{1}{3}y = 40$

What are the whole numbers a, e, y? ||

RESOLUTION.

3. By forming the Reſolution in like manner as in the foregoing thirteenth, fourteenth and fifteenth Queſtions it will appear, that . . . .

$$\begin{cases} y \sqsubset 31\frac{1}{8}, \\ y \sqsupset 33\frac{2}{11}, \\ e = 124 - \frac{11y}{3}, \\ a = \frac{8y}{3} - 83. \end{cases}$$

Whence 'tis manifeſt that 32 and 33 are the only whole numbers within the limits for the chuſing of the number y, but this muſt neceſſarily be a Multiple of 3, otherwiſe $\frac{11y}{3}$ and $\frac{8y}{3}$ will not be whole numbers, and conſequently the values of e and a above-expreſt cannot be whole numbers; therefore 33 is the ſole whole number that can be taken for the value of y, to wit, the number of Children, and conſequently the values of e and a above expreſt will give 3 for the number of Women, and 5 for the number of Men: which three numbers 5, 3 and 33 will ſolve the Queſtion, for their ſumm is 41; and if the firſt be multiplied by 4, the ſecond by 3, and the third by $\frac{1}{3}$, the ſumm of the three Products is 40, as was required.

QVEST. 22.

Twenty perſons, conſiſting of Men, Women, Boys and Girls ſpent at a Feaſt in the whole 94 ſhillings; whereof every man paid 6 ſhillings, every Woman 4 ſhillings, every Boy 3 ſhillings, and every Girl 1 ſhilling: It's deſired to find out the number of Men, likewiſe of Women, Boys and Girls.

The ſcope of this Queſtion is to find out four ſuch whole numbers that their ſumm may make 20; and that if the firſt be multiplied by 6, the ſecond by 4, the third by 3, and the fourth by 1, the ſumm of the four Products may make 94; therefore by putting a, e, y, u to repreſent thoſe four whole numbers, the Queſtion may be ſtated thus;

1. If . . . . $a + e + y + u = 20$
2. And . . . . $6a + 4e + 3y + u = 94$

What are the whole numbers a, e, y, u? ||

RESOLVTION.

The firſt Scope is to ſearch out limits for the number a in like manner as before in the twentieth Queſtion, *viz.*

3. By tranſpoſition of a in the firſt Equation, this ariſeth, $e + y + u = 20 - a$
4. Likewiſe by tranſpoſition of $6a$ in the ſecond Equation, there comes forth . . . . $4e + 3y + u = 94 - 6a$
5. The third Equation multiplied by 1, (to wit, the ſmalleſt of the numbers prefixt to the letters in the firſt part of the fourth Equation, where 1 is ſuppoſed to be prefixt to u,) doth produce the ſame third, *viz.* $e + y + u = 20 - a$
6. Again, the third Equation multiplied by 4, to wit, the greateſt of the numbers prefixt to the letters in the firſt part of the fourth Equation, doth produce $4e + 4y + 4u = 80 - 4a$
7. It is manifeſt that the firſt part of the fifth Equation is leſs than the firſt part of the fourth, therefore alſo the latter part of the fifth ſhall be leſs than the latter part of the fourth, *viz.* . . . . $20 - a \sqsupset 94 - 6a$
8. Therefore from the ſeventh ſtep, after due Reduction, it follows that . . . . $a \sqsupset 14\frac{4}{5}$
9. Again, for as much as the firſt part of the ſixth Equation is greater than the firſt part of the fourth, therefore alſo the latter part of the ſixth ſhall be greater than the latter part of the fourth, *viz.* . . . . $80 - 4a \sqsubset 94 - 6a$
10. Therefore from the ninth ſtep, after due Reduction, it follows, that . . . . $a \sqsubset 7$

Now ſince 'tis found by the tenth and eighth ſteps, that a, (or the number of Men,) is greater than 7, but leſs than $14\frac{4}{5}$, let ſome whole number within thoſe limits be taken for the value of a, viz.

11. Suppoſe . . . . $12 = a$
12. Then by ſetting 12 in the place of a in the firſt Equation, this ariſeth, ... $12 + e + y + u = 20$
13. Whence by equal ſubtraction of 12, there remains $e + y + u = 8$
14. And by multiplying the Equation in the eleventh ſtep by 6, it makes . . . . $72 = 6a$
15. Then by ſetting 72 in the place of $6a$ in the ſecond Equation, it gives . . . . $72 + 4e + 3y + u = 94$
16. And by ſubtracting 72 from each part of the laſt Equation, the Remainder is . . . . $4e + 3y + u = 22$
17. The Equation in the thirteenth ſtep being multiplied by 4, (which is prefixt to e in the ſixteenth,) gives $4e + 4y + 4u = 32$
18. Then by ſubtracting the Equation in the ſixteenth ſtep from that in the ſeventeenth, the letter e vaniſheth, and this Equation remains, ... $y + 3u = 10$

19. Whence

19. Whence by tranſpoſition of $3u$, this Equation ariſeth, ... $y = 10 - 3u$
20. Then by ſetting the latter part of the Equation in the nineteenth ſtep in the place of y in the thirteenth, this ariſeth, ... $e + 10 - 3u + u = 8$
21. Whence, after due Reduction, this Equation ariſeth, $e = 2u - 2$
22. From the latter part of the nineteenth Equation, it may be inferr'd that . . . . $u \sqsupset 3\frac{1}{3}$
23. And from the latter part of the twenty-firſt Equation, ... $u \sqsubset 1$

Now ſince by the twenty-ſecond and twenty-third ſteps, u (or the number of Girls) is found to fall between 1 and $3\frac{1}{3}$, let 2 be taken for the value of u, *viz.*

24. Suppoſe . . . . $u = 2$
25. Then from the nineteenth and twenty-fourth ſteps, $y = 4$ $(= 10 - 3u)$
26. And from the twenty-firſt and twenty-fourth ſteps, $e = 2$ $(= 2u - 2)$

Thus by the eleventh, twenty-ſixth, twenty-fifth and twenty-fourth ſteps, four whole numbers are diſcovered, to wit, 12, 2, 4 and 2, for the values of a, e, y and u.

Again, by taking 3 for the value of u, (which is within the limits before diſcovered) the nineteenth and twenty-firſt ſteps will diſcover 1 and 4 for the values of y and e, (a being 12, as before. Wherefore two Anſwers to the Queſtion are found out; for the number of Men being put 12, the number of Women will be 2, the number of Boys 4, and the number of Girls 2; or the number of Men being 12 as before, there will be four Women, 1 Boy and 3 Girls. Again, if 11 be put equal to a, (or the number of Men,) and the proceſs be repeated from the eleventh ſtep to the end of the Reſolution, there will be found two Anſwers more in whole numbers. In like manner, if 9, 10 and 13 be ſeverally put equal to a, three Anſwers more will be diſcovered; But if 8 and 14 be ſeverally put equal to a, although they be within the limits in the eighth and tenth ſteps, yet the work being repeated as before will not ſucceed to find e, y and u in whole numbers; ſo that there are only ſeven Anſwers, to wit, thoſe inſerted in the Table; but that every one of them will ſolve the Queſtion may eaſily be proved.

a	e	y	u
9	9	1	1
10	6	3	1
11	5	2	2
11	3	5	1
12	2	4	2
12	4	1	3
13	1	3	3

If a Queſtion of this nature be deſired that hath but one Anſwer in whole numbers, let the number of perſons be 60, and 100 the number of ſhillings ſpent; alſo let every Man ſpend 2 ſhillings, every Woman $\frac{2}{3}$ of a ſhilling, every Boy $\frac{1}{3}$ of a ſhilling, and every Girl $\frac{1}{2}$ of a ſhilling; then by forming the Reſolution as before, the number of Men will be found 46, the number of Women 3, the number of Boys 5, and the number of Girls 6.

QUEST. 23.

To divide 200 into five ſuch whole numbers, that if the firſt be multiplied by 12, the ſecond by 3, the third by 1, the fourth by $\frac{1}{2}$, and the fifth by $\frac{1}{3}$, the ſumm of the Products may alſo make 200.

This Queſtion may be reſolved like the foregoing twentyeth and twenty-ſecond, but I ſhall leave it as an exerciſe to the induſtrious Analyſt, who, (if he thinks it to be worth his pains,) may find out 6639 Anſwers to it in whole numbers, (as Monſieur *Bachet*, in the two laſt pages of his little Book before cited in *Sect.* 1. of this Chapter, doth affirm.

Nicholas Tartaglia handling this very Queſtion, (which is the laſt of the ſeventeenth Book of the firſt Part of his Arithmetick,) thought it a great matter that he had found out one ſingle Anſwer to it in theſe five whole numbers, to wit, 6, 12, 34, 52, 96, and aſſerted, That Queſtions of this ſort could not be perfectly ſolved, either by the *Algebraical* Art, or any certain Rule; but the contents of this Chapter do manifeſtly ſhew, that the Imperfection was in the Artiſt, and not in the Art.

The End of the Second BOOK.